MOTIVIC HOMOTOPY TYPES OF PROJECTIVE CURVES

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Contents

Introduction

The Motivation from Algebraic Topology. The objects mainly studied in this thesis are smooth projective algebraic curves over an arbitrary field k. If $k = \mathbb{C}$ there is a topological interpretation of such curves given in [Sha94]:
If $X/\mathbb{C}$ is a smooth projective algebraic curve, then the set of $\mathbb{C}$-rational points $X(\mathbb{C})$ is a connected and orientable surface. Furthermore there is a notion of genus for smooth projective algebraic curves which turns out to be the same as the genus for the associated orientable surface (the number of handles attached to the 2-sphere). Since the genus g of an orientable surface can be expressed using the Euler characteristic via $e(X) = 2 - 2g$, it is homotopy invariant. Furthermore the genus gives a full classification of connected orientable triangulable surfaces up to homeomorphism. But for a smooth projective algebraic curve $X/\mathbb{C}$ the associated orientable surface $X(\mathbb{C})$ is indeed triangulable. Therefore the following is true: For two smooth projective algebraic curves X, Y over $\mathbb{C}$ the associated orientable surfaces $X(\mathbb{C})$ and $Y(\mathbb{C})$ are homotopy equivalent if and only if they are homeomorphic. Since connected orientable triangulable surfaces are CW-complexes, the Whitehead Theorem provides: $X(\mathbb{C})$ and $Y(\mathbb{C})$ are weak homotopy equivalent if and only if they are homeomorphic.

A Homotopy Theory for Algebraic Varieties. In the 1990's Fabien Morel and Vladimir Voevodsky developed a homotopy theory on smooth algebraic varieties [MV99], the so-called $\mathbb{A}^1$- (or motivic) homotopy theory. One of its purposes was to give a new construction of motivic cohomology represented by a spectrum via a stable homotopy theory [Voe98]. This is analogous to the representation of reduced cohomology theories via Ω-spectra in algebraic topology.

From a naive point of view two morphisms $f, g : X \to Y$ of smooth algebraic varieties over a ground field k are called $\mathbb{A}^1$-*homotopic* if there is an $\mathbb{A}^1$-*homotopy*, that is a morphism $H : X \times \mathbb{A}^1_k \to Y$, such that

$$f = H_{|X\times 0} : X \xrightarrow{\mathrm{id}\times 0} X \times \mathbb{A}^1_k \xrightarrow{H} Y$$

and

$$g = H_{|X\times 1} : X \xrightarrow{\mathrm{id}\times 1} X \times \mathbb{A}^1_k \xrightarrow{H} Y.$$

A well-working machinery to get an unstable and a stable homotopy theory is the theory of *model categories*. The disadvantage of the naive point of view is, that it alone does not allow an application of this machinery. One demanded property of the category is the existence of all small limits and colimits. Therefore the category of smooth algebraic varieties first has to be enlarged to fulfil this property. It turns out that it is useful to use a known model category to get the new one, namely the model category of simplicial sets. Finally this machinery indeed yields a notion of homotopy and weak homotopy equivalences which generalizes the naive point of view and provides the so-called (weak) $\mathbb{A}^1$- (or motivic) homotopy equivalences.

Motivic Homotopy Types of Projective Curves. The aim of this thesis is to study the homotopy types of smooth projective curves, that is, to give an answer to the question: When are two such curves weak motivic homotopy equivalent?

Motivated by the result about the associated orientable surfaces for curves over $\mathbb{C}$, there is the following

Conjecture. Let X, Y be two smooth projective algebraic curves over an arbitrary field k. Then X and Y are weak motivic homotopy equivalent if and only if they are isomorphic as algebraic curves.

Note that the question is different from the situation of the associated orientable surfaces for $k = \mathbb{C}$: An isomorphism of algebraic curves over $\mathbb{C}$ corresponds to an isomorphism of complex manifolds between the surfaces. That is, a homeomorphism between the surfaces is not enough to get an isomorphisms between the algebraic curves. Furthermore the genus of an algebraic curve does not classify a curve up to isomorphism. For example the moduli space of the genus 1 for an algebraically closed field k is the field k itself (using the j-invariant for elliptic curves) which contains always more than one element. Nevertheless, it will be very helpful if the genus is homotopy invariant. This is true and therefore the proof of the conjecture can deal with every genus of its own.

We get the following

Theorem A. *The conjecture is true if* $g(X) > 0$ *or* $g(Y) > 0$.

Using the homotopy invariance of the genus, we can assume $g(X) = g(Y) > 0$ to prove Theorem A. It turns out that there are only trivial homotopies between curves of genus > 0. Thus there will be no identification under homotopy equivalence of such curves. The main argument of this fact is even true for abelian varieties and therefore the conjecture is also true for abelian varieties.

Unfortunately, the arguments for the curves of genus > 0 cannot work for the curves of genus 0 as we will see. Therefore we will have to use other arguments unless the field is algebraically closed because for these fields there is only one curve of genus 0 up to isomorphism: $\mathbb{P}^1$. That is, Theorem A implies that the conjecture is true for all algebraically closed fields.

Including the curves of genus 0, we get the following

Theorem B. *The conjecture is true for all fields of characteristic* 0.

Due to Theorem A, only the case of genus 0 is left to prove Theorem B. There is a criterion about motivic equivalence of Brauer-Severi varieties in Chow motives due to Nikita A. Karpenko [Kar00] which implies that two curves of genus 0 are motivic equivalent if and only if they are isomorphic as algebraic curves. Thus we have to make a connection between motivic homotopy theory and Chow motives which should yield: If two curves of genus 0 are weak motivic homotopy equivalent, then they are also motivic equivalent. Unfortunately, this connection needs a hypothesis on the field k. We will prove that this hypothesis holds for all fields of characteristic 0, but it seems to hold for all perfect fields. This is the reason for the assumption characteristic 0.

Therefore we will only be able to prove the conjecture for all algebraically closed fields and fields fulfilling a certain hypothesis which are at least the fields of characteristic 0.

Organization

Throughout the whole thesis the theory of categories as it can be found in [Mac98] as well as the basic theory of algebraic geometry as it can be found in [Liu02] are assumed.

In Section 1 we will introduce the objects which are mainly studied in this thesis: Smooth projective curves, abelian varieties, and Brauer-Severi varieties. Furthermore we will describe the main properties of these objects which are necessary to study homotopy types.

In Section 2 we will introduce the notion of (co)simplicial objects which is central for the further theory.

In Section 3 we will introduce the language and theory of model categories, in particular simplicial model categories in subsection 3.5 and Bousfield localization in subsection 3.6. This language is needed to introduce motivic homotopy theory.

In Section 4 we will introduce the language of Grothendieck sites, in particular the notion of (pre)sheaves. Furthermore we will need the associated sheaf functor.

In Section 5 we will introduce several homotopy theories on simplicial (pre)sheaves on an arbitrary Grothendieck site which is the basis for motivic homotopy theory.

In Section 6 we will introduce the Nisnevich site on smooth schemes to use this together with Section 5 and the Bousfield localization of Section 3 to get motivic homotopy theory. Furthermore several important properties of this model category will be outlined, in particular the functoriality in changing the base field. The last subsection 6.3 will deal with the so-called $\mathbb{A}^1$-rigid schemes whose homotopical behavior is the key to understand the homotopy types of curves of genus > 0 and abelian varieties.

The aim of Section 7 is to establish a connection between motivic homotopy theory and Chow motives. Therefore we will introduce Chow motives and Voevodsky's triangulated category of effective motives. We will see that the Chow motives embed fully faithfully in this category of effective motives for all fields of characteristic 0. To see this, we will have to use motivic cohomology. In the last subsection 7.5 we will establish a connection between the motivic homotopy category and the triangulated category of effective motives via a functor which will require a lot of work.

Finally in Section 8 we will study the homotopy types of smooth projective curves, abelian varieties, and Brauer-Severi varieties. In particular we will prove the homotopy invariance of the genus of a curve. The curves of genus $>$ 0 and abelian varieties will be understood through the homotopical behavior of the $\mathbb{A}^1$-rigid schemes. The curves of genus 0 as well as arbitrary Brauer-Severi varieties will be understood using the connection to Chow motives as developed in Section 7.

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1. Projective Curves, Abelian and Brauer-Severi Varieties

The aim of this section is to introduce the objects which will be studied in this thesis: Projective curves, abelian varieties and Brauer-Severi varieties.

1.1. **Basic Notation and Results.** First of all, we have to fix notation and our notion of varieties and curves.

Notation 1.1.1. Let k be a field. Denote by Sch/k the category of k-schemes of finite type.

Definition 1.1.2. Let k be a field and $\overline{k}$ its algebraic closure. A $X \in Sch/k$ is called a *k-variety* if it is a separated k-scheme which is geometrically integral, i.e. $X_{\overline{k}} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ is reduced and irreducible.

Remark 1.1.3. It follows immediately from EGA IV (4.5.1) and (4.6.1), that for every k-variety V and every field extension L/k V_L is also integral over L.

Definition 1.1.4. A k-variety V is called a *group variety* if V is a group scheme over k, i.e. there is given a factorization of the represented functor $\mathrm{Hom}_k(-, V) = \overline{V}$ through the category of groups $\mathcal{G}r$:

$$\begin{tikzcd} & \mathcal{G}r \arrow[d, "F"] \\ (Sch/k)^{op} \arrow[ur, "\overline{V}"] \arrow[r, "\overline{V}"] & \mathcal{S}et \end{tikzcd}$$

where F is the forgetful functor.

Definition 1.1.5. A group variety A over k is called an *abelian variety* if A is complete over k, i.e. it is proper.

Remark 1.1.6. The group structure of an abelian variety is commutative (cf. [Mil86a, Corollary 2.4]).

Definition 1.1.7. We call a k-variety C a *curve* if $\dim(C) = 1$ and C is smooth, i.e. $C_{\overline{k}}$ is regular, and complete over k.

Remark 1.1.8. Let L/k be a field extension of k and C be a curve. Then C_L is also a curve over L since properness and smoothness are preserved by base change and geometrical integrality is preserved by Remark 1.1.3.

Remark 1.1.9. Due to [Liu02, Exercise 7.5.4] every curve C is projective since C is proper by definition. On the other hand every projective curve C/k is proper. Therefore we are going to study exactly the smooth projective curves.

Notation 1.1.10. Let k be a field. The following notations denote full subcategories of Sch/k:

- Red/k reduced k-schemes
- Sm/k smooth, separated k-schemes
- $SmProj/k$ disjoint unions of smooth, projective k-varieties
- Var/k varieties over k
- Gr/k group varieties over k
- Ab/k abelian varieties over k

Remark 1.1.11. The following inclusions hold:

$$\begin{tikzcd}
Ab/k \arrow[r, hook] \arrow[d, hook, "\iota_1"'] & Gr/k \arrow[r, hook] \arrow[d, hook, "\iota_2"'] & Var/k \arrow[r, hook] & Red/k \\
SmProj/k \arrow[r, hook] & Sm/k \arrow[urr, hook, "\iota_3"'] & &
\end{tikzcd}$$

where the inclusions ι_1, ι_2, ι_3 are well known facts and the other inclusions are clear by definition.

The next proposition is a general result about reduced k-schemes which implies, that the equality of two given morphisms between those schemes can be checked on $\overline{k}$-rational points.

Proposition 1.1.12. *Let k be a field and $\overline{k}$ its algebraic closure. Then the functor*

$$\begin{array}{ccc} Red/k & \longrightarrow & \mathcal{S}et \\ X & \mapsto & X(\overline{k}) \end{array}$$

taking $\overline{k}$-rational points is faithful.

Proof. It suffices to check this for affine $X = \mathrm{Spec}(R)$, such that R is a finitely generated k-algebra with $\sqrt{0} = 0$. So let $f, g : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ be two morphisms in Red/k which correspond to morphisms $f, g : S \to R$, such that the induced maps $f_*, g_* : \mathrm{Spec}(R)(\overline{k}) \to \mathrm{Spec}(S)(\overline{k})$ coincide. We have to show that $f = g$. But $\mathrm{Spec}(R)(\overline{k}) = \mathrm{Hom}_k(R, \overline{k})$ and the same for S, hence the induced morphisms $f^*, g^* : \mathrm{Hom}_k(R, \overline{k}) \to \mathrm{Hom}_k(S, \overline{k})$ coincide. Consider the morphism $f - g : S \to R$ of abelian groups. Denote $R = k[X_1, \dots, X_n]/(\gamma_1, \dots, \gamma_m)$. We have to show that $(f - g)(h) = 0 \in R$ for all $h \in S$. By assumption $(f - g)(h)(\alpha_1, \dots, \alpha_n) = 0$ for all $(\alpha_1, \dots, \alpha_n) \in \overline{k}^n$ such that $\gamma_i(\alpha_j) = 0$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$ because such an $(\alpha_1, \dots, \alpha_n)$ is the same as a morphism $R \to \overline{k}$ and $(S \xrightarrow{f-g} R \to \overline{k}) = 0$ for all morphisms $R \to \overline{k}$ because $f^* = g^*$. According to the Hilbert Nullstellensatz [Mat86, Theorem 5.4] $(f - g)(h)^r = 0$ for an $r \geq 0$, i.e. $(f - g)(h) \in \sqrt{0} = 0$. $\square$

1.2. **The Genus and the Jacobian of a Curve.** In this section the genus and the Jacobian variety of a curve will be introduced.

The genus is a quite coarse invariant for curves which will allow us to study every genus for itself in principle. In particular, we are interested in curves of genus 0 which are called (smooth) *conics* and the curves of genus 1 are called *elliptic curves*.

The Jacobian variety of a curve is an abelian variety which carries a lot of geometric information about the curve. For example, the genus of a curve coincides with the dimension of its Jacobian.

Definition 1.2.1 (Genus of a curve)**.** Let C be a curve over k. Then denote by

$$g = g(C) := \dim_k H^1(C, \mathcal{O}_C)$$

the *genus* of the curve C where $\mathcal{O}_C$ is the structure sheaf of C.

Remark 1.2.2. The genus is invariant under arbitrary field extensions, i.e. $g(C_L) = g(C)$ for every field extension L/k (cf. [Liu02, Definition 7.3.19]).

Lemma 1.2.3. *Let $f : X \to Y$ be a finite morphism of curves. Then $g(X) \geq g(Y)$.*

Proof. First of all, our curves are geometrically reduced by definition. Furthermore they are normal because they are smooth and therefore [Liu02, Corollary 7.4.19 and Proposition 7.4.21] give the claim. □

Now we are going to introduce the Jacobian variety of a curve. To do this, we have to define the relative Picard functor.

Definition 1.2.4. Let C be a curve over k. The functor

$$P^0_C(T) = \{\mathcal{L} \in \mathrm{Pic}(C \times_k T) \mid deg(\mathcal{L}_t) = 0 \ \forall t \in T\}/q^*\mathrm{Pic}(T)$$

on Sch/k is called the *relative Picard functor* where $q : C \times T \to T$ is the second projection, $\mathrm{Pic}(-)$ the Picard group, and $\mathcal{L}_t$ the fibre of the line bundle $\mathcal{L}$ over the point t.

Lemma 1.2.5. *Let C be a curve over k. Then there is a finite separable field extension L/k, such that $C(L) \neq \emptyset$.*

Proof. Due to [Liu02, Proposition 3.2.20] $C(k^s) \neq \emptyset$, where k^s is the separable closure of k, since C is geometrically reduced. C is also of finite type over k and therefore a k^s-rational point locally looks like $k[X_1, \ldots, X_n]/I \to k^s$, i.e. it corresponds to $(a_1, \ldots, a_n) \in (k^s)^n$ which are roots of $I = (f_1, \ldots, f_k)$. Let us take $L := k(a_1, \ldots, a_n)$. Then L/k is finite and separable and we have an L-rational point $k[X_1, \ldots, X_n]/I \to L$. □

Theorem 1.2.6 (Existence of the Jacobian variety)**.** *Let C be a curve over k. Then there is a finite separable field extension L/k, such that the relative Picard functor $P^0_{C_L}$ on Sch/L is representable. Furthermore the representing L-scheme is an abelian variety which is called the* Jacobian variety *of C and denoted by* Jac_C.

Proof. Due to the previous lemma there is a finite separable field extension L/k, such that $C(L) \neq \emptyset$. Then the existence and the properties of the Jacobian variety are covered in [Mil86b, §4]. □

Here comes a very powerful property of the Jacobian variety: It provides the coincidence of the genus if the Picard group is the same.

Proposition 1.2.7. *Let X, Y be curves over k, such that $\mathrm{Pic}(X_L) \cong \mathrm{Pic}(Y_L)$ as abstract groups for all finite separable field extensions L/k. Then $g(X) = g(Y)$.*

Proof. Due to Theorem 1.2.6 we can assume that the Jacobians of X and Y exist. Take a number n such that $(\mathrm{char}(k), n) = 1$ and a finite separable field extension L/k such that all rational points of $\mathrm{Jac}_X[n]$ and $\mathrm{Jac}_Y[n]$ exist (these are finitely many which live in separable field extensions by [Mil86a, Remark 8.4]). It is clear from the definition that $\mathrm{Jac}_{C_L}(L) = \mathrm{Pic}^0(C_L)$. Furthermore $\dim(\mathrm{Jac}_C) = g(C)$ by [Mil86b, Proposition 2.1] and

$$\mathrm{Pic}^0(C_L)[n] = \mathrm{Jac}_C[n](L) \cong (\mathbb{Z}/n\mathbb{Z})^{2g(C)}$$

for $C = X, Y$ by [Mil86a, Remark 8.4]. But $\mathrm{Pic}(C_L) \cong \mathrm{Pic}^0(C_L) \oplus \mathbb{Z}$ and therefore $\mathrm{Pic}^0(C_L)[n] = \mathrm{Pic}(C_L)[n]$. Hence we have

$$(\mathbb{Z}/n\mathbb{Z})^{2g(X)} \cong \mathrm{Pic}(X_L)[n] \cong \mathrm{Pic}(Y_L)[n] = (\mathbb{Z}/n\mathbb{Z})^{2g(Y)}$$

and therefore $g(X) = g(Y)$. □

1.3. **Curves of Genus > 0 and Abelian Varieties.** For our purpose, the main tool to understand curves of genus > 0 and abelian varieties is the following proposition.

Proposition 1.3.1. *Let $X \in Sm/k$ be a curve of genus $g > 0$ or an abelian variety. Then every morphism $\mathbb{A}^1 \to X$ is constant.*

Proof. First of all, due to [Mil86a, Lemma 3.2] every morphism $\mathbb{A}^1 \to X$ extends to a morphism

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r] & X \\ \mathbb{A}^1 \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

because every curve and every abelian variety over k is complete over k by definition. Therefore it suffices to show that every morphism $\mathbb{P}^1 \to X$ has to be constant. This is true for abelian varieties by [Mil86a, Corollary 3.8]. So let X be a curve of genus $g > 0$. Assume that there is a non-constant morphism $\mathbb{P}^1 \to X$ of curves. Then f is finite according to [Liu02, Lemma 7.3.10]. Using Lemma 1.2.3 it follows that $0 = g(\mathbb{P}^1) \geq g(X)$ which is impossible. □

1.4. **Brauer-Severi Varieties.** In this section we will introduce Brauer-Severi varieties which are essentially the same as Azumaya algebras. This correspondence is the key to understand curves of genus 0 which will be outlined in the next subsection. The approach presented here is mainly taken from [Ser79] and [Jah00], but is also appears in [Ker90, Ch.30].

Definition 1.4.1. Let $V \in Sch/k$. Then V is called a *Brauer-Severi variety of dimension n* if there is a finite, separable field extension L/k, such that $V_L \cong \mathbb{P}^n_L$. Any L with this property is called a *splitting field* for X.

Remark 1.4.2. Note that every Brauer-Severi variety is indeed a smooth projective k-variety. Furthermore, because of the existence of the normal hull, a splitting field can always be chosen to be a finite Galois extension L/k.

Proposition 1.4.3. *Let V be a Brauer-Severi variety over k. Then*

$$V(L) \neq \emptyset \iff L/k \text{ is a splitting field for } V.$$

Proof. This is due to Châtelet (cf. [Ser79, Ch.X §6 Exc. 1] or cf. [Jah00, Theorem 4.5]). □

Our first aim is to quote a correspondence between isomorphism classes of Brauer-Severi varieties and certain non-abelian group cohomology classes. Confer [Ser79] for the theory of non-abelian group cohomology.

Notation 1.4.4. Denote by BS_n^k the isomorphism classes of Brauer-Severi varieties of dimension n over k. Furthermore for L/k a field extension $\mathrm{BS}_n^{L/k} \subseteq \mathrm{BS}_n^k$ denotes the elements, such that L is a splitting field.

Remark 1.4.5. It is clear by definition that

$$\mathrm{BS}_n^k = \bigcup_{\substack{L/k \\ \text{fin.sep.}}} \mathrm{BS}_n^{L/k}.$$

The following theorem is given in [Ser79, Ch.X §6] (or cf. [Jah00, Theorem 4.5 and Lemma 4.6]).

Theorem 1.4.6. *Let L/k be a finite Galois extension and $G := \mathrm{Gal}(L/k)$ the Galois group. Then there is a natural bijection of pointed sets*

$$\alpha_{n-1}^{L/k} : \mathrm{BS}_{n-1}^{L/k} \xrightarrow{\cong} H^1(G, \mathrm{PGL}_n(L))$$

which is natural in extensions $L'/L/k$ where L'/k is also Galois.

Corollary 1.4.7. *Let k^s be the separable closure of k and $G_k = \mathrm{Gal}(k^s/k)$ the absolute Galois group. Then there is a natural bijection*

$$\alpha_{n-1}^{k} : \mathrm{BS}_{n-1}^{k} \xrightarrow{\cong} H^1(G_k, \mathrm{PGL}_n(k^s))$$

Proof. This follows immediately from the previous theorem and the definition of cohomology of profinite groups if $\mathrm{PGL}_n(k^s)^{\mathrm{Gal}(k^s/k')} = \mathrm{PGL}_n(k')$ for every intermediate field $k \subseteq k' \subseteq k^s$. But this is true since the exact sequence

$$1 \to (k^s)^* \to \mathrm{GL}_n(k^s) \to \mathrm{PGL}_n(k^s) \to 1$$

induces the exact sequence

$$1 \to (k')^* \to \mathrm{GL}_n(k') \to \mathrm{PGL}_n(k^s)^{\mathrm{Gal}(k^s/k')} \to H^1(\mathrm{Gal}(k^s/k'), (k^s)^*)$$

in cohomology $H^i(\mathrm{Gal}(k^s/k'), -)$ where the last entry vanishes by Hilbert's Theorem 90. □

Definition 1.4.8. Let k be a field. A finte dimensional central simple algebra A over k is called an *Azumaya algebra*. Every field L/k with the property that $A \otimes_k L \cong M_n(L)$ is called a *splitting field* for A.

Remark 1.4.9. Note that these algebras do not have to be commutative in contrast to the k-algebras appearing in algebraic geometry. Recall from the theory of central simple algebras (cf. [Ser79, Ch.X §5 Proposition 7]) that there is always a finite separable splitting field L/k. If $A \otimes_k L \cong M_n(L)$, then A is said to have the *dimension* n^2 over k and the *degree* n. Furthermore the structure Theorem of Wedderburn yields an unique skew field D, such that $A \cong M_n(D)$. Denote by $\mathrm{ind}(A) := \dim_k(D)$ the *index* of A. Note that there is always a splitting field L/k, such that $\mathrm{ind}(A) = \dim_k(L)$.

Azumaya algebras over a field k lead to the notion of the Brauer group of the same field k. Its properties can be found in [Ker90, Ch.6].

Definition 1.4.10 (Brauer group)**.** Let k be a field. Define an equivalence relation on the set of Azumaya algebras over k by: For A, B Azumaya algebras over k define

$$A \sim B :\Longleftrightarrow \text{there are } r, s \in \mathbb{N} \text{ such that } A \otimes_k M_r(k) \cong B \otimes_k M_s(k)$$

Now define the *Brauer group of* k denoted by $\mathrm{Br}(k)$ as the set of the equivalence classes of Azumaya algebras over k with respect to this equivalence relation. The group law is given by

$$[A] + [B] = [A \otimes_k B].$$

The neutral element is $[k] = \{M_n(k) \mid n \in \mathbb{N}\}$ and the inverse of a class $[A]$ is given by the class of the opposite algebra $[A^{op}]$.

Our next aim is to quote a correspondence between isomorphism classes of Azumaya algebras and the same non-abelian group cohomology classes as for the Brauer-Severi varieties. This will provide the demanded correspondence between Brauer-Severi varieties and Azumaya algebras.

Notation 1.4.11. Denote by Az_n^k the isomorphism classes of Azumaya algebras of dimension n^2 over k. Furthermore for L/k a field extension $\mathrm{Az}_n^{L/k} \subseteq \mathrm{Az}_n^k$ denotes the elements such that L is a splitting field.

Remark 1.4.12. It follows immediately that

$$\mathrm{Az}_n^k = \bigcup_{\substack{L/k \\ \text{fin.sep.}}} \mathrm{Az}_n^{L/k}.$$

The following theorem is given in [Ser79, Ch.X §5 Proposition 8] (or cf. [Jah00, Theorem 3.6 and Lemma 3.7]).

Theorem 1.4.13. *Let L/k be a finite Galois extension and $G := \mathrm{Gal}(L/k)$ the Galois group. Then there is a natural bijection of pointed sets*

$$\alpha_{n-1}^{L/k} : \mathrm{Az}_n^{L/k} \xrightarrow{\cong} H^1(G, \mathrm{PGL}_n(L))$$

which is also natural in extensions $L'/L/k$ where L'/k is also Galois.

With the same argument as for the BS we have

Corollary 1.4.14. *Let k^s be the separable closure of k and $G_k = \mathrm{Gal}(k^s/k)$ the absolute Galois group. Then there is a natural bijection*

$$\alpha_{n-1}^{k} : \mathrm{Az}_n^{k} \xrightarrow{\cong} H^1(G_k, \mathrm{PGL}_n(k^s))$$

Bringing it all together (cf. [Jah00, Theorem 5.1]) we can conclude

Proposition 1.4.15. *There is a bijection*

$$\begin{array}{ccc} \mathrm{Az}_n^k & \xrightarrow{\cong} & \mathrm{BS}_{n-1}^k \\ A & \mapsto & X_A \end{array}$$

with the following properties.

1. *For k'/k a field extension we have $X_{A\otimes_k k'} \cong (X_A)_{k'}$*
2. *L/k is a splitting field for A if and only if it is a splitting field for X_A*

1.5. **Curves of Genus** 0. The key to understand curves of genus 0 is their characterization as Brauer-Severi varieties which are associated to Azumaya algebras as we have seen. To prove that curves of genus 0 are indeed Brauer-Severi varieties we need the following proposition which is [Liu02, Proposition 7.4.1].

Proposition 1.5.1. *Let C be a curve of genus 0. Then*

$$C(k) \neq \emptyset \Longleftrightarrow C \cong \mathbb{P}^1$$

Corollary 1.5.2. *Curves of genus 0 are exactly the Brauer-Severi varieties of dimension 1.*

Proof. Let V be a Brauer-Severi variety of dimension 1, i.e. V is a curve, then $V_L \cong \mathbb{P}^1$ for a field extension L/k. Hence V has genus 0 over L and therefore also genus 0 over k since the genus is invariant under change of the base field.

Now let C be a curve of genus 0. Due to Lemma 1.2.5 there is a finite separable field extension L/k, such that $C_L(L) \neq \emptyset$. Therefore $C_L \cong \mathbb{P}^1_L$ by Proposition 1.5.1. □

Remark 1.5.3. The corollary immediately implies that BS_1^k consists exactly of the isomorphism classes of curves of genus 0 over k.

Using Proposition 1.4.15 we get

Lemma 1.5.4. *There is a bijection*

$$\{C \mid C \text{ curve over } k,\ g(C) = 0\}/_{\cong} \longrightarrow \mathrm{Az}_2^k,$$

i.e. two curves of genus 0 are isomorphic if and only if their associated Azumaya algebras of dimension 4 are isomorphic.

The next question is: How do these Azumaya algebras look like? In fact they are well known, namely they are quaternion algebras as we will see.

