NOTES ON ROOT SYSTEMS (TWO ROOTS)

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Contents

Date: October 24, 2024.

Preface

This text contains just some explicit descriptions of the position of two roots, with 4 types of arrangements.

It was written in September 2024 as a prelude to some longer "Notes on root systems".

So there is more to come, hopefully, but who knows.

§1. Two roots

Reference: Serre 2001 (1966) [2, Chapter V. Root systems, p. 24].

1.1. Introduction.

Werte der trigonometrischen Funktionen für 0°, 30°, 45°, 60° und 90°.

(nach Bronstein-Semendjajew[1])

Winkel	Bogen	sin	cos	tan	cot	sec	csc
0°	0	0	1	0	$\mp\infty$	1	$\mp\infty$
30°	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{1}{4}\pi$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{3}\pi$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{1}{2}\pi$	1	0	$\pm\infty$	0	$\pm\infty$	1

1.2. Setup.

Let a, b be two roots in a root system and let

$$m = a^*(b)b^*(a) \in \{0, 1, 2, 3, 4\}$$

(c^* is the coroot of a root c). The case $b = \pm a$ is excluded. When appropriate, the roots are denoted by a_m, b_m to indicate the value of m.

Let $\langle\ ,\ \rangle$ be a scalar product with

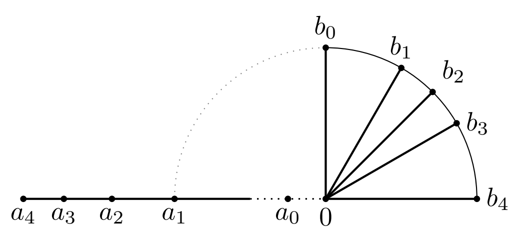
$$c^*(d) = \frac{2\langle c, d\rangle}{\langle c, c\rangle}$$

We assume that the angle between a and b is obtuse (or right). The two roots are drawn in Euclidean space $\mathbf{R}^2$ with a negative on the first axis and b in the upper half plane. Thus

$$a^*(b) \le 0, \qquad a \in \big((-\infty, 0), 0\big), \qquad b \in \big(*, [0, \infty)\big)$$

It follows that b lies in the first quadrant.

1.3. The case $\langle b, b\rangle = 1$, a long, b short.

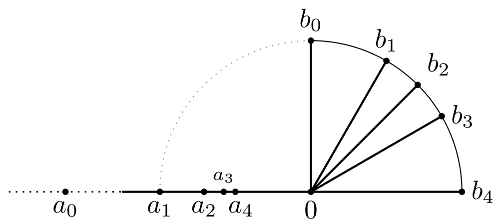


$$b_m^*(a_m) = -m \qquad\qquad a_m^*(b_m) = \begin{cases} 0 & (m = 0) \\ -1 & (m \neq 0) \end{cases}$$

$$b_m = \left(\frac{\sqrt{m}}{2}, \frac{\sqrt{4-m}}{2}\right) \qquad\qquad a_m = \begin{cases} (*, 0) & (m = 0) \\ (-\sqrt{m}, 0) & (m \neq 0) \end{cases}$$

[1]From [1, p. 79]. In a recent edition, the table got split: [3, p. 59, p. 65]

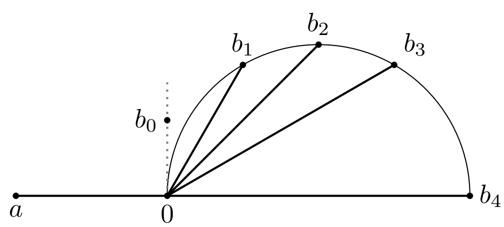
1.4. **The case $\langle b,b\rangle = 1$, a short, b long.** Same as the previous case (Section 1.3), but with reciprocal first coordinate of a.



$$a_m^*(b_m) = -m \qquad b_m^*(a_m) = \begin{cases} 0 & (m=0) \\ -1 & (m \neq 0) \end{cases}$$

$$b_m = \left(\frac{\sqrt{m}}{2}, \frac{\sqrt{4-m}}{2} \right) \qquad a_m = \begin{cases} (*,0) & (m=0) \\ (-\sqrt{m}^{-1}, 0) & (m \neq 0) \end{cases}$$

1.5. **The case $\langle a,a\rangle = 1$, a short, b long.** A customary display of two roots in $\mathbf{R}^2$ is to take $a = (1,0)$ for the short root and b in the upper half plane. Similarly here, but with $a = (-1,0)$.