Definition 1.5.5. Let $\mathrm{char}(k) \neq 2$ and $a, b \in k^*$. A *quaternion algebra* $Q = (a, b)$ over k is a 4-dimensional k-algebra with the presentation

$$Q = (a, b) = \langle u, v \mid u^2 = a, v^2 = b, uv = -vu \rangle.$$

Now let $\mathrm{char}(k) = 2$ and $d \in k, a \in k^*$. A *quaternion algebra* $Q = [d, a)$ over k is a 4-dimensional k-algebra with the presentation

$$Q = [d, a) = \langle u, v \mid u^2 + u = d, v^2 = a, vu + uv = v \rangle.$$

Remark 1.5.6. All quaternion algebras can be realized as subalgebras of matrix algebras as follows:
For $\mathrm{char}(k) \neq 2$ let $a, b \in k^*$. If a is a square in k, $Q = (a, b) = M_2(k)$ splits. If a is no square in k, let $L = k(x)$ where $x^2 - a = 0$. This is a Galois extension with Galois group $G = \mathbb{Z}/2\mathbb{Z} = \{\mathrm{id}, c\}$ where $c(l) = \bar{l}$ is the conjugation. Note that $x + \bar{x} = 0$ since $(t - x) \cdot (t - \bar{x}) = t^2 - a \in L[t]$. Then $Q = (a, b)$ can be realized as the full subalgebra of $M_2(L)$ generated by the two matrices

$$u = \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}, v = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$

For $\mathrm{char}(k) = 2$ and given $d \in k, a \in k^*$ $Q = [d, a)$ can also be realized as follows: If $d \in \mathcal{P}(k) := \{\gamma^2 + \gamma \mid \gamma \in k\}$ then $Q = M_2(k)$ splits (cf. [Sch85, Ch.8 §11]). If $d \notin \mathcal{P}(k)$ then $L = k(x)$ where $x^2 + x + d = 0$ is a Galois extension with Galois group $G = \mathbb{Z}/2\mathbb{Z} = \{\mathrm{id}, c\}$ where $c(l) = \overline{l}$ is the conjugation. Note that $x+\overline{x} = -1 = 1$ since $(t-x)\cdot(t-\overline{x}) = t^2+t+d \in L[t]$. Then Q can be realized as the full subalgebra of $M_2(L)$ generated by the two matrices

$$u = \begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix}, v = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}.$$

Lemma 1.5.7. *An Azumaya algebra of dimension* 4 *over* k *is the same as a quaternion algebra over* k.

Proof. Let Q be a quaternion algebra. [Sch85, Ch.8 Examples 12.3 and Theorem 12.2] implies that Q is a central simple algebra. Now let A be a 4-dimensional Azumaya algebra. Then it is shown in [Sch85, Ch.8 §11] that A is a quaternion algebra. □

Lemma 1.5.8. *For all quaternion algebras* Q

$$Q^{op} \cong Q$$

where $(-)^{op}$ *denotes the opposite algebra.*

Proof. Let first be $\mathrm{char}(2) \neq 0$, i.e. $Q = (a, b)$. Then [Sch85, Ch.8 Examples 12.3 and Theorem 12.7] implies that $(a, b) \otimes (a, b) \sim (a, b^2)$ where $\sim$ is the relation building up the Brauer group $\mathrm{Br}(k)$. Furthermore [Sch85, Lemma 12.6] gives us that (a, b^2) splits, i.e. is zero in $\mathrm{Br}(k)$, because $N_{L/k}(b) = b\overline{b} = b^2$ and therefore $b^2 \in N_{L/k}(L^*)$. But $(a, b)^{op}$ is the inverse of (a, b) in $\mathrm{Br}(k)$ and therefore $(a, b) \sim (a, b)^{op}$. But since these two central simple algebras have the same dimension it follows that $(a, b) \cong (a, b)^{op}$ (cf. [Ser79, Ch.X §5]). The same argument works for $\mathrm{char}(k) = 2$, i.e. $Q = [d, a)$, since [Sch85, Ch.8 Examples 12.3 and Theorem 12.7] also implies that $[d, a) \otimes [d, a) \sim [d, a^2)$ and $[d, a^2)$ again splits because of [Sch85, Lemma 12.6]. □

Remark 1.5.9. Note that there is another proof of this lemma: First of all, a quaternion algebra does not split if and only if it is a skew field, that is, $\mathrm{ind}(Q) = 2$ if Q does not split and $\mathrm{ind}(Q) = 1$ if it splits. Now the result follows using [Ker90, Satz 15.1].

Proposition 1.5.10. *Let* C_1, C_2 *be two curves of genus* 0 *over* k *with associated quaternion algebras* Q_1 *and* Q_2. *Then*

$$C_1 \cong C_2 \Longleftrightarrow Q_1 \cong Q_2 \text{ or } Q_1 \cong Q_2^{op}.$$

Proof. This follows immediately from the three preceding lemmas. □

2. Simplicial Objects

2.1. **Basic Notation and Results.**

Definition 2.1.1. Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is called *complete* if all small limits exist. Dually it is called *cocomplete* if all small colimits exist. Finally $\mathcal{C}$ is called *bicomplete* if it is both complete and cocomplete.

Remark 2.1.2. Every bicomplete category has an initial and a terminal object, since the initial object is $\operatorname{colim}_{\emptyset}$ and the terminal object is $\lim_{\emptyset}$. Denote an initial object by 0 and a terminal object by $*$.

Example 2.1.3. It is well known that the category of sets $\mathcal{S}et$ is bicomplete.

Definition 2.1.4. Denote by Δ the category which is defined by:

$$\mathrm{Ob}(\Delta) = \{[n] = (0 < \ldots < n) \mid n \in \mathbb{N}\}$$

are ordered finite sets and a morphism $f : [m] \to [n]$ is a weak monotone map of ordered sets, i.e. $i < j \Rightarrow f(i) \leq f(j)$. Denote by $\delta_i : [n] \to [n+1]$ the unique injective montone map whose image does not contain i.

Definition 2.1.5. Let $\mathcal{C}$ be a category. Then a *simplicial object X in $\mathcal{C}$* is a functor $X : \Delta^{op} \longrightarrow \mathcal{C}$. Denote $X([n]) = X_n$. Simplicial objects form a category, namely a functor category which is denoted by $s\mathcal{C}$. Furthermore a *cosimplicial object C in $\mathcal{C}$* is a functor $C : \Delta \longrightarrow \mathcal{C}$. Denote $C([n]) = C_n$. The category of cosimplicial objects is denoted by $c\mathcal{C}$. A (co)simplicial object X is called *discrete* if $X_n = X_0$ for all $n \in \mathbb{N}$ and the induced morphisms $X_n \to X_m$ are always the identity. Furthermore denote the image of δ_i under a (co)simplicial object X by $\partial_i := X(\delta_i)$.

Remark 2.1.6. There are canonical fully faithful embeddings

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & s\mathcal{C} \\ \mathcal{C} & \hookrightarrow & c\mathcal{C} \end{array}$$

where an object $X \in \mathcal{C}$ is mapped to the canonical discrete (co)simplicial object.

Remark 2.1.7. If $\mathcal{C}$ is bicomplete, then so are $s\mathcal{C}$ and $c\mathcal{C}$: Functor categories with a bicomplete target are again bicomplete since all limits and colimits can be formed objectwise.

Example 2.1.8. The categories of simplicial sets $s\mathcal{S}et$ and simplicial abelian groups $s\mathcal{A}b$ are bicomplete.

Notation 2.1.9. Let K be a simplicial set. Denote by $\mathbb{Z}[K]$ the composition

$$\Delta^{op} \xrightarrow{K} \mathcal{S}et \xrightarrow{\mathbb{Z}[-]} \mathcal{A}b$$

called the *associated free simplicial abelian group.*

Remark 2.1.10. Of course we get an adjoint functor pair

$$\mathbb{Z}[-] : s\mathcal{S}et \rightleftarrows s\mathcal{A}b : F$$

where F is the forgetful functor.

2.2. **Realization and Simplicial Complex.**

Notation 2.2.1. Denote by $\Delta^n := \mathrm{Hom}_{\Delta}(-, [n])$ the standard n-simplex in $s\mathcal{S}et$. Furthermore denote by $\partial\Delta^n \subset \Delta^n$ the boundary of Δ^n and by $\Lambda^n_k \subset \Delta^n \in s\mathcal{S}et$ the k-th horn which is the boundary of Δ^n without the k-th face.

Let $\mathcal{C}$ be a bicomplete category and $C : \Delta \longrightarrow \mathcal{C}$ a cosimplicial object. Then C induces a realization functor

$$|-| : s\mathcal{S}et \longrightarrow \mathcal{C}$$

where $|K|$ is the coequalizer in $\mathcal{C}$ of the canonical diagram

$$\coprod_{f:[n]\to[m]} K_n \times C_m \rightrightarrows \coprod_n K_n \times C_n$$

where $S \times K$ for a set S and an object $X \in \mathcal{C}$ means the discrete product, that is the coproduct $\amalg_{i \in S} X$. The two maps are induced by the maps $K(f)$ and $C(f)$ for $f : [n] \to [m]$ via

$$K_m \times C_n \xrightarrow{K(f)\times \mathrm{id}} K_n \times C_n$$

and

$$K_m \times C_n \xrightarrow{\mathrm{id}\times C(f)} K_m \times C_m.$$

Furthermore C induces a simplicial function complex functor

$$S : \mathcal{C} \longrightarrow s\mathcal{S}et$$

where $S(X)_n = \mathrm{Hom}_{\mathcal{C}}(C_n, X)$. It is easy to see that $|\Delta^n| = C_n$ and that

$$|-| : s\mathcal{S}et \rightleftarrows \mathcal{D} : S$$

is an adjoint functor pair.

Remark 2.2.2. Think about the geometric realization $|K|$ as the objects of the cosimplicial object glued together via the construction plan contained in the simplicial set K.

Notation 2.2.3. Denote by $\mathcal{T}op$ the category of topological spaces and by ∇^n the standard n-simplex in $\mathcal{T}op$.

Remark 2.2.4. The cosimplicial object $C : \Delta \longrightarrow \mathcal{T}op$ with $C(n) = \nabla^n$ induces the well-known geometric realization functor $|-|$ and the simplicial complex S which give the adjoint functor pair

$$|-| : s\mathcal{S}et \rightleftarrows \mathcal{T}op : S$$

with the property that $|\Delta^n| = \nabla^n$.

3. Model Categories

In this section the concept of model categories will be introduced. This is a method to describe and understand notions of homotopy and weak equivalences in several categories in the same language, for example:

- $\mathcal{T}op$: Topological spaces with the usual homotopy relation via $I = [0,1]$ and weak homotopy equivalences.
- $s\mathcal{S}et$: Simplicial sets with the homotopy relation via Δ^1 which is not in general an equivalence relation and weak equivalences via the geometric realization functor $|-|$.
- CH_R: Chain complexes of R-modules with the usual homotopy relation and quasi-isomorphisms as weak equivalences.

In particular, this method gives a good description for the homotopy category for all objects, even if the homotopy relation is not in general an equivalence relation.

Furthermore this is a method to do homotopy theory. That is, if there is the structure of a model category, this machinery provides a notion of homotopy and weak equivalences.

We will need this to introduce a homotopy theory on Sm/k, the so-called motivic (or $\mathbb{A}^1$-) homotopy theory.

The concept of model categories was first introduced by Quillen in [Qui67] but the approach presented here is mainly taken from [Hov99] up to the subsection about simplicial model categories which is taken from [GJ99] and the subsection about Bousfield localization which is taken from [Hir03].

3.1. Basic Notation and Results.

Definition 3.1.1. Let $\mathcal{C}$ be a category.

(1) A map $f \in \mathcal{C}$ is called a *retract* of a map $g \in \mathcal{C}$ if this is the case as objects in the category of arrows $\mathrm{Ar}(\mathcal{C})$ where the morphisms are commutative squares.

(2) Let $i : A \to B$ and $p : X \to Y$ be morphisms in $\mathcal{C}$. Then i has the *left lifting property with respect to* p and p has the *right lifting property with respect to* i if for every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{C}$ there is a *lift*, i.e. a morphism $h : B \to Y$ which makes both triangles commutative.

Definition 3.1.2 (Model category)**.** Let $\mathcal{C}$ be a bicomplete category (this axiom is called **MC1**). $\mathcal{C}$ is called a *model category* if it is equipped with three classes of morphisms, called cofibrations, fibrations and weak equivalences which are all closed under composition and contain all identities and satisfy 4 axioms:

MC2 (2 out of 3) Consider a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

in $\mathcal{C}$. If two of f, g, gf are weak equivalences then so is the third.

MC3 (Retracts) The classes of cofibrations, fibrations and weak equivalences are closed under retracts.

MC4 (Lifting) The acyclic cofibrations have the left lifting property with respect to all fibrations and the acyclic fibrations have the right lifting property with respect to all cofibrations.

MC5 (Factorization) Every morphism $f : X \to Y$ can be functorially factored in two ways:

- $f = p \circ i$ where p is a fibration and i an acyclic cofibration.

- $f = q \circ j$ where q is an acyclic fibration and j a cofibration.

where a morphism is called an *acyclic (co)fibration* if it is both a weak equivalence and a (co)fibration. Such three classes of morphisms of a bicomplete category $\mathcal{C}$ satisfying these 4 axioms are called a *model structure* on $\mathcal{C}$.

Remark 3.1.3. A functorial factorization is described in ([Hov99, Definition 1.1.1]) which essentially states that the factorization is functorial in the category of arrows, that is for a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ B \arrow[r, "g"] & D \end{tikzcd}$$

we get a factorization of f and g through objects A' and B' and an induced morphism $A' \to B'$ which makes the obvious diagram commutative. Further the construction is functorial concerning the induced morphisms.

Notation 3.1.4. Weak equivalences are denoted by $\xrightarrow{\sim}$, cofibrations by $\rightarrowtail$ and fibrations by $\twoheadrightarrow$.

Remark 3.1.5. For a model category $\mathcal{C}$ the opposite category $\mathcal{C}^{op}$ has a canonical model structure, namely take the weak equivalences in $\mathcal{C}^{op}$ as the opposite morphisms of the weak equivalences in $\mathcal{C}$. Furthermore the fibrations in $\mathcal{C}^{op}$ are the opposite morphisms of the cofibrations in $\mathcal{C}$ and the cofibrations in $\mathcal{C}^{op}$ are the opposite morphisms of the fibrations in $\mathcal{C}$.

The following lemma is [Hov99, Lemma 1.1.10].

Lemma 3.1.6. *Let $\mathcal{C}$ be a model category. Then we have the following characterizations of (acyclic) (co)fibrations:*

- *The cofibrations are exactly the maps with the left lifting property with respect to all acyclic fibrations.*
- *The acyclic cofibrations are exactly the maps with the left lifting property with respect to all fibrations.*
- *The fibrations are exactly the maps with the right lifting property with respect to all acyclic cofibrations.*
- *The acyclic fibrations are exactly the maps with the right lifting property with respect to all cofibrations.*

Remark 3.1.7. The lemma immediately implies that a model structure is uniquely determined by the weak equivalences and the fibrations or the weak equivalences and the cofibrations respectively, because the third class is characterized by a lifting property with respect to the two other classes. Therefore sometimes only two classes will be given if a model structure is declared and the third is understood via the lifting property. Of course, it is better to know the third class more explicitly.

Example 3.1.8. The category $\mathcal{T}op$ of topological spaces together with the classes

- w.e. = weak homotopy equivalences
- fib. = Serre fibrations

is a model category (cf. [Hov99, Ch.2.4]).

Example 3.1.9. The bicomplete category $s\mathcal{S}et$ of simplicial sets together with the classes

- w.e. = w.e. in $\mathcal{T}op$ after geometric realization
- fib. = Kan fibrations, i.e. the maps which have the right lifting property with respect to all inclusions $\Lambda^n_k \hookrightarrow \Delta^n$
- cof. = monomorphisms in $s\mathcal{S}et$

is a model category (cf. [Hov99, Ch.3]).

Example 3.1.10. The bicomplete category $s\mathcal{A}b$ of simplicial abelian groups together with the classes

- w.e. =w.e. of the underlying simplicial sets
- fib. =fib. of the underlying simplicial sets

is a model category (cf. [GJ99, III Theorem 2.8]).

Definition 3.1.11. Let $\mathcal{C}$ be a model category and X an object of $\mathcal{C}$. Then X is called

- *fibrant* if the canonical morphism $X \to *$ is a fibration.
- *cofibrant* if the canonical morphism $0 \to X$ is a cofibration.

Remark 3.1.12. Unfortunately not every object of a model category is fibrant and cofibrant. But applying the functorial factorization to the map $0 \to X$ yields a factorization $0 \rightarrowtail QX \overset{\sim}{\twoheadrightarrow} X$ such that QX is cofibrant and $X \mapsto QX$ is a functor. This is called the *cofibrant replacement functor*. The dual works for the maps $X \to *$ which yields a *fibrant replacement functor* $X \mapsto RX$. Because of the 2 out of 3 axiom R and Q preserve weak equivalences.

Definition 3.1.13. Let $\mathcal{C}$ be a model category and X an object of $\mathcal{C}$.

- A *cylinder object* $X \times I$ for X is a factorization of the fold map $X \amalg X \to X$ into a cofibration $X \amalg X \rightarrowtail X \times I$ followed by a weak equivalence $X \times I \to X$.
- A *path object* X^I for X is a factorization of the diagonal map $X \to X \times X$ into a weak equivalence $X \to X^I$ followed by a fibration $X^I \twoheadrightarrow X \times X$.

Remark 3.1.14. Because of the factorization axiom for every object X a cylinder and a path object always exist. Let $f, g : X \to Y$ be two maps in a model category $\mathcal{C}$. Analogous to the situation with topological spaces cylinder objects lead to the notation of *left homotopies* $H : X \times I \to Y$ and path objects lead to the notation of *right homotopies* $H : X \to Y^I$ between f and g (cf. [Hov99, Definition 1.2.4]). Note that different cylinder resp. path objects can lead to different homotopy types. Finally two maps are called *homotopic* if and only if they are both left and right homotopic.

The next proposition (which is [Hov99, Corollary 1.2.6]) describes the good behavior of the homotopy relations for good objects.

Proposition 3.1.15. *Let $\mathcal{C}$ be a model category and let X be a cofibrant and Y a fibrant object of $\mathcal{C}$. Then the left homotopy and right homotopy relations* *on* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ *coincide and do not depend on a choice of a cylinder or a path object. Furthermore, they are equivalence relations on* $\mathrm{Hom}_{\mathcal{C}}(X,Y)$*.*

The next theorem is [Hov99, Proposition 1.2.8] and generalizes the Whitehead Theorem for topological spaces.

Theorem 3.1.16 (Whitehead)**.** *Let* $f : X \to Y$ *be a morphism of a model category* $\mathcal{C}$*, such that* X *and* Y *are both fibrant and cofibrant. Then* f *is a weak equivalence if and only if it is a homotopy equivalence.*

Definition 3.1.17 (Proper model categories)**.** Let $\mathcal{C}$ be a model category.

- $\mathcal{C}$ is called *left proper* if weak equivalences are preserved by cobase change along cofibrations.
- $\mathcal{C}$ is called *right proper* if weak equivalences are preserved by base change along fibrations.
- $\mathcal{C}$ is called *proper* if it is both left and right proper.

The following lemma is given in [GJ99, II Lemma 8.12].

Lemma 3.1.18 (Glueing Lemma)**.** *Let* $\mathcal{C}$ *be a proper model category and*

$$\begin{tikzcd} X \arrow[d, "\sim"'] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"'] & Y \arrow[d, "\sim"] \\ X' & A' \arrow[l] \arrow[r, tail] & Y' \end{tikzcd}$$

a commutative diagram such that all vertical maps are weak equivalences and the horizontal maps on the righthand are cofibrations as indicated. Then the induced map

$$X \amalg_A Y \longrightarrow X' \amalg_{A'} Y'$$

on pushouts is a weak equivalence.

3.2. **The Homotopy Category.** The machinery of model categories provides an explicit (and set theoretically well behaved) description of the localized category $(w.e.)^{-1}\mathcal{C}$ for $\mathcal{C}$ a model category and $w.e.$ the class of weak equivalences.

Definition 3.2.1. (Homotopy category) Let $\mathcal{C}$ be a model category. Then denote by $\mathrm{Ho}(\mathcal{C})$ the *homotopy category* of $\mathcal{C}$ where the objects are the same as the objects of $\mathcal{C}$ but

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(RQX, RQY)/\sim$$

where $\sim$ is the homotopy relation. Denote by $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ the canonical functor called the *localization functor*. It is the identity on objects and a morphism $f : X \to Y$ is mapped to the homotopy class of $RQ(f) : RQX \to RQY$.

Remark 3.2.2. As Proposition 3.1.15 implies, $\sim$ is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(RQX, RQY)$ since X and Y are replaced by objects which are fibrant and cofibrant. Furthermore, by [Hov99, Lemma 1.2.2 and Theorem 1.2.10] one has that $\gamma : \mathcal{C} \longrightarrow \mathrm{Ho}(\mathcal{C})$ is indeed the localization of C with respect to the weak equivalences. To motivate this, we will show that γ takes weak equivalences to isomorphisms. So let $f : X \to Y$ be a weak equivalence. Since the cofibrant and fibrant replacement is weakly equivalent to the original object and using the 2 out of 3 axiom we get a weak equivalence $RQ(f) : RQX \to RQY$. But now the Whitehead Theorem 3.1.16 implies that this is a homotopy equivalence and therefore an isomorphism in $\mathrm{Ho}(\mathcal{C})$.

The following lemma describes the morphisms in the homotopy category between certain objects and is given in [Hov99, Theorem 1.2.10].

Lemma 3.2.3. *Let X be a cofibrant object and Y be a fibrant object of a model category $\mathcal{C}$. Then*

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y)/\sim$$

Notation 3.2.4. Denote the morphisms in $\mathrm{Ho}(\mathcal{C})$ by

$$[X,Y] = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X,Y)$$

3.3. **Quillen Functors and Equivalences.** Now the good functors between model categories are introduced, i.e. they induce functors on the homotopy categories.

Definition 3.3.1 (Quillen functors)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories and

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

an adjoint functor pair.

Then F is called a *left Quillen functor* if it preserves cofibrations and acyclic cofibrations. G is called a *right Quillen functor* if it preserves fibrations and acyclic fibrations. Note that F is a left Quillen functor if and only if G is a right Quillen functor by [Hov99, Lemma 1.3.4].

Finally the pair (F,G) is called a *Quillen functor* if F is a left Quillen functor or, equivalently, G is a right Quillen functor.

Example 3.3.2. The adjoint functor pair

$$\mathbb{Z}[-] : s\mathcal{S}et \rightleftarrows s\mathcal{A}b : F$$

where $\mathbb{Z}[-]$ is the free simplicial abelian group functor and F the forgetful functor is a Quillen functor since F is obviously a right Quillen functor.

The following fundamental lemma which is a key tool for Quillen functors is given in [Hov99, Lemma 1.1.12].

Lemma 3.3.3 (Ken Brown's Lemma)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories and $F : \mathcal{C} \to \mathcal{D}$ a functor. Suppose that F takes acyclic cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes acyclic fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

Corollary 3.3.4. *Let (F,G) be a Quillen functor. Then F preserves cofibrant objects and all weak equivalences between them. Dually, G preserves fibrant objects and all weak equivalences between them.*

Proof. This follows immediately from the definition of a Quillen functor and Ken Brown's Lemma. □

Corollary 3.3.5. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen functor. Then the composition $F \circ Q$ induces a functor*

$$LF : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$$

called the total left derived functor*. Dually, the composition $G \circ R$ induces a functor*

$$RG : \mathrm{Ho}(\mathcal{D}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

called the total right derived functor*. Furthermore*

$$LF : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : RG$$

is an adjoint functor pair.

Proof. The existence of LF follows immediately from the previous corollary and the characterization $\mathrm{Ho}(\mathcal{C}) = (w.e.)^{-1}\mathcal{C}$ and $\mathrm{Ho}(\mathcal{D}) = (w.e.)^{-1}\mathcal{D}$ using the universal property of a localized category:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "Q"] \arrow[d, "\gamma"'] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[rr, dotted, "LF"] & & \mathrm{Ho}(\mathcal{D})
\end{tikzcd}$$

where Q is the cofibrant replacement functor. The existence of RG is dual via $G \circ R$ where R is the fibrant replacement functor. The adjointness of LF and RG is given in [Hov99, Lemma 1.3.10]. □

Remark 3.3.6. Note that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \arrow[d, "\gamma"] \\
\mathrm{Ho}(\mathcal{C}) \arrow[r, "LF"] & \mathrm{Ho}(\mathcal{D})
\end{tikzcd}$$

commutes up to functor isomorphism since the cofibrant replacement $QX \xrightarrow{\sim} X$ becomes an isomorphism in $\mathrm{Ho}(\mathcal{D})$. The dual holds for RG.

Definition 3.3.7 (Quillen equivalences)**.** A Quillen functor (F, G) is called a *Quillen equivalence* if (LF, RG) is an equivalence of categories.

Example 3.3.8. Due to [Hov99, Theorem 3.6.7] the adjoint functor pair of Remark 2.2.4

$$| - | : s\mathcal{S}et \rightleftarrows \mathcal{T}op : S$$

is a Quillen equivalence.

3.4. **Cofibrantly Generated Model Categories.** In this section a special kind of model categories is introduced. Especially there is a recognition thorem which makes it quite easy to identify a certain category with three classes of morphisms as a model category. Quite easy means: It is often easier than to check the original axioms of a model category.

Definition 3.4.1 (Cofibrantly generated model category)**.** Let $\mathcal{C}$ be a model category. Then $\mathcal{C}$ is called a *cofibrantly generated model category* if there exist two sets of maps: I (called the *generating cofibrations*) and J (called the *generating acyclic cofibrations*), such that the following axioms are fulfilled.

(1) I and J permit Quillen's small object argument (cf. [Hov99, Theorem 2.1.14]).
(2) A map is an acyclic fibration if and only if it has the right lifting property with respect to every map of I.
(3) A map is a fibration if and only if it has the right lifting property with respect to every map of J.

Example 3.4.2. The category $s\mathcal{S}et$ with the model structure introduced in Example 3.1.9 is a cofibrantly generated model category with generating cofibrations

$$I = \{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$$

and generating acyclic cofibrations

$$J = \{\Lambda^n_k \to \Delta^n \mid n > 0, 0 \leq k \leq n\}$$

(cf. [Hov99, Theorem 3.6.5]).

Definition 3.4.3. Let I be a class of maps in a cocomplete category $\mathcal{C}$.

(1) A map is *I-injective* if it has the right lifting property with respect to all maps in I. Denote by I-inj all I-injective maps.
(2) A map is an *I-cofibration* if it has the left lifting property with respect to all maps in I-inj. Denote by I-cof all I-cofibrations.
(3) A map is a *relative I-cell complex* if it is a transfinite composition of pushouts of maps in I (cf. [Hov99, Definition 2.1.9]). Denote by I-cell all relative I-cell complexes.

Remark 3.4.4. Of course $I \subseteq I$-cof.

Here is the theorem which yields a recognition for cofibrantly generated model categories and is done in [Hov99, Theorem 2.1.19].

Theorem 3.4.5 (Recognition Theorem)**.** *Let $\mathcal{C}$ be a bicomplete category and let W be a class of maps in $\mathcal{C}$, such that W is closed under retracts and satisfies the 2 out of 3 axiom. Let I and J be sets of maps in $\mathcal{C}$ such that*

(1) *Both I and J permit Quillen's small object argument*
(2) J-cell $\subseteq W \cap I$-cof
(3) I-inj $\subseteq W \cap J$-inj
(4) *Either* $W \cap I$-cof $\subseteq J$-cof *or* $W \cap J$-inj $\subseteq I$-inj

Then there is a cofibrantly generated model structure on $\mathcal{C}$, such that W is the class of weak equivalences, I is a set of generating cofibrations and J is a set of generating acyclic cofibrations.

Remark 3.4.6. Note that Quillen's small object argument indeed yields functorial factorizations for the factorization axiom as required.

Here comes one advantage of cofibrantly generated model categories: It is easier to check that adjunctions are Quillen functors. The following lemma is given in [Hov99, Lemma 2.1.20]

Lemma 3.4.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories such that $\mathcal{C}$ is cofibrantly generated with generating (acyclic) cofibrations I (J) and let*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

be an adjoint functor pair. Then (F, G) is a Quillen functor if and only if $F(f)$ is a cofibration for all $f \in I$ and an acyclic cofibration for all $f \in J$.

3.5. **Simplicial Model Categories.** Now we are going to introduce the simplicial model categories. Those are model categories enriched over simplicial sets, such that this enrichment is compatible with the model structure.

Definition 3.5.1 (Simplicial model category)**.** A model category $\mathcal{C}$ is called a *simplicial model category* if there is a mapping space functor

$$\mathrm{Map} : \mathcal{C}^{op} \times \mathcal{C} \to s\mathcal{S}et$$

and a product functor

$$\otimes : s\mathcal{S}et \times \mathcal{C} \to \mathcal{C}$$

such that the product is associative, i.e.

$$(L \times K) \otimes X \cong L \otimes (K \otimes X)$$

natural in $X \in \mathcal{C}$, $K, L \in s\mathcal{S}et$ and Δ^0 is a unit object, i.e. $\Delta^0 \otimes X \cong X$. Furthermore there have to be adjoint functor pairs

$$- \otimes X : s\mathcal{S}et \rightleftarrows \mathcal{C} : \mathrm{Map}(X, -)$$

and

$$K \otimes - : \mathcal{C} \rightleftarrows \mathcal{C} : (-)^K$$

for all $X, Y \in \mathcal{C}$ and $K \in s\mathcal{S}et$.

Furthermore the axiom (**SM7**) has to be fulfilled, that is for every cofibration $j : A \to B$ in $\mathcal{C}$ and every fibration $q : X \to Y$ in $\mathcal{C}$

$$\mathrm{Map}(B, X) \xrightarrow{(j^*, q_*)} \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a fibration in $s\mathcal{S}et$ which is acyclic if j or q is acyclic.

Remark 3.5.2. $\mathrm{Map}(X, Y)$ has to be the simplicial set

$$\mathrm{Map}(X, Y)_n \cong \mathrm{Hom}_{s\mathcal{S}et}(\Delta^n, \mathrm{Map}(X, Y)) \cong \mathrm{Hom}_{\mathcal{C}}(\Delta^n \otimes X, Y)$$

for a simplicial model category $\mathcal{C}$, using the Yoneda Lemma and one of the demanded adjunctions.

Example 3.5.3. The category $s\mathcal{S}et$ together with the model structure already introduced is a simplicial model category if we take

$$K \otimes X := K \times X$$

(cf. [GJ99, I Proposition 11.5]).

Furthermore the model category of simplicial abelian groups $s\mathcal{A}b$ is a simplicial model category if we take

$$K \otimes X := \mathbb{Z}[K] \otimes X$$

(cf. [GJ99, III Proposition 2.13]) where the tensor product in $s\mathcal{A}b$ is built up levelwise, that is for $A, B \in s\mathcal{A}b$ it is given by

$$(A \otimes B)_n = A_n \otimes_{\mathbb{Z}} B_n$$

Remark 3.5.4. Let Y be an object in a simplicial model category $\mathcal{C}$. Because of the natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(X, Y^K) \cong \mathrm{Hom}_{\mathcal{C}}(K \otimes X, Y) \cong \mathrm{Hom}_{s\mathcal{S}et}(K, \mathrm{Map}(X, Y))$$

there is an adjoint functor pair

$$\mathrm{Map}(-, Y) : \mathcal{C} \rightleftarrows (s\mathcal{S}et)^{op} : Y^{(-)}.$$

Remark 3.5.5. If all axioms of a simplicial model category are fulfilled perhaps up to (**SM7**), (**SM7**) is equivalent to axioms (**SM7a**) and (**SM7b**) which are formulated in terms using X^K resp. $K \otimes X$ for special $K \in s\mathcal{S}et$ using the demanded adjunctions and the fact that $s\mathcal{S}et$ is a cofibrantly generated model category (cf. [GJ99, II Corollary 3.12 and Proposition 3.13]). We will notice only the axiom (**SM7b**) for arbitrary $K \in s\mathcal{S}et$ here:
For every cofibration $j : A \to B \in \mathcal{C}$ and every cofibration $i : K \to L \in s\mathcal{S}et$ the canonical map

$$K \otimes B \amalg_{K\otimes A} L \otimes A \xrightarrow{i\Box j} L \otimes B$$

is a cofibration in $\mathcal{C}$ which is acyclic if j or i is acyclic. $j\Box i$ is called the *pushout product of j and i.*

Lemma 3.5.6. *Let $\mathcal{C}$ be a simplicial model category and $K \in s\mathcal{S}et$ a simplicial set. Then*

$$K \otimes - : \mathcal{C} \rightleftarrows \mathcal{C} : (-)^K$$

is a Quillen functor.

Furthermore let $X \in \mathcal{C}$ be a cofibrant object. Then

$$- \otimes X : s\mathcal{S}et \rightleftarrows \mathcal{C} : \mathrm{Map}(X, -)$$

is a Quillen functor.

Finally let Y be a fibrant object of $\mathcal{C}$. Then

$$\mathrm{Map}(-, Y) : \mathcal{C} \rightleftarrows (s\mathcal{S}et)^{op} : Y^{(-)}$$

is a Quillen functor.

Proof. It follows immediately from (**SM7b**) and the fact that every simplicial set is cofibrant that $K \otimes -$ is a left Quillen functor. Furthermore it follows immediately from (**SM7**) that $\mathrm{Map}(X, -)$ is a right Quillen functor for a cofibrant $Y \in \mathcal{C}$ and that $\mathrm{Map}(-, Y)$ is a left Quillen functor for a fibrant $Y \in \mathcal{C}$. □

Lemma 3.5.7. *Let $\mathcal{C}$ be a simplicial model category and let $X \in \mathcal{C}$ be a cofibrant and $Y \in \mathcal{C}$ be a fibrant object. Then $f, g : X \to Y$ are homotopic if and only if they are homotopic via a homotopy $H : \Delta^1 \otimes X \to Y$. Such a homotopy is called a* simplicial homotopy.

Proof. First of all, $\Delta^1 \otimes X$ is a cylinder object for X if X is cofibrant according to [GJ99, II Lemma 3.5]. Then Proposition 3.1.15 implies that for additionally Y fibrant the definition of a homotopy does not depend on a choice of a cylinder object and the claim follows. □

Definition 3.5.8 (Simplicial mapping cylinder)**.** Let $\mathcal{C}$ be a simplicial model category and $f : X \to Y$ a map in $\mathcal{C}$. Define the *simplicial mapping cylinder of f* $\mathrm{Cyl}(f)$ as the pushout of the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \\ \Delta^1 \otimes X & \end{tikzcd}$$

where $X \cong \Delta^0 \otimes X \xrightarrow{\iota_0 \otimes X} \Delta^1 \otimes X$ is induced by the first embedding $\iota_0 : \Delta^0 \hookrightarrow \Delta^1$, that is, the map which corresponds to the map $[0] \xrightarrow{0 \mapsto 0} [1]$, using the Yoneda Lemma.

Remark 3.5.9. Let us denote by $h : X \to \mathrm{Cyl}(f)$ the map induced by the second embedding $\iota_1 : \Delta^0 \hookrightarrow \Delta^1$, that is, it corresponds to $[0] \xrightarrow{0 \mapsto 1} [1]$, via

$$X \cong \Delta^0 \otimes X \xrightarrow{\iota_1 \otimes X} \Delta^1 \otimes X \longrightarrow \mathrm{Cyl}(f)$$

This gives a factorization of f:

$$\begin{tikzcd} & \mathrm{Cyl}(f) \arrow[dr, "p"] & \\ X \arrow[ur, "h"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

Note that the canonical map $p : \mathrm{Cyl}(f) \to Y$ is induced by the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "\mathrm{id}"] \\ \Delta^1 \otimes X \arrow[r, "f \circ \pi_X"] & Y \end{tikzcd}$$

where $\pi_X = \Delta^1 \otimes X \to \Delta^0 \otimes X \cong X$ is induced by the unique map $\Delta^1 \to \Delta^0$.

Lemma 3.5.10. *Let $\mathcal{C}$ be a simplicial model category and $f : X \to Y$ a map between cofibrant objects. Then $h : X \to \mathrm{Cyl}(f)$ is a cofibration, $\mathrm{Cyl}(f)$ is cofibrant and $p : \mathrm{Cyl}(f) \to Y$ is a simplicial homotopy equivalence.*

Proof. First of all, it follows immediately from Lemma 3.5.6 that

$$K \otimes X \xrightarrow{\iota \otimes X} L \otimes X$$

is a cofibration for every inclusion $\iota : K \hookrightarrow L$ in $s\mathcal{S}et$ since X is cofibrant. Note that $K \otimes X$ is also cofibrant for all $K \in s\mathcal{S}et$. Furthermore we have $\partial\Delta^1 \otimes X \cong X \amalg X$ induced by the maps $X \cong \Delta^0 \otimes X \xrightarrow{\iota_i \otimes X} \partial\Delta^1 \otimes X$, where $\iota_i : \Delta^0 \hookrightarrow \partial\Delta^1$ are the two inclusions, since

$$\mathrm{Hom}_{\mathcal{C}}(\partial\Delta^1 \otimes X, Y) = \mathrm{Hom}_{s\mathcal{S}et}(\partial\Delta^1, \mathrm{Map}(X,Y)) = \mathrm{Hom}_{\mathcal{C}}(X \amalg X, Y)$$

for all $Y \in \mathcal{C}$. Note that we have the pushout diagram

$$\begin{tikzcd} \partial\Delta^1 \otimes X \cong X \amalg X \arrow[r, "f \amalg X"] \arrow[d, tail] & Y \amalg X \arrow[d, tail] \\ \Delta^1 \otimes X \arrow[r] & \mathrm{Cyl}(f) \end{tikzcd}$$

and therefore

$$h = X \rightarrowtail Y \amalg X \rightarrowtail \mathrm{Cyl}(f)$$

is a cofibration as claimed. Now it is clear that $\mathrm{Cyl}(f)$ is cofibrant since the unique map $0 \to \mathrm{Cyl}(f)$ is the same as the composition of the two cofibrations $0 \rightarrowtail X \rightarrowtail \mathrm{Cyl}(f)$. It is standard to see that $p : \mathrm{Cyl}(f) \to Y$ is a simplicial homotopy equivalence since $\Delta^1 \otimes -$ gives a cylinder object. $\square$

Remark 3.5.11. Note that for X cofibrant the two cofibrations

$$X \cong \Delta^0 \otimes X \xrightarrow{\iota_i \otimes X} \Delta^1 \otimes X$$

are simplicial homotopy equivalences. This immediately implies that for a map $f : X \to Y$ between cofibrant objects the map $Y \rightarrowtail \mathrm{Cyl}(f)$ is an acyclic cofibration.

Lemma 3.5.12. *Let $\mathcal{C}$ be a simplicial model category and $f : X \to Y$ a map between cofibrant objects. Then the cofibration $X \rightarrowtail \mathrm{Cyl}(f)$ is acyclic if and only if f is a weak equivalence.*

Proof. This follows immediately from the commutative diagram

$$\begin{tikzcd} & \mathrm{Cyl}(f) \arrow[dr, two heads, "p", "\sim"'] & \\ X \arrow[ur, tail, "h"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

and the 2 out of 3 axiom for weak equivalences where the indicated properties of the maps follow from the Lemma 3.5.10. □

3.6. **Bousfield Localization.** Our next aim to introduce the concept of Bousfield localization. This provides for special model categories that it is possible to take a given model structure and add more weak equivalences. That is, it provides a localization of the homotopy category.

Notation 3.6.1. Throughout this subsection let $\mathcal{C}$ be a simplicial model category. Denote by $\mathrm{map}(X, Y) := \mathrm{Map}(QX, RY)$ where Q is the cofibrant and R the fibrant replacement functor.

Remark 3.6.2. If Y is fibrant, then $\mathrm{map}(X, Y)$ is naturally weak equivalent to $\mathrm{Map}(QX, Y)$ since $\mathrm{Map}(QX, -)$ is a right Quillen functor by Lemma 3.5.6. Dually $\mathrm{map}(X, Y)$ is naturally weak equivalent to $\mathrm{Map}(X, RY)$ if X is cofibrant.

Lemma 3.6.3. *Let E be a fibrant object of $\mathcal{C}$. Then*

$$\mathrm{map}(-, E) : \mathcal{C} \to s\mathcal{S}et$$

respects weak equivalences.

Proof. First of all, $\mathrm{map}(-, E)$ is weak equivalent to $\mathrm{Map}(Q(-), E)$ by the previous remark and $\mathrm{Map}(-, E)$ is a left Quillen functor because of Lemma 3.5.6. Therefore it preserves weak equivalences between cofibrant objects. Furthermore Q respects weak equivalences and hence the claim follows. □

Lemma 3.6.4. *A morphism $f : X \to Y \in \mathcal{C}$ is a weak equivalence if and only if $f^* : \mathrm{map}(Y, E) \to \mathrm{map}(X, E)$ is a weak equivalence in $s\mathcal{S}et$ for all fibrant $E \in \mathcal{C}$.*

Proof. One direction is the previous lemma. By the previous remark again, we are using $\mathrm{Map}(QY, X)$ for the other direction. Now suppose that $Q(f)^*$:

$\mathrm{Map}(QY, E) \to \mathrm{Map}(QX, E)$ is a weak equivalence of simplicial sets. Hence it is especially a π_0-isomorphism. Furthermore

$$\begin{aligned} \pi_0 \mathrm{Map}(QY, E) &\cong \mathrm{Map}(QY, E)_0 / \sim \\ &\cong \mathrm{Hom}_{\mathcal{C}}(QY, E)/ \sim \\ &= [QY, E] \\ &= \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(QY, E) \end{aligned}$$

using Lemma 3.5.7 and hence

$$(\gamma(Q(f)))^* : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(QY, E) \to \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(QX, E)$$

is an isomorphism for all fibrant E. Therefore the Yoneda Lemma implies that $\gamma(Q(f)) : QX \to QY$ is an isomorphism in $\mathrm{Ho}(\mathcal{C})$ because it suffices to check the Hom-isomorphisms only for fibrant objects. It follows that $Q(f)$ is a weak equivalence in $\mathcal{C}$. But $Q(f)$ is obtained from f via a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ QX \arrow[u, "\sim"] \arrow[r, "Q(f)"] & QY \arrow[u, "\sim"'] \end{tikzcd}$$

and hence f is a weak equivalence because of the 2 out of 3 property for weak equivalences. $\square$

Definition 3.6.5 (S-local objects and S-local weak equivalences)**.** Let $S \subseteq Mor\mathcal{C}$ be a class of morphisms. A fibrant object $E \in \mathcal{C}$ is called *S-local* if

$$f^* : \mathrm{map}(B, E) \to \mathrm{map}(A, E)$$

is a weak equivalence of simplicial sets for all $f : A \to B \in S$.

A morphism $g : X \to Y \in \mathcal{C}$ is called an *S-local weak equivalence* if

$$g^* : \mathrm{map}(Y, W) \to \mathrm{map}(X, W)$$

is a weak equivalence of simplicial sets for all S-local W.

Remark 3.6.6. (1) Every $f \in S$ is an S-local weak equivalence.
(2) Every weak equivalence of $\mathcal{C}$ is an S-local weak equivalence.

Proof. (1) is clear by definition of an S-local object. (2) is clear using Lemma 3.6.3. $\square$

Remark 3.6.7. Using Lemma 3.6.3 and the 2 out of 3 property for weak equivalences in $s\mathcal{S}et$ an S-local object is the same as an S'-local object if S' is obtained from S by weak equivalent replacements of the morphisms in S, notated $S \sim S'$. That is every morphism $g \in S'$ is connected to a morphism $f \in S$ by a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, rightsquigarrow] & B \arrow[d, rightsquigarrow] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

where the vertical 'arrows' are zigzag compositions of weak equivalences.

Definition 3.6.8 (Left Bousfield localization)**.** Let $S \subseteq Mor\mathcal{C}$ be a class of morphisms. A model structure $L_S\mathcal{C}$ on $\mathcal{C}$ is called the *left Bousfield localization* on $\mathcal{C}$ with respect to S if

- w.e. of $L_S\mathcal{C}$ = S-local weak equivalences
- cof. of $L_S\mathcal{C}$ = cofibrations of $\mathcal{C}$

Remark 3.6.9. For a given model category $\mathcal{C}$ and a class of morphisms S it is not clear if the left Bousfield localization exists. There is an existence Theorem of Hirschhorn for good model categories which is given at the end of the subsection. The existence is assumed in the following results.

Lemma 3.6.10. *Let S, S' be two classes of morphisms, such that $S \sim S'$. Then $L_S\mathcal{C} = L_{S'}\mathcal{C}$.*

Proof. Of course, the cofibrations are the same. The equality of the weak equivalences follows immediately from Remark 3.6.7 since the S-local (S'-local resp.) weak equivalences only depend on the S-local (S'-local resp.) objects. □

Here is the result that the left Bousfield localization provides a localization of the homotopy category.

Proposition 3.6.11. *Let $L_S\mathcal{C}$ be the left Bousfield localization of $\mathcal{C}$ with respect to S. First of all $id : \mathcal{C} \to L_S\mathcal{C}$ is a left Quillen functor with the property that it takes* every *cofibrant replacement of a map in S into a weak equivalence in $L_S\mathcal{C}$. Furthermore this functor is universal with this property, i.e. consider a left Quillen functor $F : \mathcal{C} \to \mathcal{D}$ which takes every cofibrant replacement of a map in S into a weak equivalence in $\mathcal{D}$. Then there is exactly one Quillen functor $L_S\mathcal{C} \to \mathcal{D}$ with*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "id"'] & \mathcal{D} \\ L_S\mathcal{C} \arrow[ur, dotted] & \end{tikzcd}$$

commutes.

Proof. This is done in [Hir03, Proposition 3.3.19 (1)]. □

The next theorem shows that the left Bousfield localization has a functorial behavior with respect to Quillen functors.

Theorem 3.6.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be simplicial model categories and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ a Quillen functor. Let $L_S\mathcal{C}$ be the left Bousfield localization of $\mathcal{C}$ with respect to S and $L_{F(QR(S))}\mathcal{D}$ the left Bousfield localization of $\mathcal{D}$ with respect to $F(QR(S))$ where Q is the cofibrant and R the fibrant replacement functor of the model category $\mathcal{C}$. Then:*

1. *(F,G) is also a Quillen functor between the localizations $L_S\mathcal{C}$ and $L_{F(QR(S))}\mathcal{D}$.*
2. *If (F,G) is a Quillen equivalence between $\mathcal{C}$ and $\mathcal{D}$ then (F,G) is also a Quillen equivalence between the localizations $L_S\mathcal{C}$ and $L_{F(QR(S))}\mathcal{D}$.*

Proof. Since $QR(S)$ yields cofibrant replacements of the morphisms in S part (1) is clear applying Proposition 3.6.11 to the composition

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{id_{\mathcal{D}}} L_{F(QR(S))}\mathcal{D}$$

which is a left Quillen functor between $\mathcal{C}$ and $L_{F(QR(S))}\mathcal{D}$ by assumption and the first part of Proposition 3.6.11. Part (2) is done in [Hir03, Theorem 3.3.20 (1)(b)]. □

The next proposition yields an identification property for a left Bousfield localization.

Proposition 3.6.13. *Let $\mathcal{M}$ and $\mathcal{N}$ be two model structures on $\mathcal{C}$ and $S \subseteq Mor\mathcal{C}$ a class of morphisms. Assume $cof\mathcal{N} = cof\mathcal{M}$ and the $\mathcal{N}$-fibrant objects are exactly the objects which are S-local with respect to $\mathcal{M}$. Then: $\mathcal{N}$ is the left Bousfield localization of $\mathcal{M}$ with respect to S.*

Proof. We have to check that $w\mathcal{N} = S$-local weak equivalences with respect to $\mathcal{M}$. But this is clear using Lemma 3.6.4 and the definition of an S-local weak equivalence. □

The next theorem yields the existence of the left Bousfield localization for special model categories which includes the definition of *cellular model categories*. Since we will not have to deal with this definition explicitly, this definition will not be given in this thesis but the interested reader can find it in [Hir03, Chapter 12]. The theorem will be applied to another quoted theorem to get a new model structure. But note that the definition of cellular model categories includes that the model structure is cofibrantly generated.

Theorem 3.6.14 (Hirschhorn)**.** *Let $\mathcal{C}$ be a simplicial left proper cellular model category and $S \subseteq Mor\mathcal{C}$ a set of morphisms. Then the left Bousfield localization $L_S\mathcal{C}$ exists and it has the following properties:*

1. *$L_S\mathcal{C}$ is a simplicial model category with the same simplicial structure as $\mathcal{C}$.*
2. *The fibrant objects of $L_S\mathcal{C}$ are exactly the S-local objects.*
3. *$L_S\mathcal{C}$ is again left proper and cellular.*

Proof. This is done in [Hir03, Theorem 4.1.1]. □

4. Grothendieck Sites

The next aim is to introduce the concept of Grothendieck topologies on arbitrary categories. This is a generalization of the concept of a topology on a set X. It allows us to define presheaves and sheaves on a Grothendieck topology similar to the notion of presheaves and sheaves on a topological space.

4.1. **Basic Notation.**

Definition 4.1.1. (Small Grothendieck site) A *small Grothendieck site* $T = (\mathcal{C}, \mathrm{Cov}_T)$ consists of a small category $\mathcal{C}$ and a set of coverings Cov_T, i.e. for every $U \in \mathcal{C}$ there is a set of coverings $\mathrm{Cov}_T(U)$ consisting of families of morphisms $\{\phi_i : U_i \to U\}_{i \in I}$ such that the following axioms hold:

(1) Let $\{U_i \to U\} \in \mathrm{Cov}_T(U)$ and $V \to U \in \mathcal{C}$. Then all pullbacks $U_i \times_U V$ exist and $\{U_i \times_U V \to V\} \in \mathrm{Cov}_T(V)$.
(2) Let $\{U_i \to U\} \in \mathrm{Cov}_T(U)$ and a family $\{V_{ij} \to U_i\} \in \mathrm{Cov}_T(U_i)$ for every $i \in I$. Then the set of maps $\{V_{ij} \to U\}$ obtained by all compositions is in $\mathrm{Cov}_T(U)$.
(3) Every isomorphism $U' \to U \in \mathcal{C}$ belongs to $\mathrm{Cov}_T(U)$.

A set of coverings Cov_T satisfying these axioms is called a *Grothendieck topology on $\mathcal{C}$*.

Remark 4.1.2. Let $T = (\mathcal{C}, \mathrm{Cov}_T)$ be a small Grothendieck site and $X \in \mathcal{C}$. Then denote by $\mathcal{C} \downarrow X$ the category over X, that is the objects are morphisms $V \to X$ of $\mathcal{C}$ and the morphisms are commutative diagrams

$$\begin{tikzcd} V \arrow[rr] \arrow[dr] & & W . \arrow[dl] \\ & X & \end{tikzcd}$$

Denote by $\iota_X : \mathcal{C} \downarrow X \longrightarrow \mathcal{C}$ the forgetful functor. $\mathcal{C} \downarrow X$ is again a small Grothendieck site denoted by $T \downarrow X$ with the canonical coverings: For a family of morphisms $\phi_i : U_i \to U$ over X we define

$$\{\phi_i\} \in \mathrm{Cov}_{T\downarrow X}(U \to X) :\Longleftrightarrow \{\iota_X(\phi_i)\} \in \mathrm{Cov}_T(U)$$

Example 4.1.3. There are at least three Grothendieck topologies on the small category Sm/k as we will see in subsection 6.1.

Definition 4.1.4. (Presheaves and sheaves) Let $T = (\mathcal{C}, \mathrm{Cov}_T)$ be a small Grothendieck site. A *presheaf* P is a functor

$$P : \mathcal{C}^{op} \longrightarrow \mathcal{S}et.$$

A presheaf P is called a *sheaf* if for every $U \in \mathcal{C}$ and every $\{U_i \to U\} \in \mathrm{Cov}_T(U)$ the canonical diagram

$$P(U) \to \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_U U_j)$$

is an equalizer diagram in $\mathcal{S}et$.

Remark 4.1.5. The presheaves resp. sheaves form a category via taking the canonical categorical structure of a functor category resp. a full subcategory. Denote these categories by $\mathrm{Pre}(T)$ and $\mathrm{Shv}(T)$

Remark 4.1.6. It is clear that $\mathrm{Pre}(T)$ as a functor category with the bicomplete target category $\mathcal{S}et$ is again bicomplete because limits and colimits can be formed objectwise.

4.2. **The Associated Sheaf Functor.** The next proposition gives the existence of the associated sheaf functor (cf. [Bor94, Theorem 3.3.12]).

Proposition 4.2.1. *There is an adjoint functor pair*

$$a : \mathrm{Pre}(T) \rightleftarrows \mathrm{Shv}(T) : \iota$$

where ι is the canonical inclusion functor and a is called the associated sheaf functor. *Furthermore a does nothing on sheaves, that is $a \circ \iota \cong \mathrm{id}$, and commutes with finite limits.*

An immediate consequence of this adjunction is

Corollary 4.2.2. *The category of sheaves is also bicomplete.*

Proof. We make heavy use of the already seen fact, that $\mathrm{Pre}(T)$ is bicomplete. Since a preserves colimits as a left adjoint and does nothing on sheaves, colimits in $\mathrm{Shv}(T)$ can be formed in $\mathrm{Pre}(T)$ via ι followed by the sheafification a. Since limits in $\mathrm{Pre}(T)$ are formed objectwise, they commute objectwise with equalizers in $\mathcal{S}et$. Hence limits can directly be formed in $\mathrm{Pre}(T)$ via ι without sheafification because they are already sheaves. It follows from the adjointness of a and ι together with $a \circ \iota \cong \mathrm{id}$ that these constructions indeed yield all small limits and colimits in $\mathrm{Shv}(T)$. □

5. Homotopy Theories on Simplicial (Pre)Sheaves

In this section four model structures on simplicial (pre)sheaves on a small Grothendieck site will be introduced, the so-called (local) projective and (local) injective structure. This provides the technical background to introduce motivic homotopy theory. The model structure used by Morel and Voevodsky in [MV99] is the local injective one. But for our purpose it is better to use the local projective structure which is Quillen equivalent to the local injective one. Furthermore it will turn out in Theorem 5.4.5 that the homotopy theories on presheaves and the homotopy theories on sheaves lead to equivalent homotopy categories induced by the associated sheaf functor. That is, there is no difference between presheaves and sheaves in the homotopy categories.

5.1. **Basic Notation and Results.**

Notation 5.1.1. Throughout this section let $T := (\mathcal{C}, \mathrm{Cov}_T)$ denote a small Grothendieck site. Furthermore consider $s\mathcal{S}et$ always with the model structure introduced in Example 3.1.9.

Definition 5.1.2 (simplicial (pre)sheaves)**.** Denote by

$$s\mathrm{Pre}(T) := \mathrm{Fun}\left(\Delta^{op}, \mathrm{Pre}(T)\right) \text{ and } s\mathrm{Shv}(T) := \mathrm{Fun}\left(\Delta^{op}, \mathrm{Shv}(T)\right)$$

the simplicial objects in the categories $\mathrm{Pre}(T)$ and $\mathrm{Shv}(T)$, which are called the *simplicial (pre)sheaves on* T.

Remark 5.1.3. It is clear that $s\mathrm{Pre}(T) = \mathrm{Fun}(\mathcal{C}^{op}, s\mathcal{S}et)$, i.e. $s\mathcal{S}et$-valued presheaves on T, because both categories are equal to the bifunctor category $\mathrm{BiFun}(\Delta^{op} \times \mathcal{C}^{op}, \mathcal{S}et)$. Furthermore the sheaf-condition of the $s\mathcal{S}et$-valued presheaves is tested with equalizer diagrams in the category $s\mathcal{S}et$, hence objectwise for all $[n] \in \Delta$. Therefore it is also true, that $s\mathrm{Shv}(T)$ are the $s\mathcal{S}et$-valued sheaves on T.

It depends on the context which description of simplicial (pre)sheaves is used.

Remark 5.1.4. The categories of simplicial sheaves and presheaves are both bicomplete since $\mathrm{Pre}(T)$ and $\mathrm{Shv}(T)$ are bicomplete (cf. section 4) and functor categories with a bicomplete target category are again bicomplete because all limits and colimits can be formed objectwise. Furthermore there is again an adjoint functor pair

$$a : s\mathrm{Pre}(T) \rightleftarrows s\mathrm{Shv}(T) : \iota$$

where ι is the inclusion functor and a the associated sheaf functor which is built up objectwise, i.e. for $X \in s\mathrm{Pre}(T)$ the associated sheaf $a(X)$ is given by the composition

$$\Delta^{op} \xrightarrow{X} \mathrm{Pre}(T) \xrightarrow{a} \mathrm{Shv}(T)$$

where $a : \mathrm{Pre}(T) \to \mathrm{Shv}(T)$ is the associated sheaf functor of $\mathcal{S}et$-valued presheaves. Note that the associated sheaf functor again does nothing on simplicial sheaves, that is $a \circ \iota \cong \mathrm{id}$, and commutes with finite limits.

Notation 5.1.5. Let $X \in \mathcal{C}$. Then denote by $\overline{X} = \mathrm{Hom}_{\mathcal{C}}(-, X)$ the represented discrete simplicial presheaf.