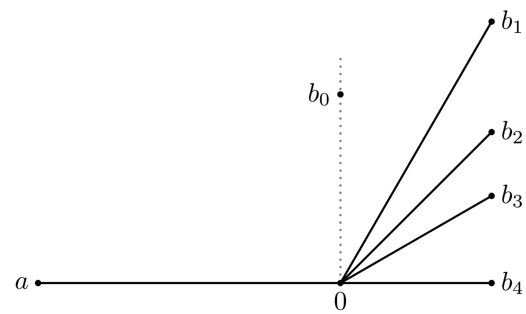
$$a^*(b_m) = -m \qquad b_m^*(a) = \begin{cases} 0 & (m=0) \\ -1 & (m \neq 0) \end{cases}$$

$$b_0 = (0,*), \quad b_m = \left(\frac{m}{2}, \frac{\sqrt{m(4-m)}}{2} \right) \qquad a = (-1,0)$$

The root b_m $(m \neq 0)$ lies on the unit circle around $-a$ since $b_m + a = s_{b_m}(a)$ is the reflection of a at $b_m^\perp$. Moreover $\langle b_m, b_m\rangle = m$. Explicitly:

$$b_0 = (0,*)\,, \quad b_1 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right), \quad b_2 = (1,1)\,, \quad b_3 = \left(\frac{3}{2}, \frac{\sqrt{3}}{2} \right), \quad b_4 = (2,0)$$

1.6. **The case $\langle a,a\rangle = 1$, a long, b short.** The tangent function shows up.



$$b_m^*(a) = -m \qquad\qquad a^*(b_m) = \begin{cases} 0 & (m=0) \\ -1 & (m \neq 0) \end{cases}$$

$$b_0 = (0,*), \quad b_m = \left(\frac{1}{2}, \frac{1}{2}\sqrt{\frac{4-m}{m}}\right) = \left(\frac{1}{2}, \frac{1}{2}\sqrt{3}^q\right) \qquad a = (-1,0)$$

$$\frac{4-m}{m} = 3^q = \begin{cases} \infty & (m=0) \\ 3 & (m=1) \\ 1 & (m=2) \\ \frac{1}{3} & (m=3) \\ 0 & (m=4) \end{cases} \qquad q = \frac{3(2-m)}{m(4-m)} = \begin{cases} +\infty & (m=0) \\ 1 & (m=1) \\ 0 & (m=2) \\ -1 & (m=3) \\ -\infty & (m=4) \end{cases}$$

Here

$$\frac{4-m}{m} = \tan^2 \sphericalangle(a,b) = \tan^2\phi$$

where

$$\phi \in \{90^\circ, 60^\circ, 45^\circ, 30^\circ, 0^\circ\} = \left\{\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, 0\right\}$$

is the acute (or right) angle between the two lines.

As for the funny computation

$$\log_3\left(\frac{4-m}{m}\right) = \frac{3(2-m)}{m(4-m)} \qquad (m \in \{0,1,2,3,4\})$$

on the 5-element set $\{0,1,2,3,4\}$: The function

$$q = \frac{3(2-m)}{m(4-m)}$$

has the correct zero and poles, with the limits $m \to 0,4$ to be taken within the interval $(0,4)$. Further, it changes sign under the involution $m \leftrightarrow 4-m$ and the constant factor ensures the correct value for $m=1$.

1.7. **Three roots with sum null.** If $a^*(b) = -1$, then

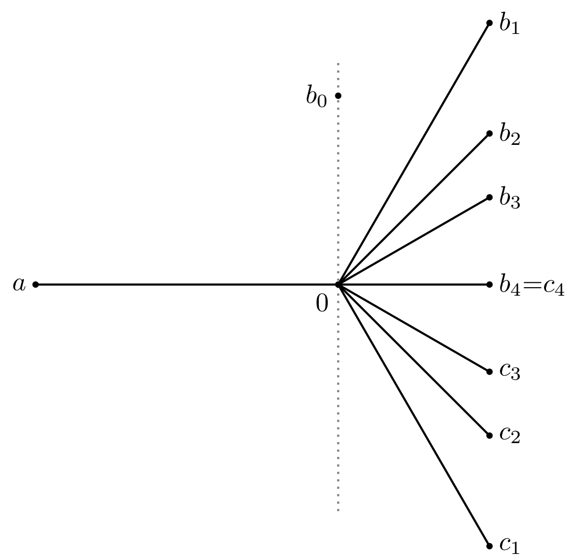
$$s_a(b) = b + a$$

is a root (the reflection of b at $a^\perp$).

With a, b_m ($m \neq 0$) as in the previous section 1.6, the root c_m with

$$a + b_m + c_m = 0$$

is shown here:



(Okay, this is not very spectacular, but "three roots with sum null" will play a role in the notes.)

References

[1] I. N. Bronstein and K. A. Semendjajew, *Taschenbuch der Mathematik: für Ingenieure und Studenten der technischen Hochschulen*, B. G. Teubner Verlagsgesellschaft, Leipzig, 1958. MR 93476 3

[2] J.-P. Serre, *Complex semisimple Lie algebras*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001, Translated from the French by G. A. Jones, Reprint of the 1987 edition. MR 1808366 3

[3] E. Zeidler (ed.), *Springer-Taschenbuch der Mathematik*, 3rd revised and expanded ed., Heidelberg: Springer Spektrum, 2013 (German), Founded by I.N. Bronstein und K.A. Semendjaew, Zbl 1276.00005. 3

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