Proposition 5.1.6. *Every simplicial presheaf is canonically isomorphic to a colimit of representable simplicial presheaves.*

Proof. The proof is due to [Mac98, Ch.III §7 Theorem 1]. It yields that there is an isomorphism

$$X \xleftarrow{\cong} \mathrm{colim}_{(U,[n],x)} \overline{U} \times \Delta^n$$

where the colimit is indexed over all $U \in \mathcal{C}$, $n \in \mathbb{N}$ and $x \in X(U)_n$ and such a triple is denoted as $(U, [n], x)$. A morphism of tripels $(U, [n], x) \to (V, [m], y)$ is just a morphism $(U, [n]) \xrightarrow{f} (V, [m])$ in $\mathcal{C} \times \Delta$ such that $X(f)(y) = x$. Now the structure morphisms of the colimit are given by: For a morphism of pairs $(U, [n]) \xrightarrow{f} (V, [m])$ take the induced morphism by the Yoneda embedding $\overline{U} \times \Delta^n \xrightarrow{\overline{f}} \overline{V} \times \Delta^m$. The claimed isomorphism is given by: Every $x \in X(U)_n$ determines a morphism $\overline{U} \times \Delta^n \xrightarrow{x} X$ by the Yoneda Lemma which yields a morphism from every factor of the colimit to X. The collection of these morphisms is of course compatible with the structure morphisms of the colimit. $\square$

Notation 5.1.7. Let $U \in T$ and $X \in s\mathrm{Pre}(T)$. Denote by $X_{|U}$ the restricted simplicial presheaf on the Grothendieck site $T \downarrow U$, that is, the composition

$$(\mathcal{C} \downarrow U)^{op} \xrightarrow{(\iota_U)^{op}} \mathcal{C}^{op} \xrightarrow{X} s\mathcal{S}et$$

where $\mathcal{C} \downarrow U$ is the category over U and ι_U the forgetful functor.

Remark 5.1.8. If X is a simplicial sheaf, then so is $X_{|U}$.

5.2. **The Closed Symmetric Monoidal Structure.** There is a closed symmetric monoidal structure on simplicial (pre)sheaves (cf. [Mac98, Ch.XI §1]).

Definition 5.2.1. Let X, Y be simplicial (pre)sheaves. Define the simplicial function complex

$$\mathrm{Map}(X, Y)_n := \mathrm{Hom}_{s\mathrm{Pre}(T)}(X \times \Delta^n, Y)$$

on presheaves and

$$\mathrm{Map}(X, Y)_n := \mathrm{Hom}_{s\mathrm{Shv}(T)}(X \times a(\Delta^n), Y)$$

on sheaves where Δ^n is taken as the constant simplicial presheaf and $a(\Delta^n)$ is the associated constant simplicial sheaf. The internal function complex is given by

$$\underline{\mathrm{Map}}(X,Y)(U) := \mathrm{Map}(X_{|U}, Y_{|U})$$

on simplicial sheaves and presheaves. A straightforward calculation shows that for X, Y simplicial sheaves $\underline{\mathrm{Map}}(X,Y)$ is again a simplicial sheaf using Lemma 5.2.4.

Remark 5.2.2. The categories of simplicial sheaves and presheaves together with the categorical product

$$- \times - : s\mathrm{Pre}(T) \times s\mathrm{Pre}(T) \longrightarrow s\mathrm{Pre}(T)$$

and

$$- \times - : s\mathrm{Shv}(T) \times s\mathrm{Shv}(T) \longrightarrow s\mathrm{Shv}(T)$$

are symmetric monoidal categories where $X \times Y$ is denoted by $X \otimes Y$. Moreover they are closed since there are adjoint functor pairs

$$- \otimes X : s\mathrm{Pre}(T) \rightleftarrows s\mathrm{Pre}(T) : \underline{\mathrm{Map}}(X,-)$$

and

$$- \otimes X : s\mathrm{Shv}(T) \rightleftarrows s\mathrm{Shv}(T) : \underline{\mathrm{Map}}(X,-).$$

Lemma 5.2.3 (Yoneda-Lemma over $s\mathcal{S}et$)**.** *Let $X \in \mathcal{C}$ and $F \in s\mathrm{Pre}(T)$. Then*

$$\mathrm{Map}(\overline{X}, F) \cong F(X) \in s\mathcal{S}et$$

Proof. This follows from the natural isomorphisms (in n)

$$\begin{aligned} \mathrm{Map}(\overline{X}, F)_n &= \mathrm{Hom}(\overline{X} \times \Delta^n, F) \\ &\cong \mathrm{Hom}(\overline{X}, \mathrm{Hom}(\Delta^n, F)) \\ &\cong \mathrm{Hom}(\overline{X}, F_n) \\ &\cong F(X)_n \end{aligned}$$

□

Lemma 5.2.4. *Let F, G be simplicial (pre)sheaves and $X \in \mathcal{C}$. Then*

$$\underline{\mathrm{Map}}(\overline{X}, G) = G(- \times X)$$

and

$$\underline{\mathrm{Map}}(F,G)(X) = \mathrm{Map}(F, G(- \times X))$$

for presheaves and for sheaves.

Proof. Here is the verification on presheaf-level: First of all note that

$$\underline{\mathrm{Map}}(F,G)(U) = \mathrm{Map}(\overline{U}, \underline{\mathrm{Map}}(F,G)) = \mathrm{Map}(\overline{U} \otimes F, G)$$

by the Yoneda Lemma (over $s\mathcal{S}et$). Then the first claim follows by the fact that $\overline{X} \otimes \overline{Y} = \overline{X \times Y}$ and again the Yoneda Lemma (over $s\mathcal{S}et$). Finally we have

$$\begin{aligned} \underline{\mathrm{Map}}(F,G)(X) &= \mathrm{Map}(F \otimes \overline{X}, G) \\ &= \mathrm{Map}(F, \underline{\mathrm{Map}}(\overline{X}, G)) \\ &= \mathrm{Map}(F, G(- \times X)) \end{aligned}$$

The argument for sheaves is similar. □

5.3. **Model Structures on Simplicial Presheaves.** Here is the definition which describes the simplicial structure on all four model structures.

Definition 5.3.1 (Simplicial structure)**.** Let X, Y be simplicial presheaves and K a simplicial set, then define

$$K \otimes X := K \times X \text{ with } K \text{ considered as constant presheaf}$$

and $\mathrm{Map}(X, Y)$ is the simplicial function complex as declared above as well as

$$X^K := \underline{\mathrm{Map}}(K, X)$$

where $\underline{\mathrm{Map}}$ is the internal function complex, also declared above. Note that $\underline{\mathrm{Map}}(K, X)(U) = \mathrm{Map}(K, X(U))$ where Map is the simplicial function complex in $s\mathcal{S}et$. Then

$$- \otimes X : s\mathcal{S}et \rightleftarrows s\mathrm{Pre}(T) : \mathrm{Map}(X, -)$$

and

$$K \otimes - : s\mathrm{Pre}(T) \rightleftarrows s\mathrm{Pre}(T) : (-)^K$$

are adjoint functor pairs.

Remark 5.3.2. Using $K \otimes X := a(K) \times X$ and $X^K := \underline{\mathrm{Map}}(a(K), X)$ for simplicial sheaves, we get a similar structure on sheaves using the closed symmetric structure for simplicial sheaves.

There is a notion of homotopy groups for simplicial presheaves which is given next.

Definition 5.3.3. Let X be a simplicial presheaf. Define $\pi_0(X)$ as the sheafified presheaf

$$U \mapsto \pi_0(X(U))$$

on the site T where $\pi_0(X(U))$ are just the connected components of the simplicial set $X(U)$.

Let $U \in \mathcal{C}$, $x \in X(U)_0$, and $n > 0$. Define $\pi_n(X_{|U}, x)$ as the sheafified presheaf

$$(V \to U) \mapsto \pi_n(X(V), x_V)$$

of homotopy groups of simplicial sets on the site $T \downarrow U$ where $x_V \in X(V)_0$ is just the image of x under the canonical map $X(U)_0 \to X(V)_0$.

Definition 5.3.4. Let $f : X \to Y$ be a map of simplicial (pre)sheaves. f is called a

- *local weak equivalence* if $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of sheaves and for all $n > 0$ and $U \in \mathcal{C}$ the induced map $f_* : \pi_n(X_{|U}, x) \to \pi_n(Y_{|U}, f(x))$ is an isomorphism of sheaves for all basepoints $x \in X(U)_0$
- *objectwise weak equivalence (resp. (co)fibration)* if for all $U \in \mathcal{C}$ $f(U) : X(U) \to Y(U)$ is a weak equivalence (resp. (co)fibration) in $s\mathcal{S}et$

Notation 5.3.5. For the (local) injective/projective model structure, we will refer to the weak equivalences, fibrations and cofibrations as the (local) injective/projective weak equivalences, fibrations and cofibrations. The same is done later with the $\mathbb{A}^1$-local model structure. This might be a little bit confusing, but it is necessary to keep apart the different model structures.

Referring to [Bla01, Theorem 1.1] there is a model structure on simplicial presheaves using the objectwise weak equivalences as weak equivalences called the *injective model structure*:

Theorem 5.3.6 (Joyal)**.** *The category of simplicial presheaves on a small Grothendieck site together with the classes*

- *w.e. = objectwise weak equivalences*
- *cof. = objectwise cofibrations*

forms a proper simplicial cofibrantly generated model category.

But using the same weak equivalences but the objectwise fibrations we also have a model structure called the *projectice model structure* referring [Bla01, Theorem 1.4]:

Theorem 5.3.7 (Hirschhorn-Bousfield-Kan-Quillen)**.** *The category of simplicial presheaves on a small Grothendieck site together with the classes*

- *w.e. = objectwise weak equivalences*
- *fib. = objectwise fibrations*

forms a proper simplicial cellular model category.

Sketch of the proof. Every cellular model category is by definition cofibrantly generated. To get the cofibrantly generated model structure you can use the Recognition Theorem 3.4.5. The generating cofibrations are

$$I := \{\partial\Delta^n \otimes \overline{X} \to \Delta^n \otimes \overline{X} \mid X \in \mathcal{C}, n \in \mathbb{N}\}$$

and the generating acyclic cofibrations are

$$J := \{\Lambda^n_k \otimes \overline{X} \to \Delta^n \otimes \overline{X} \mid X \in \mathcal{C}, n > 0, 0 \leq k \leq n\}$$

where the $\otimes$-operation was defined in Definition 5.3.1 and $\overline{X}$ is the represented discrete simplicial presheaf in $s\mathrm{Pre}(T)$. After applying the Recognition Theorem you have to check that J-inj are exactly the objectwise fibrations and that I-inj are exactly the objectwise acyclic fibrations.

To check this and to check the conditions of the Recognition Theorem one uses the fact that $s\mathcal{S}et$ is cofibrantly generated by $I = \{\partial\Delta^n \to \Delta^n \mid n \in \mathbb{N}\}$ and $J = \{\Lambda^n_k \to \Delta^n \mid n > 0, 0 \leq k \leq n\}$ introduced in Example 3.4.2 together with the adjunctions from Definition 5.3.1 and the Yoneda Lemma over $s\mathcal{S}et$ 5.2.3.

E.g. let $F \overset{\sim}{\twoheadrightarrow} G$ be an acyclic fibration, i.e. an objectwise acyclic fibration, and consider a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \otimes \overline{X} \arrow[r] \arrow[d] & F \arrow[d, two heads, "\sim"] \\
\Delta^n \otimes \overline{X} \arrow[r] & G
\end{tikzcd}$$

in $s\mathrm{Pre}(T)$. Because of the adjunctions from Definition 5.3.1 this diagram corresponds to a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Map}(\overline{X}, F) \arrow[r, "\cong"] \arrow[d] & F(X) \arrow[d, two heads, "\sim"] \\
\Delta^n \arrow[r] & \mathrm{Map}(\overline{X}, G) \arrow[r, "\cong"] & G(X)
\end{tikzcd}$$

in $s\mathcal{S}et$ where the isomorphisms come from Lemma 5.2.3. Now the second diagram has a lift and hence the first. Therefore every objectwise acyclic fibration is in I-inj. □

Now the local injective/projective model structures on simplicial presheaves are introduced both using the local weak equivalences as weak equivalences. Here is the *local injective model structure* originally proved by J. F. Jardine in [Jar87] and stated in [Bla01] as Theorem 1.2:

Theorem 5.3.8 (Jardine)**.** *The category of simplicial presheaves on a small Grothendieck site together with the classes*

- *w.e. = local weak equivalences*
- *cof. = injective (i.e. objectwise) cofibrations*

forms a proper simplicial cofibrantly generated model category.

Finally here is the *local projective model structure* introduced by Benjamin A. Blander in [Bla01] as Theorem 1.5:

Theorem 5.3.9 (Blander)**.** *The category of simplicial presheaves on a small Grothendieck site together with the classes*

- *w.e. = local weak equivalences*
- *cof. = projective cofibrations*

forms a proper simplicial cellular model category.

Remark 5.3.10. As already claimed, these two local model structures are Quillen-equivalent since

$$\mathrm{id} : (s\mathrm{Pre}(T), \mathrm{loc.proj.}) \rightleftarrows (s\mathrm{Pre}(T), \mathrm{loc.inj.}) : \mathrm{id}$$

is a Quillen equivalence.

Proof. This is a Quillen functor since every local projective cofibration is a local injective cofibration and of course this is also true for acyclic cofibrations because the weak equivalences are the same. It follows easy that this Quillen functor induces an equivalence of the homotopy categories. □

5.4. **Local Model Structures on Simplicial Sheaves.** Consider now the category of simplicial sheaves on a small Grothendieck site T as the full subcategory of simplicial presheaves. It is possible to get the local injective and the local projective model structure on simplicial sheaves from [Bla01, Theorem 1.3 and Theorem 2.1]:

Theorem 5.4.1 (Joyal)**.** *The category of simplicial sheaves on a small Grothendieck site together with the classes*

- *w.e. = local injective weak equivalences of the underlying presheaves*
- *cof. = local injective (i.e. objectwise) cofibrations of the underlying presheaves*

forms a proper simplicial cofibrantly generated model category.

Theorem 5.4.2 (Blander)**.** *The category of simplicial sheaves on a small Grothendieck site together with the classes*

- *w.e. = local projective weak equivalences of the underlying presheaves*
- *fib. = local projective fibrations of the underlying presheaves*

forms a proper simplicial cellular model category.

Remark 5.4.3. Due to the proof of [Bla01, Theorem 2.1] the local projective model structure on simplicial sheaves is cofibrantly generated by the (acyclic) cofibrations $a(I)$ resp. $a(J)$ where I and J are the cofibrantly generating (acyclic) cofibrations of the local projective model structure on simplicial presheaves and a is the associated sheaf functor.

Remark 5.4.4. Recall the adjoint functor pair

$$a : s\mathrm{Pre}(T) \rightleftarrows s\mathrm{Shv}(T) : \iota$$

where ι is the inclusion functor and a the associated sheaf functor. This functor pair is a Quillen equivalence for the local injective structure as well as for the local projective structure due to [Bla01, Theorem 2.2]. Together with Remark 5.3.10 we get the commutative diagram

$$\begin{tikzcd}
(s\mathrm{Pre}(T), \mathrm{loc.proj.}) \arrow[r, shift left, "a"] \arrow[d, shift left, "\mathrm{id}"] & (s\mathrm{Shv}(T), \mathrm{loc.proj.}) \arrow[l, shift left, "\iota"] \arrow[d, shift left, "\mathrm{id}"] \\
(s\mathrm{Pre}(T), \mathrm{loc.inj.}) \arrow[r, shift left, "a"] \arrow[u, shift left, "\mathrm{id}"] & (s\mathrm{Shv}(T), \mathrm{loc.inj.}) \arrow[l, shift left, "\iota"] \arrow[u, shift left, "\mathrm{id}"]
\end{tikzcd}$$

of Quillen equivalences: The right vertical adjunction is a Quillen functor because after the previous remark and Lemma 3.4.7 it suffices to show that the generating (acyclic) cofibrations $a(I)$ resp. $a(J)$ are preserved by the identity functor. This is true since the diagram commutes and the left vertical adjunction as well as the below horizontal adjunction are Quillen functors, i.e. the left Quillen functors id and a preserve (acyclic) cofibrations.

Therefore we get:

Theorem 5.4.5. *The following homotopy categories are equivalent:*

$$\begin{aligned}
\mathrm{Ho}((s\mathrm{Pre}(T), \mathrm{loc.proj.})) &\simeq \mathrm{Ho}((s\mathrm{Pre}(T), \mathrm{loc.inj.})) \\
&\simeq \mathrm{Ho}((s\mathrm{Shv}(T), \mathrm{loc.proj.})) \\
&\simeq \mathrm{Ho}((s\mathrm{Shv}(T), \mathrm{loc.inj.}))
\end{aligned}$$

6. Motivic Homotopy Theory

The aim of this section is to develop a homotopy theory on the category Sm/k. There is no chance of getting the structure of a model category on Sm/k itself because this category is far away from bicompleteness. The canonical way to enlarge this category such that it becomes bicomplete is to use the Yoneda embedding

$$\begin{aligned}
Sm/k &\hookrightarrow \mathrm{Fun}((Sm/k)^{op}, \mathcal{S}et) \\
X &\mapsto \overline{X} = \mathrm{Hom}_{Sm/k}(-, X)
\end{aligned}$$

But this has a big disadvantage: For a scheme $X \in Sm/k$ and open subschemes U, V, such that $X = U \cup V$ there is a pushout diagram:

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\
U \arrow[r] & X
\end{tikzcd}$$

i.e. X is the categorical union of U and V. But in general this diagram does not stay a pushout under the Yoneda embedding. So it is necessary to use a smaller category. Since

$$\mathrm{Fun}((Sm/k)^{op}, \mathcal{S}et) = \mathrm{Pre}(Sm/k)$$

for every Grothendieck topology on Sm/k, we should find a suitable Grothendieck topology, such that the Yoneda embedding factors through

$$\mathrm{Shv}(Sm/k) \hookrightarrow \mathrm{Pre}(Sm/k).$$

Moreover, these kind of diagrams (or even a generalized one) should stay categorical unions. This property is for example satisfied by the Zariski topology. Since $\mathrm{Shv}(Sm/k)$ is also bicomplete for every Grothendieck topology, it is a suitable replacement of $\mathrm{Pre}(Sm/k)$.

Furthermore any smooth pair (Z, X) should be locally equivalent to a pair of the form $(\mathbb{A}^n, \mathbb{A}^m)$ in the Grothendieck topology. This is for example fulfilled by the étale topology.

A Grothendieck topology satisfying both demanded properties is the Nisnevich topology, but we will only proof the first property since we do not need the second one for our studies.

After this we will be able to develop a homotopy theory on Sm/k because we have already seen that it is possible to get the local projective model structure on simplicial (pre)sheaves on any Grothendieck topology on Sm/k. But as we will see this is not the right homotopy theory on Sm/k. Therefore we have to change the model structure to obtain the $\mathbb{A}^1$-local model structure on simplicial (pre)sheaves. This is, up to Quillen equivalence, the same model structure as considered in [MV99].

Afterwards we will give a full classification of the homotopy types of the socalled $\mathbb{A}^1$-rigid schemes which is going to be a key to understand the homotopy types of projective curves and abelian varieties.

6.1. **The Nisnevich Site.**

The approach presented here is mainly taken from [MV99].

The small category Sm/k is equipped with three Grothendieck topologies in the following way.

Definition 6.1.1 (Coverings)**.** Consider a set of morphisms

$$\{U_i \xrightarrow{p_i} X\}_{i \in I} \ \in Sm/k$$

This set is called a

(1) *Zariski covering* if every p_i is an open embedding and $\bigcup p_i(U_i) = X$.
(2) *Nisnevich covering* if every p_i is étale and for all $x \in X$ there is an $i \in I$ and an $y \in U_i$ such that $p_i(y) = x$ and the residue fields of x and y coincide, i.e. $k(x) \cong k(y)$ via p_i.
(3) *étale covering* if every p_i is étale and $\bigcup p_i(U_i) = X$.

Remark 6.1.2. An equivalent condition for a Nisnevich covering is: All p_i are étale and for every field L every morphism $\mathrm{Spec}(L) \to X \in Sm/k$ factors through a p_i.

Proof. It is well known that for a scheme $S \in Sm/k$ a morphism $\mathrm{Spec}(L) \to S \in Sm/k$ is the same as a tuple $(x \in S, k(x) \hookrightarrow L$ over $k)$, i.e. a factorization of a morphism $\mathrm{Spec}(L) \to X \in Sm/k$ through a p_i is the same as an $y \in U_i$ and a commutative diagram

$$\begin{tikzcd} k(x) \arrow[r, hook, "p_i"] \arrow[dr, hook] & k(y) \arrow[d, hook] \\ & L \end{tikzcd}$$

over k. Hence, if for every $x \in X$ there is an $y \in U_i$ with $k(x) \cong k(y)$ via p_i a factorization exists. Conversely given an $x \in X$ take L as $k(x)$. Then by assumption a factorization exists and hence $k(x) \hookrightarrow k(y)$ has a section i.e. is surjective and therefore an isomorphism. □

Remark 6.1.3. An equivalent condition for an étale covering is: All p_i are étale and for every separably closed field L every morphism $\mathrm{Spec}(L) \to X \in Sm/k$ factors through a p_i.

Proof. Consider the diagram of the proof of 6.1.2 which is equivalent to a factorization. Assume that such a factorization always exists for L separably closed. Then we can take $L := k(x)^s$ for $x \in X$, the separable closure of $k(x)$. Then a factorization yields an $i \in I$ and an $y \in U_i$ such that $p_i(y) = x$, hence $\bigcup p_i(U_i) = X$. Conversely consider a morphism $(x \in X, k(x) \hookrightarrow L)$. By assumption there is an $i \in I$ and an $y \in U_i$ such that $p_i(y) = x$. Hence p_i yields a separable field extension $k(x) \hookrightarrow k(y)$ because p_i is étale. Finally a factorization exists because L is separably closed. □

Lemma 6.1.4. *Let $\tau \in \{Zar, Nis, ét\}$. Then*

$$(Sm/k)_\tau := (Sm/k, Cov_\tau) \text{ with } Cov_\tau(X) = \{\tau\text{-coverings of } X\}$$

is a small Grothendieck site. Furthermore we have the inclusions

$$Cov_{Zar}(X) \subset Cov_{Nis}(X) \subset Cov_{ét}(X)$$

for all $X \in Sm/k$.

Proof. It is standard that the Zariski coverings provide a Grothendieck topology on Sm/k. Using Remark 6.1.2 a straightfoward computation checks the Nisnevich case and analogous the étale case with Remark 6.1.3. The inclusions also follow with these two remarks because every Zariski covering has of course the required factorization property. □

Hence we have

$$\mathrm{Shv}((Sm/k)_{ét}) \subset \mathrm{Shv}((Sm/k)_{Nis}) \subset \mathrm{Shv}((Sm/k)_{Zar}).$$

Because of these inclusions we have the following

Lemma 6.1.5. *Every representable presheaf is an étale sheaf and therefore also a Nisnevich sheaf.*

Proof. We need two propositions from [Mil80]. The first is II 1.5:
F is an étale sheaf if and only if F is a Zariski sheaf and for all étale coverings $\{V \to U\}$ (one element!) with V, U being affine schemes the diagram

$$F(U) \to F(V) \rightrightarrows F(V \times_U V)$$

is exact in $\mathcal{S}et$.

The second is I 2.17:
Every faithfully flat morphism $f : Y \to X$ of finite type gives for all Z an exact diagram

$$\mathrm{Hom}(X, Z) \to \mathrm{Hom}(Y, Z) \rightrightarrows \mathrm{Hom}(Y \times_X Y, Z)$$

in $\mathcal{S}et$.

Now let $F = \mathrm{Hom}(-, X)$ be a representable presheaf. It is well known that F is a Zariski-sheaf. Let $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ be an étale covering. In particular, this is faithfully flat and the exactness of the diagram follows from the second proposition. □

Remark 6.1.6. By the lemma above we have a factorization of the Yoneda embedding

$$\begin{tikzcd} Sm/k \arrow[r, hook] \arrow[dr, hook, "Yoneda"'] & \mathrm{Shv}((Sm/k)_{Nis}) \arrow[d, hook] \\ & \mathrm{Pre}((Sm/k)_{Nis}) \end{tikzcd}$$

as required for a suitable enlargement of the category Sm/k.

Notation 6.1.7. From now on we will identify an $X \in Sm/k$ with its image $\overline{X} \in \mathrm{Shv}((Sm/k)_{Nis}) \subset \mathrm{Pre}((Sm/k)_{Nis})$.

Our next aim is to generalize the concept of a union diagram and to check that these diagrams are pushouts in Sm/k *and* in $\mathrm{Shv}((Sm/k)_{Nis})$. Furthermore the Nisnevich topology is generated by these kind of diagrams, i.e. the sheaf property can be tested on these diagrams. More precisely:

Definition 6.1.8 (EDS)**.** An *elementary distinguished square (EDS)* in $(Sm/k)_{Nis}$ is a pullback

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

such that i is an open embedding, p is an étale morphism and $p^{-1}(X-U) \xrightarrow{p} X-U$ is an isomorphism using the reduced structure of a closed subscheme.

Remark 6.1.9. Because of the pullback property an EDS can be rewritten as

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "i'"] \arrow[d, "p_{|p^{-1}(U)}"'] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

with i' also an open embedding and $p|_{p^{-1}(U)}$ also an étale morphism because étale morphisms and open embeddings are closed under base change.

Remark 6.1.10. Such a pair (i, p) of an EDS gives a Nisnevich covering of X.

Proof. First of all: As an open embedding i is an étale morphism. Now let $x \in X$.
Case 1. $x \in U$, i.e. $i(x) = x$. Then the residue fields clearly coincide because they are defined locally.
Case 2. $x \notin U$, then there is an $z \in p^{-1}(X - U)$ such that $p(z) = x$. Since for every closed embedding $j : W \hookrightarrow Y$ with $j(w) = y$ the residue fields of w and y conincide and $p^{-1}(X - U) \cong X - U$, the claim follows. $\square$

Here is one of the demanded properties of the EDSs which is [MV99, Proposition 3.1.4]

Proposition 6.1.11. *A presheaf F on $(Sm/k)_{Nis}$ is a sheaf if and only if F makes every EDS in a pullback diagram in $\mathcal{S}et$, i.e.*

$$\begin{tikzcd} F(X) \arrow[r, "F(p)"] \arrow[d, "F(i)"'] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \times_X V) \end{tikzcd}$$

is a pullback diagram in $\mathcal{S}et$ for all EDS (i, p).

Corollary 6.1.12. *Every EDS is a pushout diagram in Sm/k.*

Proof. Consider an EDS in Sm/k:

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

Since for all $Z \in Sm/k$ the presheaf $F := \mathrm{Hom}(-, Z)$ is by Lemma 6.1.5 a Nisnevich-sheaf, Proposition 6.1.11 implies that the diagram

$$\begin{tikzcd} \mathrm{Hom}(X, Z) \arrow[r] \arrow[d] & \mathrm{Hom}(V, Z) \arrow[d] \\ \mathrm{Hom}(U, Z) \arrow[r] & \mathrm{Hom}(U \times_X V, Z) \end{tikzcd}$$

is a pullback in $\mathcal{S}et$. I.e. if we take Z as a test object for the pushout property for the EDS, the pullback property of the second diagram is exactly the same as the pushout property of the first for the test object Z. $\square$

This means that an EDS in Sm/k is indeed an algebraic geometric generalization of a (topological) union square:

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \cup V \end{tikzcd}$$

We have to check the following lemma to get the last demanded property of these squares and the category $\mathrm{Shv}((Sm/k)_{Nis})$ as a suitable enlargement of Sm/k.

Lemma 6.1.13. *Every EDS is a pushout diagram in* $\mathrm{Shv}((Sm/k)_{Nis})$*. Especially*

$$V/(U \times_X V) \xrightarrow{\cong} X/U$$

canonically in $\mathrm{Shv}((Sm/k)_{Nis})$.

Proof. Let (i, p) be an EDS. Consider this as a diagram in $\mathrm{Shv}((Sm/k)_{Nis})$ and let $F \in \mathrm{Shv}((Sm/k)_{Nis})$ be a test object together with compatible morphisms $\phi : U \to F$ and $\psi : V \to F$. Because of Proposition 6.1.11 the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(p)"] \arrow[d, "F(i)"'] & F(V) \arrow[d] \\ F(U) \arrow[r] & F(U \times_X V) \end{tikzcd}$$

is a pullback in $\mathcal{S}et$ and because of the Yoneda Lemma ϕ and ψ resp. are compatible elements of $F(U)$ resp. $F(V)$ in this diagram. Hence the pullback property of this diagram in $\mathcal{S}et$ gives the pushout property for the EDS for the test object F in $\mathrm{Shv}((Sm/k)_{Nis}$.

Now consider the diagram

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & * \arrow[d] \\ V \arrow[r] & X \arrow[r] & X/U \end{tikzcd}$$

in $\mathrm{Shv}((Sm/k)_{Nis})$ where $*$ is the terminal object in this category represented by $\mathrm{Spec}(k)$. X/U is by definition the pushout of the right square which exists because of the bicompleteness of $\mathrm{Shv}(Sm/k)_{Nis})$. As we have seen the left square is also a pushout in $\mathrm{Shv}((Sm/k)_{Nis})$. It is well known that the composition of two pushout diagrams is again a pushout diagram, i.e. the outer square is a pushout. But by definition the pushout of the outer square is $V/(U \times_X V)$. Hence the claim follows. □

6.2. **The $\mathbb{A}^1$-local Model Structure.** The approach presented here is partly taken from [Bla01].

Troughout this section simplicial (pre)sheaves are always understood on the small Grothendieck site $(Sm/k)_{Nis}$. Using the local projective model structure of section 5 on simplicial (pre)sheaves together with the embeddings

$$Sm/k \hookrightarrow \mathrm{Pre}((Sm/k)_{Nis}) \hookrightarrow s\mathrm{Pre}((Sm/k)_{Nis})$$

or

$$Sm/k \hookrightarrow \mathrm{Shv}((Sm/k)_{Nis}) \hookrightarrow s\mathrm{Shv}((Sm/k)_{Nis})$$

where the right embeddings are taking discrete simplicial (pre)sheaves it is possible to get a homotopy theory on Sm/k. But as we will see in Proposition 6.2.4 the local projective model structure itself leads to an unteresting homotopy theory for smooth schemes.

Remark 6.2.1. Note that the results about the EDSs of the previous subsection still hold in $s\mathrm{Pre}((Sm/k)_{Nis})$ and $s\mathrm{Shv}((Sm/k)_{Nis})$.

The following lemma is the specialization of [Bla01, Lemma 4.1] and characterizes local projective fibrant objects.

Lemma 6.2.2. *An $F \in s\mathrm{Pre}((Sm/k)_{Nis})$ is local projective fibrant if and only if F is projective fibrant and it is* flasque*, i.e. for every EDS (i,p) the diagram*

$$\begin{tikzcd} F(X) \arrow[r, "F(p)"] \arrow[d, "F(i)"'] & F(V) \arrow[d] \\ F(U) \arrow[r] & F\left(U \times_X V\right) \end{tikzcd}$$

(denoted by $F(i,p)$) is homotopy cartesian in $s\mathcal{S}et$ (cf. [GJ99, II (8.14)]*).*

The next lemma describes a big advantage of the local projective model structure concerning smooth schemes.

Lemma 6.2.3. *Every $X \in Sm/k$ is local projective fibrant and cofibrant as a presheaf and as a sheaf.*

Proof. We are going to show this for X as a presheaf. Then the result follows for X as a sheaf because the fibrations for sheaves are exactly the fibrations for presheaves by definition and of course the presheaf-cofibrations are contained in the sheaf-cofibrations because the lifting property is fulfilled.

X is a cofibrant presheaf because the projective model structure is cofibrantly generated by the cofibrations $\{\partial\Delta^n \otimes X \to \Delta^n \otimes X \mid X \in Sm/k, n \in \mathbb{N}\}$ (cf. the proof of Theorem 5.3.7). To show that X is fibrant we will use the previous lemma. X is projective fibrant because every morphism of discrete simplicial sets is a fibration and hence every morphism of discrete simplicial presheaves is an objectwise fibration and therefore a projective fibration. It follows from Proposition 6.1.11 that $X(i,p)$ is a pullback diagram of discrete simplicial sets for all EDSs (i,p). But [GJ99, II Remark 8.17] and the fact that $X(U) \to X(U \times_X V)$ is already a fibration in $s\mathcal{S}et$ imply that $X(i,p)$ is homotopy cartesian in $s\mathcal{S}et$. □

By the following proposition, the local projective model structure leads to an uninteresting homotopy theory for smooth schems. That is, this model structure yields no identification of smooth schemes in the homotopy category.

Proposition 6.2.4. *Sm/k embeds fully faithfully in the local projective homotopy category on simplicial (pre)sheaves.*

Proof. Let $X, Y \in Sm/k$. Using Lemma 6.2.3 we have $\mathrm{Hom}_{\mathrm{Ho}}(X,Y) = [X,Y]_{\mathrm{loc.proj.}}$ local projective homotopy classes where Ho denotes the homotopy category of the local projective model structure on simplicial sheaves or presheaves on $(Sm/k)_{Nis}$. But since these two model structures are simplicial, Lemma 3.5.7 implies that local projective homotopies are always given via $\Delta^1 \otimes X = \Delta^1 \times X$. But X and Y are discrete simplicial presheaves hence there are only constant homotopies and therefore

$$\mathrm{Hom}_{\mathrm{Ho}}(X,Y) \cong [X,Y]_{\mathrm{loc.proj.}} = \mathrm{Hom}_{s\mathrm{Pre}}(X,Y) \cong \mathrm{Hom}_k(X,Y). \qquad \square$$

Therefore we have to enlarge the class of weak equivalences to get a larger localization of the category via the homotopy category. A good method to do this is to use a Bousfield localization as introduced in subsection 3.6. Another demanded property is, that the affine line $\mathbb{A}^1$ should play the same role as the unit interval $I = [0, 1]$ in the usual homotopy theory of topological spaces, that is, to generalize the naive point of view of the Introduction. But to satisfy this, $\mathbb{A}^1$ should be weak equivalent to $* = \mathrm{Spec}(k)$, i.e. trivial in the homotopy category. Therefore the k-rational point $0 : \mathrm{Spec}(k) \to \mathbb{A}^1$ should be a weak equivalence and also all morphisms obtained from it in Sm/k, i.e. the morphisms $U \to U \times \mathbb{A}^1$ for all $U \in Sm/k$ obtained from the point 0.

Definition 6.2.5 ($\mathbb{A}^1$-local objects and $\mathbb{A}^1$-local weak equivalences)**.** Let $Z \in s\mathrm{Pre}((Sm/k)_{Nis})$. Then Z is called $\mathbb{A}^1$-*local* if it is local projective fibrant and

$$g^* : \mathrm{Map}(U \times \mathbb{A}^1, Z) \to \mathrm{Map}(U, Z)$$

is a weak equivalence of simplicial sets for all $g \in S := \{U \xrightarrow{U\times 0} U \times \mathbb{A}^1 \mid U \in Sm/k\}$, the set of maps obtained from the k-rational point $0 : \mathrm{Spec}(k) \to \mathbb{A}^1$.

A map $f : X \to Y \in s\mathrm{Pre}((Sm/k)_{Nis})$ is called an $\mathbb{A}^1$-*local weak equivalence* if

$$(Qf)^* : \mathrm{Map}(QY, Z) \to \mathrm{Map}(QX, Z)$$

is a weak equivalence of simplicial sets for all $\mathbb{A}^1$-local Z where Q is the local projective cofibrant replacement functor.

Remark 6.2.6. In view of Remark 3.6.2 and the fact that $U \times \mathbb{A}^1$ and U are local projective cofibrant for all $U \in Sm/k$ by Lemma 6.2.3, the $\mathbb{A}^1$-local objects and $\mathbb{A}^1$-local weak equivalences are the same as the S-local objects and S-local weak equivalences for the set of maps

$$S = \{U \xrightarrow{U\times 0} U \times \mathbb{A}^1 \mid U \in Sm/k\}.$$

Thanks to Theorem 5.3.9 and Theorem 5.4.2 we can apply Theorem 3.6.14 (the existence of left Bousfield localization) to the local projective model structure on simplicial (pre)sheaves on the site $(Sm/k)_{Nis}$ with respect to the set of morphisms S defined above to get the following so-called $\mathbb{A}^1$-*local model structure.*

Theorem 6.2.7. *The categories of simplicial sheaves and presheaves on* $(Sm/k)_{Nis}$ *together with the classes*

- *w.e. =* $\mathbb{A}^1$*-local weak equivalences*
- *cof. = local projective cofibrations*

form proper simplicial cellular model categories.

Proof. It is clear that the $\mathbb{A}^1$-local weak equivalences are exactly the S-local weak equivalences since every $U \in Sm/k$ is local projective fibrant by Lemma 6.2.3. Therefore almost everything follows from Theorem 3.6.14 up to the right properness. This is done in [Bla01, Lemma 3.1]. □

Remark 6.2.8. Note that by Theorem 3.6.14 (1) the simplicial structure is again given by the same adjoints as defined in Definition 5.3.1. Especially $\Delta^1 \otimes F = \Delta^1 \times F$ for $F \in s\mathrm{Pre}((Sm/k)_{Nis})$.

Remark 6.2.9. It is clear that for every k-rational point $p : \mathrm{Spec}(k) \to \mathbb{A}^1$ the diagram

$$\begin{tikzcd} X \arrow[r, "X\times p"] \arrow[rr, bend right, "\mathrm{id}"'] & X \times \mathbb{A}^1 \arrow[r] & X \end{tikzcd}$$

commutes. Therefore for every $U \in Sm/k$ the map $U \xrightarrow{\mathrm{id}\times 0} U \times \mathbb{A}^1$ in S can be replaced by $U \xrightarrow{\mathrm{id}\times p} U \times \mathbb{A}^1$ for any other k-rational point $p : \mathrm{Spec}(k) \to \mathbb{A}^1$ or by the projection $\pi : U \times \mathbb{A}^1 \to U$ to get the same weak equivalences according to Lemma 3.6.10.

The $\mathbb{A}^1$-local model structure used in [MV99] can be obtained completely analogously with use of the local injective model structure on simplicial (pre)sheaves instead of the local projective one and the same set of morphisms S together with the left Bousfield localization.

Remark 6.2.10. Consider again the diagram of Quillen equivalences of Remark 5.4.4

$$\begin{tikzcd} (s\mathrm{Pre}((Sm/k)_{Nis}), \mathrm{loc.proj.}) \arrow[r, shift left, "a"] \arrow[d, shift left, "\mathrm{id}"] & (s\mathrm{Shv}((Sm/k)_{Nis}), \mathrm{loc.proj.}) \arrow[l, shift left, "\iota"] \arrow[d, shift left, "\mathrm{id}"] \\ (s\mathrm{Pre}((Sm/k)_{Nis}), \mathrm{loc.inj.}) \arrow[r, shift left, "a"] \arrow[u, shift left, "\mathrm{id}"] & (s\mathrm{Shv}((Sm/k)_{Nis}), \mathrm{loc.inj.}) \arrow[l, shift left, "\iota"] \arrow[u, shift left, "\mathrm{id}"] \end{tikzcd}$$

Now we have $a(QR(S)) \sim S$, where Q is the cofibrant and R the fibrant replacement functor of the local projective model structure: Every presheaf involved in a morphism of S is a sheaf, local projective fibrant, and cofibrant as a presheaf and as a sheaf by Lemma 6.2.3 and a is a left Quillen functor on the local projective model structure. For the same reason $\mathrm{id}(QR(S)) \sim S$ on presheaves and on sheaves. Furthermore all simplicial presheaves are local injective cofibrant and a as a Quillen functor preserves weak equivalences between cofibrant objects. Hence $a(QR(S)) \sim a(RS) \sim a(S) = S$. Therefore Theorem 3.6.12 (2) together with Lemma 3.6.10 imply that this diagram gives also a diagram of Quillen equivalences between the $\mathbb{A}^1$-local model structures.

Therefore we have:

Theorem 6.2.11. *The following homotopy categories are equivalent:*

$$\begin{aligned} \mathrm{Ho}_{\mathbb{A}^1}((s\mathrm{Pre}((Sm/k)_{Nis}), \mathrm{l.proj.})) &\simeq \mathrm{Ho}_{\mathbb{A}^1}((s\mathrm{Pre}((Sm/k)_{Nis}), \mathrm{l.inj.})) \\ &\simeq \mathrm{Ho}_{\mathbb{A}^1}((s\mathrm{Shv}((Sm/k)_{Nis}), \mathrm{l.proj.})) \\ &\simeq \mathrm{Ho}_{\mathbb{A}^1}((s\mathrm{Shv}((Sm/k)_{Nis}), \mathrm{l.inj.})) \end{aligned}$$

where $\mathrm{Ho}_{\mathbb{A}^1}$ *denotes the* $\mathbb{A}^1$*-local structure.*

From now on we will only consider the $\mathbb{A}^1$-local model structure obtained from the local projective model structure. This model structure gives us the so-called *motivic homotopy theory* on Sm/k.

Definition 6.2.12 (Motivic homotopy category)**.** Let

$$\mathcal{H}(k) := \mathrm{Ho}_{\mathbb{A}^1}(s\mathrm{Pre}((Sm/k)_{Nis})) \simeq \mathrm{Ho}_{\mathbb{A}^1}(s\mathrm{Shv}((Sm/k)_{Nis}))$$

be the homotopy category of the $\mathbb{A}^1$-local model structure on simplicial (pre)sheaves. This category is called the *motivic homotopy category of k*.

Remark 6.2.13. Being only interested in homotopy types, i.e. isomorphism classes in the homotopy category, there is no difference between taking presheaves or sheaves for our purpose.

Definition 6.2.14. There is a canonical functor

$$h_k : Sm/k \to \mathcal{H}(k)$$

obtained from the composition

$$Sm/k \hookrightarrow \mathrm{Shv}((Sm/k)_{Nis}) \hookrightarrow s\mathrm{Shv}((Sm/k)_{Nis}) \xrightarrow{\gamma} \mathcal{H}(k)$$

or

$$Sm/k \hookrightarrow \mathrm{Pre}((Sm/k)_{Nis}) \hookrightarrow s\mathrm{Pre}((Sm/k)_{Nis}) \xrightarrow{\gamma} \mathcal{H}(k)$$

where the last functor γ is the localization functor into the homotopy category of the $\mathbb{A}^1$-local model structure. The index k of h_k is omitted if the ground field is clear.

One advantage of the $\mathbb{A}^1$-local projective model structure is the functoriality of the whole construction in changing the base field as it was developed for example by Oliver Röndigs in [Rön05].

Proposition 6.2.15. *Let L/k be a finite separable field extension. Then $f : k \hookrightarrow L$ induces a functor*

$$f^* : \mathcal{H}(k) \longrightarrow \mathcal{H}(L)$$

such that the diagram

$$\begin{tikzcd} Sm/k \arrow[r, "f^*"] \arrow[d, "h_k"'] & Sm/L \arrow[d, "h_L"] \\ \mathcal{H}(k) \arrow[r, "f^*"] & \mathcal{H}(L) \end{tikzcd}$$

commutes up to functor-isomorphism where $f^*(X) = X_L$ for $X \in Sm/k$ is just the base change functor.

Proof. First of all, we take the morphism $f : \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ instead of f. Note that f is smooth. Hence f induces the functor $f_* : Sm/L \longrightarrow Sm/k$ via precomposition with f which induces

$$f^* : s\mathrm{Pre}((Sm/k)_{Nis}) \longrightarrow s\mathrm{Pre}((Sm/L)_{Nis})$$

via precomposition with the opposite of f_*. Hence the diagram

$$\begin{tikzcd} Sm/k \arrow[r, "f^*"] \arrow[d, hook] & Sm/L \arrow[d, hook] \\ s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[r, "f^*"] & s\mathrm{Pre}((Sm/L)_{Nis}) \end{tikzcd}$$

commutes up to functor-isomorphism because there is the isomorphism

$$\mathrm{Hom}_k(f_*(-), X) \cong \mathrm{Hom}_L(-, f^*(X))$$

which is natural in X. This isomorphism follows immediately from the universal property of the base change $f^*(X)$. Furthermore f^* on presheaves has the right adjoint

$$f_* : s\mathrm{Pre}((Sm/L)_{Nis}) \longrightarrow s\mathrm{Pre}((Sm/k)_{Nis})$$

which is defined by precomposition with the base change functor on Sm/k (cf. [Rön05, Lemma 2.4]). Finally [Rön05, Proposition 2.15] implies that (f^*, f_*) is a Quillen functor on the $\mathbb{A}^1$-local model structure since Röndigs also uses the local projective model structure. Now we take the total left derived functor $f^* := Lf^* : \mathcal{H}(k) \to \mathcal{H}(L)$ (cf. Corollary 3.3.5) and we get the claimed diagram. □

Now we have to notice some important technical properties of this model structure to be able to continue our studies.

Lemma 6.2.16. *Every $X \in Sm/k$ is $\mathbb{A}^1$-local cofibrant.*

Proof. Since every $X \in Sm/k$ is local projective cofibrant by Lemma 6.2.3 and the $\mathbb{A}^1$-local cofibrations are exactly the local projective cofibrations the claim follows. □

Remark 6.2.17. The generating cofibrations of the $\mathbb{A}^1$-local model structure are of course the same as the generating projective cofibrations that is the maps $X \otimes (\partial\Delta^n \hookrightarrow \Delta^n)$ where $X \in Sm/k$ and $n \geq 0$.

The generating $\mathbb{A}^1$-local acyclic cofibrations are a little bit mysterious but there is a subset J of them which fulfil some of their properties in special situations.

Definition 6.2.18. Define a set J of maps in $s\mathrm{Pre}((Sm/k)_{Nis})$ consisting of the following maps:

- $(\Lambda^n_i \hookrightarrow \Delta^n) \otimes X$ for all $X \in Sm/k$, $n \geq 1$ and $0 \leq i \leq n$.
- For every EDS

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, hook, "i"] & X \end{tikzcd}$$

  the pushout products of the map

$$U \amalg_Y \mathrm{Cyl}(j) \rightarrowtail \mathrm{Cyl}(U \amalg_Y \mathrm{Cyl}(j) \to X)$$

  with the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n \geq 0$ where $\mathrm{Cyl}(-)$ is the simplicial mapping cylinder.
- For every projection $X \times \mathbb{A}^1 \to X$ where $X \in Sm/k$, the pushout products of the map

$$X \times \mathbb{A}^1 \rightarrowtail \mathrm{Cyl}(X \times \mathbb{A}^1 \to X)$$

  with the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n \geq 0$.

Remark 6.2.19. Note that all $U \amalg_Y \mathrm{Cyl}(j)$ and X are $\mathbb{A}^1$-local cofibrant: According to Lemma 3.5.10 and Remark 3.5.11 $\mathrm{Cyl}(j)$ is $\mathbb{A}^1$-local cofibrant and $U \rightarrowtail U \amalg_Y \mathrm{Cyl}(j)$ is a $\mathbb{A}^1$-local cofibration since U and Y are cofibrant by Lemma 6.2.16. Therefore the indicated cofibrations $U \amalg_Y \mathrm{Cyl}(j) \rightarrowtail$

$\mathrm{Cyl}(U \amalg_Y \mathrm{Cyl}(j) \to X)$ and $X \times \mathbb{A}^1 \rightarrowtail \mathrm{Cyl}(X \times \mathbb{A}^1 \to X)$ are indeed $\mathbb{A}^1$-local cofibrations by Lemma 3.5.10. Furhermore they are $\mathbb{A}^1$-local weak equivalences by the following lemma and therefore acyclic cofibrations. Hence the pushout products with the maps $\partial\Delta^n \hookrightarrow \Delta^n$ are acyclic cofibrations because of the simplicial model category axiom (**SM7b**). For the same reason the maps $(\Lambda^n_i \hookrightarrow \Delta^n) \otimes X$ are acyclic cofibrations since every $X \in Sm/k$ is cofibrant and the maps $\Lambda^n_i \hookrightarrow \Delta^n$ are clearly objectwise acyclic cofibrations.

Lemma 6.2.20. *All maps*

$$U \amalg_Y \mathrm{Cyl}(j) \rightarrowtail \mathrm{Cyl}(U \amalg_Y \mathrm{Cyl}(j) \to X)$$

and

$$X \times \mathbb{A}^1 \rightarrowtail \mathrm{Cyl}(X \times \mathbb{A}^1 \to X)$$

of the previous definition are $\mathbb{A}^1$-local weak equivalences.

Proof. According to Lemma 3.5.12 it suffices to show that $U \amalg_Y \mathrm{Cyl}(j) \to X$ and $X \times \mathbb{A}^1 \to X$ are $\mathbb{A}^1$-local weak equivalences since all involved objects are $\mathbb{A}^1$-local cofibrant. Because of the commutative diagram

$$\begin{tikzcd} X \arrow[r, "X\times 0"] \arrow[rr, bend right, "\mathrm{id}"'] & X \times \mathbb{A}^1 \arrow[r] & X \end{tikzcd}$$

and the 2 out of 3 property for $\mathbb{A}^1$-local weak equivalences this is clear for the second type of maps. For the first type note that there is a commutative diagram

$$\begin{tikzcd} U \amalg_Y V \arrow[d, tail, "\sim"'] \arrow[dr] & \\ U \amalg_Y \mathrm{Cyl}(j) \arrow[r] & X \end{tikzcd}$$

where the vertical map is an acyclic cofibration since this is the case for $V \to \mathrm{Cyl}(j)$. Because of the 2 out of 3 property for $\mathbb{A}^1$-local weak equivalences it suffices to show that $U \amalg_Y V \to X$ are $\mathbb{A}^1$-local weak equivalences. For this recall that every EDS is a pushout diagram in the category of simplicial sheaves by Lemma 6.1.13. Furthermore colimits of sheaves are computed as the sheafified colimits of the underlying presheaves, that is, the maps in question read as $U \amalg_Y V \to a(U \amalg_Y V)$. This a local weak equivalence by [Jar87, Lemma 2.6]. $\square$

The following lemma describes the role of this set J for fibrations and is given in [DRØ03, Lemma 2.15].

Lemma 6.2.21. *A morphism $f : X \to Y$, such that Y is $\mathbb{A}^1$-local fibrant, is an $\mathbb{A}^1$-local fibration if and only if it has the right lifting property with respect to J.*

Furthermore this set of maps J admits the small object argument with respect to $\mathbb{N}$:

Lemma 6.2.22. *J admits the small object argument with respect to $\mathbb{N}$* [Hov99, Theorem 2.1.14]*, that is the domains of the maps of J are small relative to J-cell with respect to $\mathbb{N}$ (cf.* [Hov99, Definition 2.1.3]*).*

Proof. This follows immediately from [DRØ03, Lemma 2.5]. □

Remark 6.2.23. This means that the objects of J-cell for the factorization can be chosen as sequential colimits

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$$

indexed over the natural numbers $\mathbb{N}$ where the f_i are pushouts of coproducts of maps in J.

The following lemma characterizes the $\mathbb{A}^1$-local fibrant simplicial (pre)-sheaves.

Lemma 6.2.24. *Let F be a simplicial (pre)sheaf. Then: F is $\mathbb{A}^1$-local fibrant if and only if F is local projective fibrant and $F(g) : F(U \times \mathbb{A}^1) \to F(U)$ is a weak equivalence of simplicial sets for all $g \in S$.*

Proof. Using Theorem 3.6.14 (2) the $\mathbb{A}^1$-local fibrant objects are exactly the $\mathbb{A}^1$-local objects. Hence F is $\mathbb{A}^1$-local fibrant if and only if F is local projective fibrant and

$$g^* : \mathrm{Map}(U \times \mathbb{A}^1, F) \to \mathrm{Map}(U, F)$$

is a weak equivalence of simplicial sets for all $g \in S$. But using Lemma 5.2.3 there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}(U \times \mathbb{A}^1, F) \arrow[r, phantom, "\cong"] \arrow[d, "g^*"'] & F(U \times \mathbb{A}^1) \arrow[d, "F(g)"] \\
\mathrm{Map}(U, F) \arrow[r, phantom, "\cong"] & F(U)
\end{tikzcd}$$

and hence using 2 out of 3 g^* is a weak equivalence if and only if $F(g)$ is a weak equivalence. □

$\mathbb{A}^1$-local weak equivalences between $\mathbb{A}^1$-local fibrant objects are easy to understand since they are objectwise weak equivalences by the following lemma.

Lemma 6.2.25. *Let F and G be $\mathbb{A}^1$-local fibrant (pre)sheaves. Then: $f : F \to G$ is an $\mathbb{A}^1$-local weak equivalence if and only if f is a projective (i.e. objectwise) weak equivalence.*

Proof. If f is a projective weak equivalence it is clearly an $\mathbb{A}^1$-local weak equivalence because $\sim_{\text{proj.}} \subset \sim_{\text{loc.proj.}} \subset \sim_{\mathbb{A}^1-\text{local}}$. Now let f be an $\mathbb{A}^1$-local weak equivalence. Because of Lemma 6.2.16 every $U \in Sm/k$ is $\mathbb{A}^1$-local cofibrant and therefore Lemma 3.5.6 implies that $\mathrm{Map}(U, -)$ is a right Quillen functor because the $\mathbb{A}^1$-local structure is simplicial and therefore it preserves weak equivalences between fibrant objects. Hence $f_* : \mathrm{Map}(U, F) \to \mathrm{Map}(U, G)$ is a weak equivalence for all $U \in Sm/k$. But $\mathrm{Map}(U, F) = F(U)$ by Lemma 5.2.3 and the same for G. Therefore $f(U) : F(U) \to G(U)$ is a weak equivalence in $sSet$ for all $U \in Sm/k$ and hence f is an objectwise, i.e. projective, weak equivalence. □

6.3. **$\mathbb{A}^1$-rigid Schemes.** Due to the Whitehead Theorem 3.1.16 the cofibrant and fibrant schemes are quite good objects to study homotopy types of, because $\mathbb{A}^1$-local weak equivalences and homotopy equivalences are the same between those objects. As we already know every smooth scheme is cofibrant. So the question is: Which ones are fibrant?

Definition 6.3.1 ($\mathbb{A}^1$-rigid scheme)**.** Let $X \in Sm/k$. X is called *$\mathbb{A}^1$-rigid* if

$$\mathrm{Hom}_k(U \times \mathbb{A}^1, X) \to \mathrm{Hom}_k(U, X)$$

is bijective for all $U \in Sm/k$ where the map is induced by $U \to U \times \mathbb{A}^1$ obtained from the k-rational point $0 : \mathrm{Spec}(k) \to \mathbb{A}^1$.

Remark 6.3.2. The maps inducing the isomorphisms for the $\mathbb{A}^1$-rigid property are exactly the same maps of the set S which is used to get the $\mathbb{A}^1$-local model structure via Bousfield localization.

Lemma 6.3.3. *Let $X \in Sm/k$. The following statements are equivalent:*

(1) *X is $\mathbb{A}^1$-rigid.*
(2) $\mathrm{Hom}_k(U \times \mathbb{A}^1, X) \xrightarrow{(\mathrm{id}\times 0)^*} \mathrm{Hom}_k(U, X)$ *is injective for all $U \in Sm/k$.*
(3) $\mathrm{Hom}_k(U, X) \xrightarrow{\pi^*} \mathrm{Hom}_k(U \times \mathbb{A}^1, X)$ *is surjective for all $U \in Sm/k$ where $\pi : U \times \mathbb{A}^1 \to U$ is the projection.*

Proof. Note that the diagram

$$\begin{tikzcd} \mathrm{Hom}_k(U, X) \arrow[r, "\pi^*"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Hom}_k(U \times \mathbb{A}^1, X) \arrow[r, "(\mathrm{id}\times 0)^*"] & \mathrm{Hom}_k(U, X) \end{tikzcd}$$

commutes and π^* is always injective. This gives the claim. ☐

Example 6.3.4. $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid.

Proof. Note that $\mathbb{G}_m = k[X, X^{-1}]$ as a represented presheaf induces the functor $\mathbb{G}_m(U) \cong \mathcal{O}_U(U)^*$ on Sm/k. Being also a Zariski-sheaf the $\mathbb{A}^1$-rigidity can be checked on affine $U \in Sm/k$. But such an affine U has the form $U = \mathrm{Spec}(R)$ where R is a finitely generated k-algebra and hence $U \times \mathbb{A}^1 = \mathrm{Spec}(R[X])$. Furthermore the diagram

$$\begin{tikzcd} \mathrm{Hom}_k(U \times \mathbb{A}^1, \mathbb{G}_m) \arrow[d] & = & \mathbb{G}_m(U \times \mathbb{A}^1) & \cong & R[X]^* \arrow[d, "(X\mapsto 0)"] \\ \mathrm{Hom}_k(U, \mathbb{G}_m) & = & \mathbb{G}_m(U) & \cong & R^* \end{tikzcd}$$

commutes and $(X \mapsto 0) : R[X]^* \to R^*$ is clearly an isomorphism and hence also the left vertical arrow as claimed. ☐

Example 6.3.5. $\mathbb{P}^1$ is not $\mathbb{A}^1$-rigid because the map

$$\mathrm{Hom}_k(\mathrm{Spec}(k), \mathbb{P}^1) \xrightarrow{\pi^*} \mathrm{Hom}_k(\mathrm{Spec}(k) \times \mathbb{A}^1, \mathbb{P}^1)$$

is not surjective: Every image $\mathbb{A}^1 \to \mathrm{Spec}(k) \to \mathbb{P}^1$ is a constant map, but there are non-constant embeddings $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$.

Here is the answer to the question: Which smooth schemes are fibrant?

Lemma 6.3.6. *Let $X \in Sm/k$. Then: X is an $\mathbb{A}^1$-rigid scheme if and only if X is $\mathbb{A}^1$-local fibrant.*

Proof. Lemma 6.2.3 implies that every $X \in Sm/k$ is local projective fibrant. Now using Lemma 6.2.24 X is $\mathbb{A}^1$-local fibrant if and only if $g^* : X(U \times \mathbb{A}^1) \to X(U)$ is a weak equivalence of simplicial sets for all $g \in S$. But these simplicial sets are all discrete ones (which are fibrant and cofibrant simplicial sets). Hence these maps are weak equivalences if and only if they are homotopy equivalences if and only if they are isomorphisms. □

One could think that studying homotopy types of $\mathbb{A}^1$-rigid schemes should be nice because we can focus on homotopy equivalences. But everything is boring because nothing happens there, that is, there are only trivial homotopies between those schemes. This is motivated by the fact that for an $\mathbb{A}^1$-rigid scheme Y and a naive homotopy $H : X \times \mathbb{A}^1 \to Y$ between two maps $f, g : X \to Y$ these two maps have to be the same which follows immediately from the $\mathbb{A}^1$-rigidity of Y for X. The formal proof is given in the next theorem.

Theorem 6.3.7. *The full subcategory of $\mathbb{A}^1$-rigid schemes in Sm/k embeds fully faithfully via h into the motivic homotopy category of k. Especially:*

$$h(X) \cong h(Y) \text{ in } \mathcal{H}(k) \Longleftrightarrow X \cong Y \text{ in } Sm/k$$

for X and Y $\mathbb{A}^1$-rigid schemes.

Proof. Let X and Y be $\mathbb{A}^1$-rigid schemes. Then $\mathrm{Hom}_{\mathcal{H}(k)}(X, Y) = [X, Y]_{\mathbb{A}^1}$ ($\mathbb{A}^1$-local homotopy classes) because X and Y are $\mathbb{A}^1$-local fibrant and cofibrant by the previous lemma and Lemma 6.2.16. According to Lemma 3.5.7 $\mathbb{A}^1$-local homotopies are always given via $\Delta^1 \otimes X = \Delta^1 \times X$ because this model structure is simplicial. But X as well as Y are discrete simplicial presheaves hence there are only constant homotopies and it follows that

$$\mathrm{Hom}_{\mathcal{H}(k)}(h(X), h(Y)) = \mathrm{Hom}_{s\mathrm{Pre}}(X, Y) = \mathrm{Hom}_k(X, Y).$$ □

7. Categories of Motives

The aim of this section is to introduce the category of Chow motives first considered by Grothendieck and the triangulated category of effective motives given by Vladimir Voevodsky in [Voe00]. The approach presented here is mainly taken from [MVW06]. For the language and theory of abelian, derived and triangulated categories confer [Wei94] and [Nee01].

7.1. Chow Motives and $SmCor(k)$.

Recall that $SmProj/k$ denotes the category of disjoint unions of smooth projective k-varieties.

Definition 7.1.1 (Chow motives)**.** Define the category of *Chow motives* $\mathrm{CH}(k)$ as follows: The objects are exactly the objects of $SmProj/k$ but for $X, Y \in SmProj/k$ the morphisms are defined by

$$\mathrm{Hom}_{\mathrm{CH}(k)}(X, Y) := \bigoplus_{Y_i} \mathrm{CH}^{\dim Y_i}(X \times Y_i)$$

where the Y_i run through the connected components of Y and $\mathrm{CH}^n(Z)$ are the cycles of codimension n of Z modulo rational equivalence (cf. [Ful98]).

The composition of morphisms is given by: Let $X, Y, Z \in SmProj/k$. Then there is a bilinear homomorphism

$$\begin{array}{ccccc} \mathrm{CH}^*(X \times Y) & \otimes & \mathrm{CH}^*(Y \times Z) & \longrightarrow & \mathrm{CH}^*(X \times Z) \\ (\alpha & , & \beta) & \mapsto & \beta \circ \alpha := \pi_{XZ*}((\alpha \times Z) \cdot (X \times \beta)) \end{array}$$

where $\pi_{XZ} : X \times Y \times Z \to X \times Z$ is the projection and $\cdot$ is the multiplication of the graded Chow ring $\mathrm{CH}^*(X \times Y \times Z)$. This induces an associative bilinear homomorphism

$$\mathrm{Hom}_{\mathrm{CH}(k)}(X, Y) \otimes \mathrm{Hom}_{\mathrm{CH}(k)}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathrm{CH}(k)}(X, Z)$$

(cf. [Ful98, Definition 16.1.1, Proposition 16.1.1 and Example 16.1.1]). Note that the identity morphism is given by the diagonal $\Delta_X \in \mathrm{CH}^{\dim X}(X \times X)$ if X is connected.

Remark 7.1.2. There is a canonical functor

$$SmProj/k \xrightarrow{\Gamma} \mathrm{CH}(k)$$

defined by $\Gamma(X) = X$ on objects. The action on morphisms is $\Gamma(X \xrightarrow{f} Y) = \Gamma_f$, the rational equivalence class of the graph of f if X is connected. Note that the diagram

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times Y \arrow[d, "f \times \mathrm{id}"] \\ Y \arrow[r, hook, "\Delta_Y"] & Y \times Y \end{tikzcd}$$

is a pullback diagram and the diagonal morphism Δ_Y is a closed embedding since Y is a separated scheme. Therefore the graph Γ_f is indeed a closed embedding because closed embeddings are closed under base change. Furthermore if we take the reduced structure of a closed subscheme, Γ_f becomes integral and therefore a subvariety of $X \times Y$ because X is already irreducible. Hence Γ_f lives indeed in $\mathrm{Hom}_{\mathrm{CH}(k)}(X, Y)$. Furthermore it is finite and surjective over X if X is connected since the composition

$$X \xrightarrow{\Gamma_f} X \times Y \xrightarrow{\pi_X} X$$

is the identity on X.

Definition 7.1.3 (Finite correspondences)**.** Let $X, Y \in Sm/k$ such that X is connected. An *elementary finite correspondence from X to Y* is a closed integral subset $W \subseteq X \times Y$ whose associated integral subscheme (i.e. taking the reduced structure) is finite and surjective over X. If X is not connected we understand by an elementary finite correspondence from X to Y one from a connected component of X to Y. The group of *finite correspondences from X to Y $Cor(X, Y)$* is defined as the free abelian group generated by the elementary correspondences from X to Y.

Example 7.1.4. Note that

$$Cor(X, \mathrm{Spec}(k)) = \bigoplus_{X_i} \mathbb{Z}$$

where the X_i run through the connected components of X since these are the elementary correspondences $X_i \subseteq X_i = X_i \times \mathrm{Spec}(k)$.

Definition 7.1.5 ($SmCor(k)$)**.** Define the Suslin-Voevodsky category of *smooth correspondences* $SmCor(k)$ as follows: The objects are exactly the smooth k-schemes as in Sm/k but for $X, Y \in Sm/k$ the morphisms are defined as the finite correspondences $Cor(X, Y)$ from X to Y. The composition of morphisms is defined similarly as in $\mathrm{CH}(k)$ but without passing to rational equivalence classes. This is well defined (cf. [MVW06, Lecture 1]).

Remark 7.1.6. There is a canonical faithful functor

$$Sm/k \xrightarrow{[-]} SmCor(k)$$

defined by $[X] = X$ on objects. The action on morphisms is $[X \xrightarrow{f} Y] = \Gamma_f$ (the graph of f) if X is connected. Since the arguments of Remark 7.1.2 work for all $X \in Sm/k$, we get that Γ_f is an elementary finite correspondence from X to Y if X is connected. The faithfulness of $[-]$ is clear since a morphism $f : X \to Y \in Sm/k$ is determined by its graph via $f = X \xrightarrow{\Gamma_f} X \times Y \xrightarrow{\pi_Y} Y$.

Therefore we should think about the category $SmCor(k)$ as the category Sm/k endowed with more morphisms.

Remark 7.1.7. $SmCor(k)$ is an additive category with direct sum

$$[X] \oplus [Y] = [X \amalg Y]$$

and zero object $\emptyset$. The universal property of the direct sum follows immediately from the fact that $Cor(\coprod X_i, Y) = \bigoplus Cor(X_i, Y)$.

Definition 7.1.8 (Symmetric monoidal structure)**.** For $X, Y \in SmCor(k)$ let

$$X \otimes Y := [X \times Y]$$

be the tensor product. The functoriality is given via: Let V be an elementary correspondence from X to X' and W an elementary correspondence from Y to Y'. Then $V \times W$ gives a finite correspondence via $\sum n_i Z_i \in Cor(X \otimes Y, X' \otimes Y')$ where the Z_i are the irreducible components of $V \times W$ and $n_i = l_{\mathcal{O}_{Z_i, V\times W}}(\mathcal{O}_{Z_i, V \times W})$ is the geometric multiplicity of Z_i in $V \times W$ (cf. [Ful98, Ch.1.5]).

This gives $SmCor(k)$ a symmetric monoidal structure with unit $\mathrm{Spec}(k)$.

Remark 7.1.9. Note that the direct sum commutes with the tensor product, that is

$$Y \otimes (\bigoplus_i X_i) \cong \bigoplus_i (Y \otimes X_i).$$

Therfore $SmCor(k)$ is an *additive symmetric monoidal category* in the sense of [MVW06, Definition 8A.3].

7.2. **(Pre)Sheaves with Transfers.** Now we will introduce the objects that will lead to the objects of the triangulated category of effective motives.

Definition 7.2.1 ((Pre)Sheaves with transfers)**.** A *presheaf with transfers X on Sm/k* is an additive functor

$$X : SmCor(k)^{op} \longrightarrow \mathcal{A}b$$

which is called a *Nisnevich sheaf with transfers* if the composition

$$(Sm/k)^{op} \xrightarrow{[-]^{op}} SmCor(k)^{op} \xrightarrow{X} \mathcal{A}b$$

is an abelian sheaf on the Grothendieck site $(Sm/k)_{Nis}$.

Notation 7.2.2. Denote the category of presheaves with transfers on Sm/k by $\mathrm{Pre}^{tr}(k)$ and the category of Nisnevich sheaves with transfers on Sm/k by $\mathrm{Shv}^{tr}_{Nis}(k)$. Furthermore denote by

$$V : \mathrm{Pre}^{tr}(k) \longrightarrow \mathrm{Pre}((Sm/k)_{Nis})$$

the forgetful functor $V(X) = F \circ X \circ [-]$ where $F : \mathcal{A}b \to \mathcal{S}et$ is the forgetful functor from abelian goups to sets.

The next proposition follows immediately from [MVW06, Theorem 13.1].

Proposition 7.2.3. *There is an adjoint functor pair*

$$(-)_{Nis} : \mathrm{Pre}^{tr}(k) \rightleftarrows \mathrm{Shv}^{tr}_{Nis}(k) : \iota$$

where ι is the forgetful functor. The functor $(-)_{Nis}$ is called the associated sheaf functor. *Furthermore the abelian sheaf $X_{Nis} \circ [-]$ is nothing else than the sheafified abelian presheaf $X \circ [-]$. Therefore $(-)_{Nis} \circ \iota \cong \mathrm{id}$ and $(-)_{Nis}$ commutes with finite limits.*

Remark 7.2.4. There is of course again an extension of this adjunction on simplicial (pre)sheaves with transfers

$$(-)_{Nis} : s\mathrm{Pre}^{tr}(k) \rightleftarrows s\mathrm{Shv}^{tr}_{Nis}(k) : \iota$$

built up objectwise, that is for a simplicial presheaf with transfers $X : \Delta^{op} \to \mathrm{Pre}^{tr}(k)$ take X_{Nis} as the composition

$$\Delta^{op} \xrightarrow{X} \mathrm{Pre}^{tr}(k) \xrightarrow{(-)_{Nis}} \mathrm{Shv}^{tr}_{Nis}(k)$$

Corollary 7.2.5. *Both $\mathrm{Pre}^{tr}(k)$ and $\mathrm{Shv}^{tr}_{Nis}(k)$ are bicomplete abelian categories and the sheafification $(-)_{Nis}$ is an exact functor.*

Proof. The bicompleteness is analogous to Corollary 4.2.2: In $\mathrm{Pre}^{tr}(k)$ do everything objectwise since all objectwise built small limits and colimits provide additive functors. The limits in $\mathrm{Shv}^{tr}_{Nis}(k)$ are the same as the limits of the underlying presheaves and the colimits are the sheafified colimits of the underlying presheaves. The rest is standard since an additive functor category with abelian target category is again abelian if you do everything objectwise which provides the abelian structure for $\mathrm{Pre}^{tr}(k)$. The abelian structure on $\mathrm{Shv}^{tr}_{Nis}(k)$ follows. $\square$

The next theorem is a standard result on additive functor categories where the target category is an abelian category with enough projective and injective objects. It is stated as [MVW06, Theorem 2.3].

Theorem 7.2.6. *The abelian category $\mathrm{Pre}^{tr}(k)$ of presheaves with transfers has enough projectives and injectives.*

Notation 7.2.7. Let $X \in Sm/k$. Denote by $\mathbb{Z}_{tr}(X)$ the represented presheaf with transfers, that is

$$\mathbb{Z}_{tr}(X)(U) = Cor(U, X).$$

Furthermore denote $\mathbb{Z} := \mathbb{Z}_{tr}(\mathrm{Spec}(k))$ which is nothing else than the constant Zariski-sheaf $\mathbb{Z}$ on Sm/k since $Cor(X, \mathrm{Spec}(k)) = \bigoplus_{X_i} \mathbb{Z}$ where the X_i run through the connected components of X.

Remark 7.2.8. By the Yoneda Lemma

$$\mathrm{Hom}_{\mathrm{Pre}^{tr}(k)}(\mathbb{Z}_{tr}(X), F) \cong F(X)$$

for all $F \in \mathrm{Pre}^{tr}(k)$. Therefore

$$\mathrm{Hom}_{\mathrm{Pre}^{tr}(k)}(\mathbb{Z}_{tr}(X), -) : \mathrm{Pre}^{tr}(k) \longrightarrow \mathcal{A}b$$

is an exact functor and $\mathbb{Z}_{tr}(X)$ is a projective object in $\mathrm{Pre}^{tr}(k)$.

The following lemma follows immediately from [MVW06, Lemma 6.2].

Lemma 7.2.9. *Let $X \in Sm/k$. Then the represented presheaf with transfers $\mathbb{Z}_{tr}(X)$ is a Nisnevich sheaf with transfers.*

Definition 7.2.10. Let (X, x) be a pointed scheme in Sm/k, that is, there is given a k-rational point $x : \mathrm{Spec}(k) \to X$. Then define $\mathbb{Z}_{tr}(X, x)$ as the cokernel of the map $x_* : \mathbb{Z} \to \mathbb{Z}_{tr}(X)$ in $\mathrm{Pre}^{tr}(k)$ associated to x since $\mathbb{Z}$ was $\mathbb{Z}_{tr}(\mathrm{Spec}(k))$.

Remark 7.2.11. x_* clearly splits the canonical map $\mathbb{Z}_{tr}(X) \to \mathbb{Z}$ induced by the canonical map $X \to \mathrm{Spec}(k)$. Hence there is a natural splitting $\mathbb{Z}_{tr}(X) \cong \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x)$.

Definition 7.2.12. Let (X_i, x_i) for $i = 1, \ldots n$ be pointed schemes in Sm/k. Then define $\mathbb{Z}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n)) = \mathbb{Z}_{tr}(X_1 \wedge \ldots \wedge X_n)$ as the cokernel of the map

$$\bigoplus_i \mathbb{Z}_{tr}(X_1 \times \ldots \times \widehat{X_i} \times \ldots \times X_n) \xrightarrow{\mathrm{id} \times \ldots \times x_i \times \ldots \mathrm{id}} \mathbb{Z}_{tr}(X_1 \times \ldots \times X_n)$$

where $\widehat{X_i}$ means that X_i is replaced by $\mathrm{Spec}(k)$.

Notation 7.2.13. For a pointed scheme (X, x) and $q > 0$ denote

$$\mathbb{Z}_{tr}((X, x)^{\wedge q}) := \mathbb{Z}_{tr}(\underbrace{(X, x) \wedge \ldots \wedge (X, x)}_{q \text{ times}}).$$

Furthermore denote $\mathbb{Z}_{tr}((X, x)^{\wedge 0}) := \mathbb{Z}$ and $\mathbb{Z}_{tr}((X, x)^{\wedge q}) := 0$ for $q < 0$.

Remark 7.2.14. Of course $\mathbb{Z}_{tr}((X, x)^{\wedge 1}) = \mathbb{Z}_{tr}(X, x)$.

The following lemma generalizes the fact that $\mathbb{Z}_{tr}(X) \cong \mathbb{Z} \oplus \mathbb{Z}_{tr}(X, x)$ and is given in [MVW06, Lemma 2.13].

Lemma 7.2.15. *For pointed schemes (X_i, x_i) in Sm/k, where $i = 1, \ldots, n$, the presheaf with transfers $\mathbb{Z}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ is a direct summand of $\mathbb{Z}_{tr}(X_1 \times \ldots \times X_n)$.*

This lemma together with the already known facts about the represented presheaves with transfers imply the following

Corollary 7.2.16. *For pointed schemes (X_i, x_i) in Sm/k, with $i = 1, \ldots, n$, the presheaf with transfers $\mathbb{Z}_{tr}((X_1, x_1) \wedge \ldots \wedge (X_n, x_n))$ is a Nisnevich sheaf with transfers and a projective object of $\mathrm{Pre}^{tr}(k)$.*

Following [MVW06, Lecture 8] there is a tensor product

$$\otimes^{tr} : \mathrm{Pre}^{tr}(k) \times \mathrm{Pre}^{tr}(k) \longrightarrow \mathrm{Pre}^{tr}(k)$$

with the property that

$$\mathbb{Z}_{tr}(X \otimes Y) = \mathbb{Z}_{tr}(X) \otimes^{tr} \mathbb{Z}_{tr}(Y)$$

for all $X, Y \in SmCor(k)$. Furthermore there is an internal function complex

$$\underline{\mathrm{Hom}}(F, G)(X) = \mathrm{Hom}_{\mathrm{Pre}^{tr}(k)}(F \otimes^{tr} \mathbb{Z}_{tr}(X), G)$$

such that for every $F \in \mathrm{Pre}^{tr}(k)$

$$- \otimes^{tr} F : \mathrm{Pre}^{tr}(k) \rightleftarrows \mathrm{Pre}^{tr}(k) : \underline{\mathrm{Hom}}(F, -)$$

is an adjoint functor pair (cf. [MVW06, Lemma 8.3]).

There is also a tensor product on sheaves

$$\otimes^{tr}_{Nis} : \mathrm{Shv}^{tr}_{Nis}(k) \times \mathrm{Shv}^{tr}_{Nis}(k) \longrightarrow \mathrm{Shv}^{tr}_{Nis}(k)$$

defined by $F \otimes^{tr}_{Nis} G := (\iota(F) \otimes^{tr} \iota(G))_{Nis}$, the sheafification of the tensor product on presheaf-level. Since $\mathbb{Z}_{tr}(X \otimes Y)$ is already a Nisnevich sheaf we again get the property

$$\mathbb{Z}_{tr}(X \otimes Y) = \mathbb{Z}_{tr}(X) \otimes^{tr}_{Nis} \mathbb{Z}_{tr}(Y)$$

The internal function complex for sheaves is given by

$$\underline{\mathrm{Hom}}(F, G)(X) = \mathrm{Hom}_{\mathrm{Shv}^{tr}_{Nis}(k)}(F \otimes^{tr}_{Nis} \mathbb{Z}_{tr}(X), G)$$

since a straightforward calculation shows that $\underline{\mathrm{Hom}}(F, G)$ is a sheaf for sheaves F and G using the following lemma. It is clear that

$$- \otimes^{tr}_{Nis} F : \mathrm{Shv}^{tr}_{Nis}(k) \rightleftarrows \mathrm{Shv}^{tr}_{Nis}(k) : \underline{\mathrm{Hom}}(F, -)$$

is again an adjoint functor pair.

Lemma 7.2.17. *Let F, G be (pre)sheaves with transfers. Then there are natural isomorphisms*

$$\underline{\mathrm{Hom}}(\mathbb{Z}_{tr}(X), G) \cong G(- \otimes X)$$

and

$$\underline{\mathrm{Hom}}(F, G)(X) \cong \mathrm{Hom}(F, G(- \otimes X))$$

for presheaves and for sheaves.

Proof. Here is the verification on presheaf-level: The first claim follows by the Yoneda Lemma and the fact that $\mathbb{Z}_{tr}(X) \otimes^{tr} \mathbb{Z}_{tr}(Y) = \mathbb{Z}_{tr}(X \otimes Y)$. Then we have

$$\begin{aligned}
\underline{\mathrm{Hom}}(F, G)(X) &= \mathrm{Hom}_{\mathrm{Pre}^{tr}(k)}(F \otimes^{tr} \mathbb{Z}_{tr}(X), G) \\
&\cong \mathrm{Hom}_{\mathrm{Pre}^{tr}(k)}(F, \underline{\mathrm{Hom}}(\mathbb{Z}_{tr}(X), G)) \\
&\cong \mathrm{Hom}_{\mathrm{Pre}^{tr}(k)}(F, G(- \otimes X))
\end{aligned}$$

The argumentation on sheaves is similar. □

The previous constructions lead to the following proposition.

Proposition 7.2.18. *Both* $(\mathrm{Pre}^{tr}(k), \otimes^{tr}, \mathbb{Z})$ *and* $(\mathrm{Shv}^{tr}_{Nis}(k), \otimes^{tr}_{Nis}, \mathbb{Z})$ *are closed symmetric monoidal categories.*

Remark 7.2.19. Since the monoidal structure on (pre)sheaves with transfers is closed, these categories are additive symmetric monoidal.

Remark 7.2.20. There is an extension of the closed symmetric structures of the (pre)sheaves with transfers on simplicial (pre)sheaves with transfers: The tensor products

$$\otimes^{tr} : s\mathrm{Pre}^{tr}(k) \times s\mathrm{Pre}^{tr}(k) \longrightarrow s\mathrm{Pre}^{tr}(k)$$

and

$$\otimes^{tr}_{Nis} : s\mathrm{Shv}^{tr}_{Nis}(k) \times s\mathrm{Shv}^{tr}_{Nis}(k) \longrightarrow s\mathrm{Shv}^{tr}_{Nis}(k)$$

are given by $(X \otimes^{tr} Y)_n = X_n \otimes^{tr} Y_n$ and $(X \otimes^{tr}_{Nis} Y)_n = X_n \otimes^{tr}_{Nis} Y_n$.

The internal simplicial function complexes have to be

$$\underline{\mathrm{Hom}}(F, G)(U)_n \cong \mathrm{Hom}_{s\mathrm{Pre}^{tr}(k)}(F \otimes^{tr} \mathbb{Z}_{tr}(U) \otimes^{tr} \mathbb{Z}[\Delta^n], G)$$

on presheaves and

$$\underline{\mathrm{Hom}}(F, G)(U)_n \cong \mathrm{Hom}_{s\mathrm{Shv}^{tr}_{Nis}(k)}(F \otimes^{tr}_{Nis} \mathbb{Z}_{tr}(U) \otimes^{tr}_{Nis} \mathbb{Z}[\Delta^n], G)$$

on sheaves.

Proposition 7.2.21. *There are adjoint functor pairs*

$$(-)_{tr} : s\mathrm{Pre}((Sm/k)_{Nis}) \rightleftarrows s\mathrm{Pre}^{tr}(k) : V$$

and

$$(-)_{tr} : s\mathrm{Shv}((Sm/k)_{Nis}) \rightleftarrows s\mathrm{Shv}^{tr}_{Nis}(k) : V$$

where V is the forgetful functor, determined by the property that on presheaf-level

$$(U \times \Delta^n)_{tr} = \mathbb{Z}_{tr}(U) \otimes^{tr} \mathbb{Z}[\Delta^n]$$

for all $U \in Sm/k$. Furthermore the functors $(-)_{tr}$ are strict symmetric monoidal. For a simplicial (pre)sheaf X we call X_{tr} the associated simplicial (pre)sheaf with transfers.

Proof. Let X be a simplicial presheaf. Proposition 5.1.6 yields that X can canonically be written as a colimit of representable functors

$$X = \mathrm{colim}_{(U,[n],x)} U \times \Delta^n$$

Define

$$X_{tr} := \mathrm{colim}_{(U,[n],x)} \mathbb{Z}_{tr}(U) \otimes^{tr} \mathbb{Z}[\Delta^n]$$

This gives the demanded adjunction: First of all

$$\mathrm{Hom}_{s\mathrm{Pre}^{tr}}(\mathbb{Z}_{tr}(U), G) \cong G(U) \cong \mathrm{Hom}_{s\mathrm{Pre}}(U, V(G))$$

which is natural in U. Then we get a chain of natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{s\mathrm{Pre}^{tr}}(\mathbb{Z}_{tr}(U) \otimes^{tr} \mathbb{Z}[\Delta^n], Y) &\cong \mathrm{Hom}_{s\mathrm{Pre}^{tr}}(\mathbb{Z}_{tr}(U), \underline{\mathrm{Hom}}(\mathbb{Z}[\Delta^n], Y)) \\
&\cong \mathrm{Hom}_{s\mathrm{Pre}}(U, V\underline{\mathrm{Hom}}(\mathbb{Z}[\Delta^n], Y)) \\
&\cong \mathrm{Hom}_{s\mathrm{Pre}}(U, \underline{\mathrm{Map}}(\Delta^n, V(Y))) \\
&\cong \mathrm{Hom}_{s\mathrm{Pre}}(U \times \Delta^n, V(Y))
\end{aligned}$$

and therefore we are done.

Now let X be a simplicial sheaf. Define $X_{tr} := (\iota(X)_{tr})_{Nis}$ where ι was the forgetful functor from simplicial sheaves to simplicial presheaves. Of course $\iota \circ V = V \circ \iota$ and therefore

$$\begin{aligned}
\mathrm{Hom}_{s\mathrm{Shv}^{tr}_{Nis}}((\iota(X)_{tr})_{Nis}, Y) &= \mathrm{Hom}_{s\mathrm{Pre}^{tr}}(\iota(X)_{tr}, \iota(Y)) \\
&= \mathrm{Hom}_{s\mathrm{Pre}}(\iota(X), V(\iota(Y))) \\
&= \mathrm{Hom}_{s\mathrm{Pre}}(\iota(X), \iota(V(Y))) \\
&= \mathrm{Hom}_{s\mathrm{Shv}}(X, V(Y))
\end{aligned}$$

which verifies the second demanded adjunction.

It follows immediately from the definitions that both functors $(-)_{tr}$ are strict symmetric monoidal, that is, there is a natural isomorphism $(X \times Y)_{tr} \cong X_{tr} \otimes^{tr} Y_{tr}$ for simplicial presheaves X and Y resp. $(X \times Y)_{tr} \cong X_{tr} \otimes^{tr}_{Nis} Y_{tr}$ for simplicial sheaves X and Y. $\square$

Remark 7.2.22. It is clear that $X_{tr} = \mathbb{Z}_{tr}(X)$ for all $X \in Sm/k$. Furthermore $K_{tr} = \mathbb{Z}[K]$ for every simplicial set K considered as a constant simplicial presheaf, since $\mathbb{Z}[-]$ commutes with all colimits as a left adjoint.

Remark 7.2.23. Summarizing the previous results we have a commutative diagram

$$\begin{tikzcd}
s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[r, shift left, "(-)_{tr}"] \arrow[d, shift left, "a"] & s\mathrm{Pre}^{tr}(k) \arrow[l, shift left, "V"] \arrow[d, shift left, "(-)_{Nis}"] \\
s\mathrm{Shv}((Sm/k)_{Nis}) \arrow[r, shift left, "(-)_{tr}"] \arrow[u, shift left, "\iota"] & s\mathrm{Shv}^{tr}_{Nis}(k) \arrow[l, shift left, "V"] \arrow[u, shift left, "\iota"]
\end{tikzcd}$$

since for a simplicial presheaf with transfers X we have $a(V(X)) = V(X_{Nis})$ by Proposition 7.2.3 and of course $V \circ \iota = \iota \circ V$. Furthermore $(-)_{tr} \circ a = (-)_{Nis} \circ (-)_{tr}$ since both functors are left adjoint to the same forgetful functor $s\mathrm{Shv}^{tr}_{Nis}(k) \to s\mathrm{Pre}((Sm/k)_{Nis})$.

Now note that there is homotopical content in the adjunction

$$(-) : s\mathrm{Pre}((Sm/k)_{Nis}) \rightleftarrows s\mathrm{Pre}^{tr}(k) : V$$

that is, there is a model structure on $s\mathrm{Pre}^{tr}(k)$ such that this adjoint functor pair becomes a Quillen functor. This is done for the projective as well as for the $\mathbb{A}^1$-local structure in [RØ06]. We only need the projective structure which is given in [RØ06, Theorem 2.3].

Theorem 7.2.24. *The category of simplicial presheaves with transfers together with the classes*

- *w.e. = projective (i.e., objectwise) weak equ. after applying V*
- *fib. = projective (i.e., objectwise) fibrations after applying V*

forms a proper simplicial cofibrantly generated model category. The generating cofibrations are given by

$$I = \{\mathbb{Z}_{tr}(U) \otimes^{tr} \mathbb{Z}[\partial\Delta^n \hookrightarrow \Delta^n] \mid U \in Sm/k, n \geq 0\}$$

and the generating acyclic cofibrations are given by

$$J = \{\mathbb{Z}_{tr}(U) \otimes^{tr} \mathbb{Z}[\Lambda^n_k \hookrightarrow \Delta^n] \mid U \in Sm/k, n > 0, 0 \leq k \leq n\}$$

Remark 7.2.25. It follows immediately from the definitions of the projective model structure on simplicial persheaves with transfers that the adjoint functor pair

$$(-) : s\mathrm{Pre}((Sm/k)_{Nis}) \rightleftarrows s\mathrm{Pre}^{tr}(k) : V$$

is indeed a Quillen functor since V is obviously a right Quillen functor.

Furthermore the simplicial structure is given by

$$K \otimes X = \mathbb{Z}[K] \otimes^{tr} X$$

for $K \in s\mathcal{S}et$ and $X \in s\mathrm{Pre}^{tr}(k)$.

7.3. **The Triangulated Category of Effective Motives.**

Definition 7.3.1. For an abelian category $\mathcal{A}$ denote by $\mathrm{Ch}^{-}(\mathcal{A})$ the abelian category of bounded above cochain complexes in $\mathcal{A}$ and by $\mathrm{Ch}^{\leq 0}(\mathcal{A})$ the full subcategory with the objects $X \in \mathrm{Ch}(\mathcal{A})$ such that $X^n = 0$ for $n > 0$. A morphism $f : X \to Y$ in $\mathrm{Ch}^{-}(\mathcal{A})$ is called a *quasi-isomorphism* if it induces isomorphisms in cohomology, that is

$$H^n(f) : H^n(X) \to H^n(Y)$$

is an isomorphism for all $n \in \mathbb{Z}$ where

$$H^n(X) = \mathrm{Ker}(X^n \xrightarrow{\partial} X^{n+1})/\mathrm{Im}(X^{n-1} \xrightarrow{\partial} X^n)$$

is the usual cohomology of a cochain complex.

Denote by $D^{-}(\mathcal{A})$ the derived category, that is, the localization of $\mathrm{Ch}^{-}(\mathcal{A})$ with respect to the quasi-isomorphisms (cf. [Wei94, Ch.10]).

Remark 7.3.2. There is a canonical inclusion

$$\iota : \mathcal{A} \hookrightarrow \mathrm{Ch}^{-}(\mathcal{A})$$

where $\iota(A)$ is the cochain complex with A concentrated in degree 0.

Remark 7.3.3. Note that the derived category $D^{-}(\mathcal{A})$ of an abelian category $\mathcal{A}$ has a canonical structure of a triangulated category (cf. [Wei94, Ch.10]): The shift functor is given by $X \mapsto X[1]$ where $X[1]^n = X^{n+1}$. The exact triangles are given by the triangles isomorphic to the strict triangles

$$X \xrightarrow{f} Y \to \mathrm{Cone}(f) \to X[1]$$

where $\mathrm{Cone}(f)$ is the mapping cone (cf. [Wei94, Ch.1.5]).

Remark 7.3.4. Recall that the category $\mathrm{Ch}^{\leq 0}(\mathcal{A})$ for an abelian category $\mathcal{A}$ can be understood as the full subcategory of the functor category $\mathcal{A}^{\mathbb{N}}$ of additive functors valued in $\mathcal{A}$. Hence it is an abelian category, since we can do everything objectwise, and we have the following coincidence:

The categories $\mathrm{Ch}^{\leq 0}(\mathrm{Pre}^{tr}(k))$ and $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}(k))$ are nothing else than the categories of presheaves resp. sheaves with transfers valued in $\mathrm{Ch}^{\leq 0}(\mathcal{A}b)$. That is, they are additive functors $X : SmCor(k)^{op} \to \mathrm{Ch}^{\leq 0}(\mathcal{A}b)$ for presheaves which additionally fulfil the usual equalizer diagrams for the Nisnevich coverings in $\mathrm{Ch}^{\leq 0}(\mathcal{A}b)$ if they are sheaves.

Furthermore the categories $\mathrm{Ch}^{-}(\mathrm{Pre}^{tr}(k))$ and $\mathrm{Ch}^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ are fully faithful contained in the categories of presheaves resp. sheaves with transfers valued in $\mathrm{Ch}^{-}(\mathcal{A}b)$, but they are not the same.

Proposition 7.3.5. *The adjoint functor pair*
$$(-)_{Nis} : \mathrm{Pre}^{tr}(k) \rightleftarrows \mathrm{Shv}^{tr}_{Nis}(k) : \iota$$
induces an adjoint functor pair
$$(-)_{Nis} : \mathrm{Ch}^{-}(\mathrm{Pre}^{tr}(k)) \rightleftarrows \mathrm{Ch}^{-}(\mathrm{Shv}^{tr}_{Nis}(k)) : \iota$$
where ι is the forgetful functor, with the property that $(-)_{Nis}$ maps quasi-isomorphisms to quasi-isomorphisms.

Proof. The sheafification is just defined objectwise: For a cochain complex of presheaves with transfers X the sheafification X_{Nis} is the cochain complex with $(X_{Nis})^n = (X^n)_{Nis}$ and boundary morphisms $\delta_{Nis} : (X^n)_{Nis} \to (X^{n+1})_{Nis}$. Since the cohomology $H^*(X)$ of a sheaf complex X is computed as the sheafified cohomology presheaf with transfers
$$SmCor(k)^{op} \xrightarrow{X} \mathrm{Ch}^{-}(\mathcal{A}b) \xrightarrow{H^*} \mathcal{A}b$$
it is clear that $(-)_{Nis}$ respects quasi-isomorphisms. □

Remark 7.3.6. There is a canonical tensor product on $\mathrm{Ch}^{-}(\mathrm{Pre}^{tr}(k))$: For two bounded above cochain complexes X and Y their tensor product $X \otimes^{tr} Y$ is given by finite direct sums
$$(X \otimes^{tr} Y)^n = \bigoplus_{i+j=n} X^i \otimes^{tr} Y^i$$
and differential maps induced by the maps
$$X^i \otimes^{tr} Y^j \xrightarrow{\partial_X \otimes \mathrm{id} + (-1)^j \mathrm{id} \otimes \partial_Y} (X^{i-1} \otimes^{tr} Y^j) \oplus (X^i \otimes^{tr} Y^{j-1})$$
which yields again a bounded above cochain complex.

The tensor product on $\mathrm{Ch}^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ can be defined analogously, but note that there is also the description $X \otimes^{tr}_{Nis} Y = (\iota(X) \otimes^{tr} \iota(Y))_{Nis}$ since $(-)_{Nis}$ commutes with coproducts as a left adjoint.

Our next aim is to extend the tensor product on the derived category $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ such that it becomes a tensor triangulated category in the sense of [MVW06, Definition 8A.1]. Note that this definition of a tensor triangulated category includes that the shift functor commutes with the tensor product and that for an exact triangle
$$A \to B \to C \to A[1]$$
the induced triangles
$$A \otimes D \to B \otimes D \to C \otimes D \to (A \otimes D)[1]$$
and
$$D \otimes A \to D \otimes B \to D \otimes C \to (D \otimes A)[1]$$
are again exact triangles.

Definition 7.3.7 (The total derived tensor product)**.** Let X and Y be two bounded above complexes of presheaves with transfers. There are quasi-isomorphisms $P \xrightarrow{\simeq} X$ and $Q \xrightarrow{\simeq} Y$ where P and Q are complexes of projective objects since $\mathrm{Pre}^{tr}(k)$ has enough projectives. Then define
$$X \otimes^{tr}_{L} Y := P \otimes^{tr} Q$$

The definition for sheaves is

$$X \otimes^{tr}_{L,Nis} Y := (\iota(X) \otimes^{tr}_{L} \iota(Y))_{Nis}.$$

Remark 7.3.8. Note that the definition does not depend on the choice of the complexes P and Q up to chain homotopy, that is, it is well-defined in the derived category. There is a natural morphism

$$X \otimes^{tr}_{L} Y \to X \otimes^{tr} Y$$

which induces

$$X \otimes^{tr}_{L,Nis} Y \to X \otimes^{tr}_{Nis} Y.$$

This yields indeed the demanded structure (cf. [MVW06, Definition 14.2]).

Proposition 7.3.9. *The triangulated category $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ together with the total derived tensor product $\otimes^{tr}_{L,Nis}$ is a tensor triangulated category.*

Now we are going to define the triangulated category of effective motives $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.

Definition 7.3.10. Define $\mathcal{E}_{\mathbb{A}^1}$ as the smallest thick localizing subcategory of the triangulated category $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ which contains the cone of

$$\mathbb{Z}_{tr}(X \otimes \mathbb{A}^1) \xrightarrow{\mathbb{Z}_{tr}(\pi)} \mathbb{Z}_{tr}(X)$$

for every projection $\pi : X \times \mathbb{A}^1 \to X$. That is, $\mathcal{E}_{\mathbb{A}^1}$ is generated by these cones, arbitrary direct sums and by the properties:

- If $A \to B \to C \to A[1]$ is an exact triangle in $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$, then if two out of A, B, C are in $\mathcal{E}_{\mathbb{A}^1}$ then so is the third.
- If $A \oplus B$ is in $\mathcal{E}_{\mathbb{A}^1}$ then so are A and B

Definition 7.3.11 ($\mathrm{DM}^{\mathrm{eff}}_{-}(k)$)**.** Define the *triangulated category of effective motives* $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ as

$$\mathrm{DM}^{\mathrm{eff}}_{-}(k) := D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))/\mathcal{E}_{\mathbb{A}^1}$$

the Verdier localization of the triangulated category $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ with respect to the thick localizing subcategory $\mathcal{E}_{\mathbb{A}^1}$.

Remark 7.3.12. The existence of the Verdier localization is proved in [Nee01, Theorem 2.1.8] and yields a canonical triangulated localization functor

$$l : D^{-}(\mathrm{Shv}^{tr}_{Nis}(k)) \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k)$$

which is the identity on objects. The triangulated structure on $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ is just given as follows: The exact triangles are the triangles which are isomorphic to the triangles which come from the exact triangles of $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ via l. The shift functor is just the same on objects. The localization functor l has the property that $l(X) \cong 0$ in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ for all $X \in \mathcal{E}_{\mathbb{A}^1}$. Hence every map $\mathbb{Z}_{tr}(X \otimes \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$ becomes an isomorphism under l since the cone becomes zero and isomorphisms can be detected by their cones in triangulated categories. That is, a morphism is an isomorphism if its cone is zero.

Remark 7.3.13. The category $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ can be understood as a localization of $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ in the following way:

A morphism f in $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ is called an $\mathbb{A}^1$-*weak equivalence* if its cone lies in $\mathcal{E}_{\mathbb{A}^1}$. Denote all $\mathbb{A}^1$-weak equivalences by $W_{\mathbb{A}^1}$. Then

$$\mathrm{DM}^{\mathrm{eff}}_{-}(k) = D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))[W_{\mathbb{A}^1}^{-1}]$$

is the localization with respect to the $\mathbb{A}^1$-weak equivalences.

$\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ has similar good properties as $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k))$ (cf. [MVW06, Lecture 14]):

Proposition 7.3.14. *The triangulated category of effective motives* $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ *together with the total derived tensor product* $\otimes^{tr}_{L,Nis}$ *is a tensor triangulated category.*

Notation 7.3.15. Denote the tensor product in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ just by $\otimes$.

Definition 7.3.16. Define the functor $M : Sm/k \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k)$ as the composition

$$\begin{array}{l} Sm/k \\ \big\downarrow{\scriptstyle \mathbb{Z}_{tr}(-)} \\ \mathrm{Shv}^{tr}_{Nis}(k) \hookrightarrow \mathrm{Ch}^{-}(\mathrm{Shv}^{tr}_{Nis}(k)) \longrightarrow D^{-}(\mathrm{Shv}^{tr}_{Nis}(k)) \xrightarrow{l} \mathrm{DM}^{\mathrm{eff}}_{-}(k) \end{array}$$

$M(X)$ is called the *motive of* X and is nothing else than the complex with the represented Nisnevich sheaf with transfers $\mathbb{Z}_{tr}(X)$ concentrated in degree 0.

Remark 7.3.17. Note that

$$M(X \times Y) \cong M(X) \otimes M(Y) \text{ in } \mathrm{DM}^{\mathrm{eff}}_{-}(k)$$

because

$$\begin{aligned} \mathbb{Z}_{tr}(X) \otimes^{tr}_{L,Nis} \mathbb{Z}_{tr}(Y) &= (\mathbb{Z}_{tr}(X) \otimes^{tr}_{L} \mathbb{Z}_{tr}(Y))_{Nis} \\ &= (\mathbb{Z}_{tr}(X) \otimes^{tr} \mathbb{Z}_{tr}(Y))_{Nis} \\ &= \mathbb{Z}_{tr}(X \otimes Y) \end{aligned}$$

since $\mathbb{Z}_{tr}(X)$ and $\mathbb{Z}_{tr}(Y)$ are projective presheaves with transfers.

7.4. **Motivic Cohomology.** The aim of this subsection is to define motivic cohomology using an approach of Voevodsky which was given in [Voe00] and to give an application to Chow motives which provides a fully faithful embedding $\mathrm{CH}(k) \hookrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k)$. Note that there are other possibilities to define motivic cohomology as given in [FSV00], [MVW06], and [Voe98]. The approach in [Voe98] uses stable motivic homotopy theory and the analogue of the Eilenberg-MacLane spectrum. But these approaches will not be discussed here.

Definition 7.4.1. Define the cosimplicial object $\Delta^{\bullet}$ in Sm/k as

$$\Delta^n = \mathrm{Spec}(k[X_0, \ldots, X_n]/\sum_{i=0}^{n} X_i = 1)$$

The structure maps are given by: For $f : [n] \to [m] \in \Delta$ define $f^* : \Delta^n \to \Delta^m$ by

$$f^*(X_i) = \sum_{j \in f^{-1}(i)} X_j$$

for $i = 0, \ldots, m$ where the empty sum is 0. This means in particular that $f^*(X_i) = 0$ for all $i \notin f([n])$.

Remark 7.4.2. Note that $\Delta^n \cong \mathbb{A}^n$.

Definition 7.4.3. Let X be a presheaf with transfers. Then define the cochain complex $C^*X \in \mathrm{Pre}^{tr}(k)$ as

$$C^{-n}X := X(- \times \Delta^n)$$

for $n \geq 0$ and $C^{-n}X = 0$ for $n < 0$. The differential maps

$$C^{-(n+1)}X = X(- \times \Delta^{n+1}) \xrightarrow{\delta} X(- \times \Delta^n) = C^{-n}X$$

are the alternating sum of the maps induced by the face maps $\partial_i : \Delta^n \to \Delta^{n+1}$.

Remark 7.4.4. For a $X \in Sm/k$ the complex $C^*(\mathbb{Z}_{tr}(X))$ is called the *Suslin-complex of* X. Further the following fact is straightforward: If X is a Nisnevich sheaf with transfers, than C^*X is a complex of Nisnevich sheaves with transfers.

Definition 7.4.5 (Motivic complexes)**.** Take $\mathbb{G}_m$ as the pointed scheme $(\mathbb{A}^1 - 0, 1)$. Then for every $q \geq 0$ the *motivic complex* $\mathbb{Z}(q) \in \mathrm{Ch}^-(\mathrm{Shv}^{tr}_{Nis}(k))$ is defined as

$$\mathbb{Z}(q) := C^*\mathbb{Z}_{tr}(\mathbb{G}^{\wedge q})[-q]$$

$\mathbb{Z}(1)$ is called the *Tate object*.

Remark 7.4.6. $\mathbb{Z}(q)$ is indeed an object of $\mathrm{Ch}^-(\mathrm{Shv}^{tr}_{Nis}(k))$:

It is a bounded above cochain complex of presheaves with transfers since $\mathbb{Z}(q)^i = C^{i-q}\mathbb{Z}_{tr}(\mathbb{G}^{\wedge q})$ vanishes for $i > q$ by definition of C^*. Furthermore $\mathbb{Z}_{tr}(\mathbb{G}^{\wedge q})$ is a Nisnevich sheaf with transfers by Corollary 7.2.16 for all $q \geq 0$ and therefore $\mathbb{Z}(q)$ is a complex of Nisnevich sheaves with transfers using the previous remark.

Definition 7.4.7 (Motivic cohomology)**.** Let $X \in Sm/k$. Define the *motivic cohomology* $H^{p,q}_{mot}(X, \mathbb{Z})$ *with integral coefficients of* X as the abelian group

$$H^{p,q}_{mot}(X, \mathbb{Z}) := \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_-(k)}(M(X), \mathbb{Z}(q)[p]).$$

Motivic cohomology can be compared with Chow groups. This is done by the next proposition that follows from [MVW06, Proposition 14.16 and Corollary 19.2].

Proposition 7.4.8. *Let k be a perfect field. Then we have a natural isomorphism*

$$H^{2i,i}_{mot}(X, \mathbb{Z}) \cong \mathrm{CH}^i(X)$$

Notation 7.4.9. Let $X \in \mathrm{DM}_{-}^{\mathrm{eff}}(k)$ be a complex of Nisnevich sheaves with transfers. Denote by

$$X(1) := X \otimes \mathbb{Z}(1)$$

the twist with the Tate object via the tensor product in $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$.

Voevodsky has proven in [Voe02] the following *Cancellation Theorem*.

Theorem 7.4.10 (Cancellation)**.** *Let k be a perfect field, $X, Y \in \mathrm{DM}_{-}^{\mathrm{eff}}(k)$. Then tensoring with $\mathbb{Z}(1)$ induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}(X, Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}(X(1), Y(1))$$

Proposition 7.4.11. *Let k be a field of characteristic 0. Then there is a fully faithful embedding*

$$\mathrm{CH}(k) \hookrightarrow \mathrm{DM}_{-}^{\mathrm{eff}}(k)$$

with the property that it acts on objects by $X \mapsto M(X)$.

Proof. First of all note that by the Cancellation Theorem for every $l \geq 0$ and $n \in Z$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}(X(l)[n], Y(l)[n]) \cong \mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}(X, Y)$$

Now let X and Y be connected smooth projective varieties and let $d = \dim(Y)$. Then we have a chain of natural isomorphisms

$$\begin{aligned}
&(1) \qquad \mathrm{CH}^d(X \times Y) &&= H^{2d,d}_{mot}(X \times Y, \mathbb{Z}) \\
&(2) &&= \mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}(M(X \times Y), \mathbb{Z}(d)[2d]) \\
&(3) &&= \mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}(M(X) \otimes M(Y), \mathbb{Z}(d)[2d]) \\
&(4) &&= \mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}(M(X)(d)[2d], M(Y)(d)[2d]) \\
&(5) &&= \mathrm{Hom}_{\mathrm{DM}_{-}^{\mathrm{eff}}(k)}(M(X), M(Y))
\end{aligned}$$

where equation (1) follows from the previous proposition. Equation (2) is our definition of motivic cohomology and equation (4) is a duality which follows from [MVW06, Theorem 16.24] since Y is proper.

These natural isomorphisms define the functor on morphisms. □

Remark 7.4.12. Note that all equations of the proof are valid for a perfect field k up to the dualizability of the motives of smooth projective varieties $M(X)$ in $\mathrm{DM}_{-}^{\mathrm{eff}}(k)$ where characteristic 0 is needed. There are preprints like [RØ06] which imply that this dualizability holds for all perfect fields k using the dualizability of these objects in the *stable motivic homotopy category* $\mathcal{SH}(k)$. But the author does not know the details.

Remark 7.4.13. Note that this proposition and its proof appears also in [MVW06, Proposition 20.1], but there is a little mistake in the proof given there: The number $d = \dim(X)$ is used instead of $d = \dim(Y)$.

7.5. **The Functor $\mathcal{H}(k) \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k)$.** The aim of this subsection is to construct a functor $H : \mathcal{H}(k) \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k)$ such that the following diagram commutes up to functor isomorphism:

$$\begin{tikzcd} Sm/k \arrow[r, "h"] \arrow[dr, "M"'] & \mathcal{H}(k) \arrow[d, "H"] \\ & \mathrm{DM}^{\mathrm{eff}}_{-}(k) \end{tikzcd}$$

Recall that the motivic homotopy category $\mathcal{H}(k)$ can be obtained as the $\mathbb{A}^1$-local homotopy category of $s\mathrm{Pre}((Sm/k)_{Nis})$. Therefore the first step is to construct a functor $H' : s\mathrm{Pre}((Sm/k)_{Nis}) \longrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k)$ such that the diagram

$$\begin{tikzcd} Sm/k \arrow[r, hook] \arrow[dr, "M"'] & s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[d, "H'"] \\ & \mathrm{DM}^{\mathrm{eff}}_{-}(k) \end{tikzcd}$$

commutes up to functor isomorphism. Then we make use of the fact that the homotopy category $\mathcal{H}(k)$ is the localization of $s\mathrm{Pre}((Sm/k)_{Nis})$ with respect to the $\mathbb{A}^1$-local weak equivalences. That is we have to check that H' maps every $\mathbb{A}^1$-local weak equivalence to an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ to get the induced functor H:

$$\begin{tikzcd} Sm/k \arrow[r, hook] \arrow[rr, bend left, "h"] \arrow[dr, "M"'] & s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[r] \arrow[d, "H'"] & \mathcal{H}(k) \arrow[dl, dotted, "H"] \\ & \mathrm{DM}^{\mathrm{eff}}_{-}(k) & \end{tikzcd}$$

A few words of caution: It is quite canonical to construct the functor H', but it requires a lot of work to check the demanded property that H' maps every $\mathbb{A}^1$-local weak equivalence to an ismorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.

Before we are going through the technical details, note that the essential reasons for this property of H' are only two properties of the Nisnevich site and the category $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ as it will turn out in Lemma 7.5.14:

(1) Every EDS

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, hook, "i"] & X \end{tikzcd}$$

is a pushout diagram in the category of Nisnevich sheaves as we have seen in Lemma 6.1.13.

(2) Every projection $\pi : X \times \mathbb{A}^1 \to X$ becomes an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ since this is exactly done by the Verdier localization.

One necessary step for the construction of H' is the translation of simplicial objects to negative cochain complexes. This is a well known theory which is quoted next.

Definition 7.5.1. Let $\mathcal{A}$ be an abelian category. Denote by

$$N : s\mathcal{A} \longrightarrow \mathrm{Ch}^{\leq 0}(\mathcal{A})$$

the *normalized cochain complex functor*, that is for a simplicial object A in $\mathcal{A}$ take $N(A)^0 = A_0$ and for $n \geq 1$

$$N^{-n}(A) = \bigcap_{i=0}^{n-1} \mathrm{Ker}(\partial_i : A_n \to A_{n-1})$$

and the differential $d : N^{-n}(A) \to N^{-(n-1)}(A)$ is given by $d = (-1)^n \partial_n$.

Remark 7.5.2. Note that N is part of an equivalence of categories known as the *Dold-Kan correspondence* (cf.[Wei94, Ch.8.4] or [GJ99, III Theorem 2.5]). Furthermore for a discrete simplicial object $A \in \mathcal{A}$ the normalized cochain complex $N(A)$ is nothing else than the cochain complex with A concentrated in degree 0.

Remark 7.5.3. Note that the Dold-Kan correspondence acts on the abelian categories $\mathrm{Pre}^{tr}(k)$ and $\mathrm{Shv}^{tr}_{Nis}(k)$ in the following way:
Let $X : SmCor(k)^{op} \to s\mathcal{A}b$ be a simplicial (pre)sheaf with transfers. Then $N(X)$ is nothing else than the composition

$$SmCor(k)^{op} \xrightarrow{X} s\mathcal{A}b \xrightarrow{N} \mathrm{Ch}^{\leq 0}(\mathcal{A}b)$$

since (pre)sheaf kernels and intersections are limits and therefore they are computed objectwise.

The next proposition describes the homotopical behavior of the Dold-Kan correspondence for $\mathcal{A} = \mathcal{A}b$, the abelian category of abelian groups, and is given in [GJ99, III Corollary 2.7].

Proposition 7.5.4. *Let A be a simplicial abelian group. Then there is a natural isomorphism*

$$\pi_n(A, 0) \cong H^{-n}(NA).$$

From this we immediately get the following

Corollary 7.5.5. *Let $f : A \to B$ be a weak equivalence of simplicial abelian groups, that is, f is a weak equivalence on the underlying simplicial sets. Then $N(f) : NA \to NB$ is a quasi-isomorphism.*

Unfortunately, the Dold-Kan correspondence is not strict symmetric monoidal, but the maps in question are quasi-isomorphisms:

Proposition 7.5.6. *Let X and Y be simplicial (pre)sheaves with transfers. Then there is a natural quasi-isomorphism*

$$N(X \otimes^{tr} Y) \simeq N(X) \otimes^{tr} N(Y)$$

for presheaves and

$$N(X \otimes^{tr}_{Nis} Y) \simeq N(X) \otimes^{tr}_{Nis} N(Y)$$

for Nisnevich sheaves.

Proof. It suffices to show this for presheaves since the sheafification $(-)_{Nis}$ commutes with all constructions and respects quasi-isomorphisms. The claim for presheaves follows immediately from the Eilenberg-Zilber Theorem (cf. [Wei94, Theorem 8.5.1] or [GJ99, IV Theorem 2.4]) applied to the abelian category $\mathrm{Pre}^{tr}(k)$ and the fact that the Moore complex C used there is naturally quasi-isomorphic to the normalized complex N. □

Since the objects of the triangulated category of effective motives $\mathrm{DM}_-^{\mathrm{eff}}(k)$ are bounded above cochain complexes in Nisnevich sheaves with transfers $\mathrm{Shv}_{Nis}^{tr}(k)$, we have to map the simplicial presheaves to these objects. Therefore we have to use the functor $(-)_{tr}$ to get simplicial (pre)sheaves with transfers, the Dold-Kan correspondence to get negative cochain complexes, and the sheafification. A question that arises naturally is the following: At which stage should we sheafify? According to the following remark, it does not depend on the stage.

Remark 7.5.7. Both squares of the diagram

$$\begin{tikzcd} s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[r, "a"] \arrow[d, "(-)_{tr}"'] & s\mathrm{Shv}((Sm/k)_{Nis}) \arrow[d, "(-)_{tr}"] \\ s\mathrm{Pre}^{tr}(k) \arrow[r, "(-)_{Nis}"] \arrow[d, "N"'] & s\mathrm{Shv}_{Nis}^{tr}(k) \arrow[d, "N"] \\ \mathrm{Ch}^{\leq 0}(\mathrm{Pre}^{tr}(k)) \arrow[r, "(-)_{Nis}"] & \mathrm{Ch}^{\leq 0}(\mathrm{Shv}_{Nis}^{tr}(k)) \end{tikzcd}$$

commute. This is already known for the upper square and the lower square commutes since $(-)_{Nis}$ commutes with finite limits of presheaves with transfers.

Notation 7.5.8. Denote this composition of functors of the previous remark by $\widehat{H}$.

Remark 7.5.9. Since it is a composition of left adjoints, $\widehat{H}$ is also a left adjoint and preserves all colimits. Furthermore the diagram

$$\begin{tikzcd} Sm/k \arrow[rr] \arrow[d, "{[-]}"'] & & s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[d, "\widehat{H}"] \\ SmCor(k) \arrow[r, "\mathbb{Z}_{tr}(-)"] & \mathrm{Shv}_{Nis}^{tr}(k) \arrow[r] & \mathrm{Ch}^{\leq 0}(\mathrm{Shv}_{Nis}^{tr}(k) \end{tikzcd}$$

commutes since $N(\mathbb{Z}_{tr}(X))$ is the cochain complex with $\mathbb{Z}_{tr}(X)$ concentrated in degree 0 and $\mathbb{Z}_{tr}(X)$ is already a Nisnevich sheaf with transfers.

Definition 7.5.10 (The functor H')**.** Define the functor $\overline{H}$ as the composition

$$\overline{H} := s\mathrm{Pre}((Sm/k)_{Nis}) \xrightarrow{\widehat{H}} \mathrm{Ch}^{\leq 0}(\mathrm{Shv}_{Nis}^{tr}(k)) \to \mathrm{DM}_-^{\mathrm{eff}}(k)$$

and the functor H' as the composition

$$H' := s\mathrm{Pre}((Sm/k)_{Nis}) \xrightarrow{Q} s\mathrm{Pre}(Sm/k)_{Nis}) \xrightarrow{\overline{H}} \mathrm{DM}_-^{\mathrm{eff}}(k)$$

where Q is the $\mathbb{A}^1$-local cofibrant replacement functor.

To avoid confusions here is an overview of these three functors:

$$\begin{tikzcd} s\mathrm{Pre}^{tr}(k) \arrow[d, "N"'] & s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[l, "(-)_{tr}"'] \arrow[d, "\widehat{H}"'] \arrow[drr, "\overline{H}"] & & s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[ll, "Q"'] \arrow[d, "H'"] \\ \mathrm{Ch}^{\leq 0}(\mathrm{Pre}^{tr}(k)) \arrow[r, "(-)_{Nis}"] & \mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k)) \arrow[rr] & & \mathrm{DM}^{\mathrm{eff}}_{-}(k) \end{tikzcd}$$

Lemma 7.5.11. *Let $f : X \hookrightarrow Y \in s\mathrm{Pre}((Sm/k)_{Nis})$ be an $\mathbb{A}^1$-local cofibration. Then $\widehat{H}(f)$ is a monomorphism in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$.*

Proof. Since sheafification is exact and the Dold-Kan correspondence N is an equivalence of categories, it suffices to show this for f_{tr} in $s\mathrm{Pre}^{tr}(k)$. The functor $(-)_{tr}$ preserves colimits as a left adjoint and monomorphisms are closed under retracts and cobase change. Furthermore every cofibration is a retract of a map in I-cell where $I = \{(\partial\Delta^n \hookrightarrow \Delta^n) \otimes U\}$ and monomorphism are the same as objectwise monomorphisms in the category of abelian groups $\mathcal{A}b$. Thus it suffices to show this for the maps in I. Recall that $\mathbb{Z}_{tr}(U)(V) = Cor(V, U)$ is a free abelian group for all $V \in Sm/k$ and thus it is flat. Hence we have the equality

$$\begin{aligned} (((\partial\Delta^n \hookrightarrow \Delta^n) \otimes U)_{tr})(V)_m &= (\mathbb{Z}_{tr}(U) \otimes^{tr} (\mathbb{Z}[\partial\Delta^n \hookrightarrow \Delta^n])(V)_m \\ &= \mathbb{Z}_{tr}(U)(V) \otimes_{\mathbb{Z}} (\mathbb{Z}[\partial\Delta^n_m] \hookrightarrow \mathbb{Z}[\Delta^n_m]) \end{aligned}$$

for all $V \in Sm/k$ and $n \in \mathbb{N}$ which shows that all maps of I become monomorphisms. $\square$

Corollary 7.5.12. *Consider a pushout diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "j"'] & C \arrow[d, tail, "f"] \\ B \arrow[r] & D \end{tikzcd}$$

in $s\mathrm{Pre}((Sm/k)_{Nis}$ where the two vertical maps are $\mathbb{A}^1$-local cofibrations as indicated. Then the cones $\mathrm{Cone}(\widehat{H}(j))$ and $\mathrm{Cone}(\widehat{H}(j))$ are quasi-isomorphic in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$.

Proof. By the lemma above, every cofibration is mapped to a monomorphism in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$ via $\widehat{H}$. Furthermore $\widehat{H}$ respects pushouts, that is, we get the induced pushout diagram

$$\begin{tikzcd} \widehat{H}(A) \arrow[r] \arrow[d, hook, "\widehat{H}(j)"'] & \widehat{H}(C) \arrow[d, hook, "\widehat{H}(f)"] \\ \widehat{H}(D) \arrow[r] \arrow[d] & \widehat{H}(D) \arrow[d] \\ \mathrm{Coker}(\widehat{H}(j)) \arrow[r, "\cong"] & \mathrm{Coker}(\widehat{H}(j)) \end{tikzcd}$$

in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$ where it is standard that the cokernels are isomorphic. Due to [Wei94, 1.5.8] the cokernels of monomorphism are quasi-isomorphic to the cones. Hence $\mathrm{Cone}(\widehat{H}(j_i))$ and $\mathrm{Cone}(\widehat{H}(f_i))$ are quasi-isomorphic. $\square$

Lemma 7.5.13. *Let $f : X \to Y$ be an objectwise weak equivalence of $\mathbb{A}^1$-local cofibrant simplicial presheaves. Then $\widehat{H}(f) : \widehat{H}(X) \to \widehat{H}(Y)$ is a quasi-isomorphism in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$.*

Proof. First of all we have to check that f_{tr} is an objectwise weak equivalence, that is $f_{tr}(U) : X_{tr}(U) \to Y_{tr}(U)$ is a weak equivalence in $s\mathcal{S}et$, for all $U \in Sm/k$. For this consider the projective model structure on simplicial presheaves, that is the weak equivalences are the objectwise weak equivalences and the cofibrations are the same as the $\mathbb{A}^1$-local cofibrations. Since we have also a projective model structure on $s\mathrm{Pre}^{tr}(k)$ by Theorem 7.2.24 and $(-)_{tr}$ is a left Quillen functor by Remark 7.2.25, f_{tr} is an objectwise weak equivalence.

Now it follows immediately from Corollary 7.5.5 and Remark 7.5.3 that $N(f_{tr})$ is a quasi-isomorphism in $\mathrm{Ch}^{\leq 0}(\mathrm{Pre}^{tr}(k))$. Finally Proposition 7.3.5 implies that $(-)_{Nis}$ respects quasi-isomorphisms and the claim follows. $\square$

Recall the set of morphism J from Definition 6.2.18 which are all $\mathbb{A}^1$-local acyclic cofibrations. Then we have the following crucial lemmas concerning this set of maps.

Lemma 7.5.14. *$\overline{H}$ maps every map $j \in J$ to an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.*

Proof. First of all, the maps of the type $(\Lambda^n_i \hookrightarrow \Delta^n) \otimes X$ for $X \in Sm/k$ are objectwise weak equivalences between cofibrant objects since they are the generating projective acyclic cofibrations. Hence these maps become quasi-isomorphisms in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$ by the previous lemma and therefore of course isomorphisms under $\overline{H}$.

For the other types of maps of J recall that for a map $f : A \to B$ between cofibrant objects the projection $p : \mathrm{Cyl}(f) \to B$ is an objectwise weak equivalence between cofibrant objects by Lemma 3.5.10 (using the projective model structure). Hence $p : \mathrm{Cyl}(f) \to B$ becomes a quasi-isomorphism in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$ under $\widehat{H}$ by the previous lemma.

Now consider a map

$$U \amalg_Y \mathrm{Cyl}(j) \rightarrowtail \mathrm{Cyl}(U \amalg_Y \mathrm{Cyl}(j) \to X)$$

for an EDS as it appears in the definition of J. This map becomes quasi-isomorphic to the image of $U \amalg_Y \mathrm{Cyl}(j) \to X$ under $\widehat{H}$. Note that we have the commutative diagram

$$\begin{tikzcd} U \amalg_Y \mathrm{Cyl}(j) \arrow[r] \arrow[d, "U\amalg_Y p"'] & X \\ U \amalg_Y V \arrow[ur] & \end{tikzcd}$$

where $U \amalg_Y p$ is also an objectwise weak equivalence between cofibrant objects (using the projective model structure again). Therefore $U \amalg_Y p$ also becomes a quasi-isomorphism under $\widehat{H}$. Furthermore $a(U \amalg_Y V) \cong X$ by

Lemma 6.1.13. Therefore we get

$$\widehat{H}(U \amalg_Y V \to X) = N((a(U \amalg_Y V \to X))_{tr}) = (\mathbb{Z}_{tr}(X) \xrightarrow{\mathrm{id}} \mathbb{Z}_{tr}(X))$$

by Remark 7.5.7 and Remark 7.5.9. Therefore the considered map becomes quasi-isomorphic to $\mathrm{id} : \mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(X)$ under $\widehat{H}$ and hence an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.

Furthermore the maps of the type

$$X \times \mathbb{A}^1 \rightarrowtail \mathrm{Cyl}(X \times \mathbb{A}^1 \to X)$$

become quasi-isomorphic to $\mathbb{Z}_{tr}(X \times \mathbb{A}^1) \to \mathbb{Z}_{tr}(X)$ in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$ under $\widehat{H}$ by Remark 7.5.9 and therefore isomorphisms in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ since this is exactly done by the Verdier localization $D^{-}(\mathrm{Shv}^{tr}_{Nis}(k)) \to \mathrm{DM}^{\mathrm{eff}}_{-}(k)$.

Recall that the maps of J are defined as the pushout products of these two types of maps with $\iota : \partial\Delta^n \hookrightarrow \Delta^n$, that is, they fit into diagrams of the type

$$\begin{tikzcd}
\partial\Delta^n \otimes A \arrow[r, "\iota\otimes A"] \arrow[d, tail, "\partial\Delta^n\otimes f"'] & \Delta^n \otimes A \arrow[d, tail] \arrow[ddr, bend left, "\Delta^n\otimes f"] & \\
\partial\Delta^n \otimes B \arrow[r] \arrow[drr, bend right, "\iota\otimes B"'] & \partial\Delta^n \otimes B \amalg_{\partial\Delta^n\otimes A} \Delta^n \otimes A \arrow[dr, "\iota\square f"] & \\
& & \Delta^n \otimes B
\end{tikzcd}$$

where $f : A \rightarrowtail B$ denotes a map of one of these two types which becomes an isomorphism under $\overline{H}$ as already seen. Because of the simplicial model structure, Lemma 3.5.6 implies that the two vertical maps in the diagram are indeed cofibrations. We have to show that $\overline{H}(\iota\square f)$ is an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$. Using Corollary 7.5.12 the cones of the images of these two vertical maps become isomorphic in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ under $\widehat{H}$. Furthermore $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ is a triangulated category where isomorphisms are detected by their cones. Therefore it suffices to show that $K \otimes A \xrightarrow{K\otimes f} K \otimes B$ becomes an isomorphism under $\overline{H}$ for an arbitrary simplicial set K. That is, we have to check that the cone of $\widehat{H}(K \otimes f)$ becomes 0 in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.

For this observe that $\widehat{H}(K \otimes f) = N(f_{tr} \otimes^{tr} \mathbb{Z}[K])_{Nis}$ since $(-)_{tr}$ is strict symmetric monoidal and therefore it is quasi-isomorphic to

$$\widehat{H}(A) \otimes^{tr}_{Nis} N(\mathbb{Z}[K])_{Nis} \xrightarrow{\widehat{H}(f)\otimes^{tr}_{Nis} N(\mathbb{Z}[K])_{Nis}} \widehat{H}(B) \otimes^{tr}_{Nis} N(\mathbb{Z}[K])_{Nis}$$

by Proposition 7.5.6. Furthermore the map $\widehat{H}(f) : \widehat{H}(A) \to \widehat{H}(B)$ can be replaced quasi-isomorphically by a map $\widehat{f} : \widehat{A} \to \widehat{B}$ between projective objects of $\mathrm{Pre}^{tr}(k)$ as described above since all representable objects $\mathbb{Z}_{tr}(X)$ are projective. Since $N(\mathbb{Z}[K])$ is a constant presheaf of cochain complexes of free abelian groups, it is a cochain complex of projective presheaves. That is, $- \otimes^{tr}_{Nis} N(\mathbb{Z}[K])_{Nis}$ respects quasi-isomorphisms since the cohomology is computed objectwise in $\mathrm{Ch}^{\leq 0}(\mathcal{A}b)$ for presheaves and sheafification respects quasi-isomorphisms. Thus $\widehat{H}(f \otimes K)$ is quasi-isomorphic to

$$\widehat{A} \otimes^{tr}_{Nis} N(\mathbb{Z}[K])_{Nis} \xrightarrow{\widehat{f}\otimes^{tr}_{Nis} N(\mathbb{Z}[K])_{Nis}} \widehat{B} \otimes^{tr}_{Nis} N(\mathbb{Z}[K])_{Nis}$$

Since all involved objects of this map are complexes of projectives, the total derived tensor product calculates as

$$\begin{aligned}\widehat{A} \otimes^{tr}_{L,Nis} N(\mathbb{Z}[K])_{Nis} &= (\widehat{A} \otimes^{tr}_{L} N(\mathbb{Z}[K]))_{Nis} \\ &= (\widehat{A} \otimes^{tr} N(\mathbb{Z}[K]))_{Nis} \\ &= \widehat{A} \otimes^{tr}_{Nis} N(\mathbb{Z}[K])_{Nis}\end{aligned}$$

and similar for $\widehat{B}$. Summarizing all this the map $\overline{H}(f \otimes K)$ is isomorphic to

$$\overline{H}(A) \otimes \overline{H}(K) \xrightarrow{\overline{H}(f) \otimes \overline{H}(K)} \overline{H}(B) \otimes \overline{H}(K)$$

in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$. Since this category is a tensor triangulated category we get $\mathrm{Cone}(\overline{H}(K \otimes f)) \cong \mathrm{Cone}(\overline{H}(f)) \otimes \overline{H}(K))$ in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ which is zero since $\mathrm{Cone}(\overline{H}(f))$ is already zero. $\square$

Lemma 7.5.15. *Let f in* $s\mathrm{Pre}((Sm/k)_{Nis})$ *as it appears in the small object argument applied to J (cf. Remark 6.2.23). Then $\overline{H}(f)$ is an isomorphism in* $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.

Proof. Recall that such a map f arises from a sequence

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots$$

via $f = (X_0 \to \mathrm{colim}_{i \geq 0} X_i)$ where every f_i is a cobase change of coproducts of maps of J. Denote such a coproduct as

$$j_i : \coprod_{j \in J_i} D(j) \xrightarrow{\coprod j} \coprod_{j \in J_i} C(j)$$

where $J_i \subseteq J$ and $D(j)$ and $C(j)$ resp. denote the (co)domain of j. First of all, we have to check that all maps $\overline{H}(f_i)$ are isomorphisms in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$. For this note that we have the pushout diagram

$$\begin{tikzcd} \coprod_{j \in J_i} D(j) \arrow[r] \arrow[d, tail, "j_i"'] & C \arrow[d, tail, "f_i"] \\ \coprod_{j \in J_i} C(j) \arrow[r] & D \end{tikzcd}$$

in $s\mathrm{Pre}((Sm/k)_{Nis}$. By Corollary 7.5.12 the cones of $\widehat{H}(j_i)$ and $\widehat{H}(f_i)$ are quasi-isomorphic. Furthermore $\widehat{H}(j_i) = \oplus_{j \in J_i} \widehat{H}(j)$ since $\widehat{H}$ respects all colimits. By Lemma 7.5.11 we get that $\widehat{H}(j_i)$ and all $\widehat{H}(j)$ for $j \in J_i$ are monomorphisms. Therefore we have quasi-isomorphisms $\mathrm{Cone}(\widehat{H}(j_i) \simeq \mathrm{Coker}(\widehat{H}(j_i)$ and $\mathrm{Cone}(\widehat{H}(j) \simeq \mathrm{Coker}(\widehat{H}(j)$ for all $j \in J_i$. Hence we have a chain of quasi-isomorphisms

$$\mathrm{Cone}(\widehat{H}(j_i)) \simeq \mathrm{Coker}(\widehat{H}(j_i)) \cong \bigoplus_{j \in J_i} \mathrm{Coker}(\widehat{H}(j)) \simeq \bigoplus_{j \in J_i} \mathrm{Cone}(\widehat{H}(j)).$$

The last quasi-isomorphism follows from the fact that cohomology of cochain complexes commutes with direct sums. By the proof of the previous lemma we get $\mathrm{Cone}(\widehat{H}(j)) = 0$ in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ for all $j \in J$. Hence $\mathrm{Cone}(\widehat{H}(f_i)) = 0$ in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$. Therefore all maps $\overline{H}(f_i)$ are isomorphisms in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ since isomorphisms in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ are detected by their cones.

Furthermore we have $\widehat{H}(f) = (\widehat{H}(X_0) \to \mathrm{colim}_i \widehat{H}(X_i))$ since $\widehat{H}$ preserves all colimits. Now we have the exact sequence

$$0 \to \bigoplus_{i \geq 0} \widehat{H}(X_i) \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i \geq 0} \widehat{H}(X_i) \to \mathrm{colim}_{i \geq 0} \widehat{H}(X_i) \to 0$$

where the shift map is induced by the maps $f_i : X_i \to X_{i+1}$. The exactness follows from the fact that sheafification is exact and that

$$0 \to \bigoplus_{i \geq 0} N((X_i)_{tr}) \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i \geq 0} N((X_i)_{tr}) \to \mathrm{colim}_{i \geq 0} N((X_i)_{tr}) \to 0$$

is an exact sequence on presheaf-level which can be checked levelwise in the category of abelian groups $\mathcal{A}b$: The right-exactness is clear since the cokernel of $(\mathrm{id} - \mathrm{shift})$ is nothing else than the colimit. Since we can work with elements in $\mathcal{A}b$, a straightforward calculation shows the injectivity of $(\mathrm{id} - \mathrm{shift})$.

Note that we get a commutative diagram

$$\begin{tikzcd} \widehat{H}(X_0) \arrow[r] \arrow[dr] & \mathrm{Cone}(\mathrm{id} - \mathrm{shift}) \arrow[d, "q"] \\ & \mathrm{colim}_{i \geq 0} \widehat{H}(X_i) \end{tikzcd}$$

in $\mathrm{Ch}^{\leq 0}(\mathrm{Shv}^{tr}_{Nis}(k))$ where q is a quasi-isomorphism by the exactness of the sequence above and [Wei94, 1.5.8]. Since every map $\overline{H}(f_i)$ is an isomorphism in the triangulated category $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$, it follows from [Nee01, Lemma 1.6.6] that $\overline{H}(X_0) \to \mathrm{Cone}(\mathrm{id} - \mathrm{shift})$ is an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$. Hence

$$\overline{H}(f) : \overline{H}(X_0) \to \overline{H}(\mathrm{colim}_i X_i)$$

is an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ as claimed. □

Lemma 7.5.16. *There is a functorial fibrant replacement functor via a factorization $X \xrightarrow{j} X^f \xrightarrow{p} *$ for every $X \in Sm/k$, such that p is a fibration, j an acyclic cofibration and $\overline{H}(X \xrightarrow{j} X^f)$ is an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$.*

Proof. Recall again the set of maps J from Definition 6.2.18. According to Lemma 6.2.22 J admits the small object argument [Hov99, Theorem 2.1.14] with respect to $\mathbb{N}$. That is, we have a a functorial factorization

$$X \xrightarrow{j} X^f \xrightarrow{p} *$$

such that j is a map as it appears in Remark 6.2.23 and in the previous lemma, and p is in J-inj. Hence p has the right lifting property with respect to all maps of J. Thus Lemma 6.2.21 implies that p is an $\mathbb{A}^1$-local fibration since a terminal object $*$ is always fibrant. The map j in J-cell is an acyclic cofibration since every map of J-cell is in J-cof and every map in J-cof is an acylic cofibration. This is the case since J-cell is in J-cof by [Hov99, Lemma 2.1.10] and all maps of J are acyclic cofibrations. Finally the map $j : X \to X^f$ becomes an isomorphism in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ via $\overline{H}$ by the previous lemma. □

Remark 7.5.17. Note that this fibrant replacement functor $X \mapsto X^f$ does not have to be the same as the canonical one $X \mapsto RX$ since we are not taking *all* generating acyclic cofibrations of the $\mathbb{A}^1$-local model structure for the small object argument.

Proposition 7.5.18. *Let $f : X \xrightarrow{\sim} Y$ be an $\mathbb{A}^1$-local weak equivalence of simplicial presheaves. Then $H'(f)$ is an isomorphism in $\mathrm{DM}_-^{\mathrm{eff}}(k)$.*

Proof. Since $H' = \overline{H} \circ Q$, where Q is the $\mathbb{A}^1$-local cofibrant replacement functor which respects $\mathbb{A}^1$-local weak equivalences, we can assume that X and Y are cofibrant and we can check the property for $\overline{H}$. By the previous lemma, we get a commutative diagram

$$\begin{tikzcd} X^f \arrow[r, "f'", "\sim"'] & Y^f \\ X \arrow[u, tail, "\sim"] \arrow[r, "f", "\sim"'] & Y \arrow[u, tail, "\sim"'] \end{tikzcd}$$

where X^f and Y^f are again cofibrant, f' is an $\mathbb{A}^1$-local weak equivalence because of the 2 out of 3 property and $\overline{H}$ maps the fibrant replacements to isomorphisms. Furthermore Lemma 6.2.25 implies that f' is an objectwise weak equivalence between cofibrant simplicial presheaves. Therefore $\widehat{H}(f)$ is a quasi-isomorphism of complexes by Lemma 7.5.13. Hence $\overline{H}$ also maps f' to an isomorphism and the claim follows. □

Corollary 7.5.19. *The diagram*

$$\begin{tikzcd} Sm/k \arrow[r, hook] \arrow[dr, "M"'] & s\mathrm{Pre}((Sm/k)_{Nis}) \arrow[d, "H'"] \\ & \mathrm{DM}_-^{\mathrm{eff}}(k) \end{tikzcd}$$

commutes up to functor isomorphism.

Proof. Recall that every $X \in Sm/k$ is $\mathbb{A}^1$-local cofibrant and $X_{tr} = \mathbb{Z}_{tr}(X)$ is already a Nisnevich sheaf with transfers. Hence

$$(N(\mathbb{Z}_{tr}(X)))_{Nis} = (\mathbb{Z}_{tr}(X))_{Nis} = \mathbb{Z}_{tr}(X),$$

the cochain complex concentrated in degree 0, since $\mathbb{Z}_{tr}(X)$ is a discrete simplicial object. This is of course the same as $M(X)$, the motive of X. Furthermore the natural map $QX \xrightarrow{\sim} X$ is an $\mathbb{A}^1$-local weak equivalence between cofibrant objects. Hence it becomes an isomorphism in $\mathrm{DM}_-^{\mathrm{eff}}(k)$ by the previous proposition and therefore $M(X) \cong H'(X)$. □

Corollary 7.5.20. *H' induces a functor $H : \mathcal{H}(k) \to \mathrm{DM}_-^{\mathrm{eff}}(k)$, such that the following diagram commutes up to functor isomorphism:*

$$\begin{tikzcd} Sm/k \arrow[r, "h"] \arrow[dr, "M"'] & \mathcal{H}(k) \arrow[d, "H"] \\ & \mathrm{DM}_-^{\mathrm{eff}}(k) \end{tikzcd}$$

Proof. This follows from the previous proposition and the fact that $\mathcal{H}(k)$ is the localization of $s\mathrm{Pre}((Sm/k)_{Nis})$ with respect to the $\mathbb{A}^1$-local weak equivalences. □

8. Homotopy Types of Projective Curves

The aim of this section is to understand what happens to curves when applying the homotopy functor

$$h : Sm/k \longrightarrow \mathcal{H}(k)$$

That is to answer the question: When are two curves C_1, C_2 motivic homotopy equivalent, that is, $h(C_1) \cong h(C_2)$?

Since the arguments are the same, we are also dealing with abelian varieties and Brauer-Severi varieties.

8.1. **Curves of Genus > 0 and Abelian Varieties.**

The main tool to answer our question for the curves of genus > 0 and abelian varieties is to see that they are $\mathbb{A}^1$-rigid.

Lemma 8.1.1. *Let $X \in Sm/k$ be a curve of genus $g > 0$ or an abelian variety. Then X is $\mathbb{A}^1$-rigid.*

Proof. Due to Lemma 6.3.3 we have to check the injectivity of $\mathrm{Hom}_k(U \times \mathbb{A}^1, X) \to \mathrm{Hom}_k(U, X)$ for all $U \in Sm/k$. Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U \times \mathbb{A}^1 \arrow[d, "f"] \\ U \times \mathbb{A}^1 \arrow[r, "g"] & X \end{tikzcd}$$

We have to show that $f = g$. Let $\overline{k}$ be the algebraic closure of k. Then the functor $(-)(\overline{k}) : Red/k \to \mathcal{S}et$ taking $\overline{k}$-rational points is faithful by Proposition 1.1.12. Hence it suffices to show that in the induced diagram of $\overline{k}$-rational points the induced maps $f(\overline{k})$ and $g(\overline{k})$ coincide. The induced diagram reads as

$$\begin{tikzcd} U(\overline{k}) \arrow[r, "y \mapsto (y,0)"] \arrow[d, "y \mapsto (y,0)"'] & U(\overline{k}) \times \mathbb{A}^1(\overline{k}) \arrow[d, "f(\overline{k})"] \\ U(\overline{k}) \times \mathbb{A}^1(\overline{k}) \arrow[r, "g(\overline{k})"] & X(\overline{k}) \end{tikzcd}$$

Now let $(y, x) \in U(\overline{k}) \times \mathbb{A}^1(\overline{k})$. We have to show that $f(\overline{k})(y, x) = g(\overline{k})(y, x)$. But $f(\overline{k})(y, x)$ can be described as the composition

$$\begin{tikzcd} \mathrm{Spec}(\overline{k}) \arrow[r, "x"] \arrow[rr, bend left, "{(x,y)}"] & \mathbb{A}^1_{\overline{k}} \arrow[r, "\text{via } y"] \arrow[rr, bend right, "\text{constant}"'] & U_{\overline{k}} \times \mathbb{A}^1_{\overline{k}} \arrow[r, "f"] & X_{\overline{k}} \end{tikzcd}$$

where the morphism $\mathbb{A}^1 \to X$ is constant because of Proposition 1.3.1. Therefore $f(\overline{k})(y, x) = f(\overline{k})(y, 0)$ and analogous $g(\overline{k})(y, x) = g(\overline{k})(y, 0)$. But

$f(\overline{k})(y,0) = g(\overline{k})(y,0)$ by the commutativity of the induced diagram and hence the claim follows. □

Here comes the answer for abelian varieties and curves of genus > 0: There is no identification under motivic homotopy equivalence.

Theorem 8.1.2. *The full subcategories of abelian varieties Ab/k and curves of genus > 0 embed via h fully faithfully in the motivic homotopy category of k. Especially:*

$$h(X) \cong h(Y) \text{ in } \mathcal{H}(k) \Longleftrightarrow X \cong Y \text{ in } Sm/k$$

for X and Y abelian varieties or X and Y curves of genus > 0.

Proof. This follows immediately from the previous lemma and Theorem 6.3.7. □

Unfortunately, this method does not work for curves of genus 0 since we have already seen that $\mathbb{P}^1$ is not $\mathbb{A}^1$-rigid.

8.2. **Homotopy Invariance of the Picard Group and the Genus.** Now we will show the homotopy invariance of the genus. This provides that no curves of different genus are motivic homotopy equivalent and we can continue with studying the curves of genus 0 for themselves.

Proposition 8.2.1. *The Picard group is homotopy invariant, i.e. for $X, Y \in Sm/k$*

$$h(X) \cong h(Y) \text{ in } \mathcal{H}(k) \Longrightarrow \mathrm{Pic}(X) \cong \mathrm{Pic}(Y) \text{ in } \mathcal{G}r$$

Proof. Let $\mathbb{P}^\infty = \mathrm{colim}_n \mathbb{P}^n$. Then $\mathbb{P}^\infty$ is a group object in $\mathcal{H}(k)$ and

$$\mathrm{Hom}_{\mathcal{H}(k)}(X, \mathbb{P}^\infty) = \mathrm{Pic}(X)$$

for every $X \in Sm/k$ by [MV99, Proposition 4.3.8]. Therefore the claim follows. □

Remark 8.2.2. Note that there is another proof of this invariance for perfect fields k using the commutative diagram

$$\begin{tikzcd} Sm/k \arrow[r, "h"] \arrow[dr, "M"'] & \mathcal{H}(k) \arrow[d, "H"] \\ & \mathrm{DM}^{\mathrm{eff}}_-(k) \end{tikzcd}$$

up to functor isomorphism of section 7.5 and the fact that

$$\mathrm{Pic}(X) = \mathrm{CH}^1(X) = H^{2,1}_{mot}(X, \mathbb{Z}) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}_-(k)}(M(X), \mathbb{Z}(1)[2])$$

for perfect fields k.

Proposition 8.2.3. *The genus of a curve is homotopy invariant, i.e. for X, Y curves*

$$h(X) \cong h(Y) \text{ in } \mathcal{H}(k) \Longrightarrow g(X) = g(Y)$$

Proof. Let L/k be a finite separable field extension. According to Proposition 6.2.15 there is a commutative diagram

$$\begin{tikzcd} Sm/k \arrow[r, "f^*"] \arrow[d, "h_k"'] & Sm/L \arrow[d, "h_L"] \\ \mathcal{H}(k) \arrow[r, "f^*"] & \mathcal{H}(L) \end{tikzcd}$$

up to functor-isomorphism and therefore $h_L(X_L) \cong h_L(Y_L)$ in $\mathcal{H}(L)$. Hence we have $\mathrm{Pic}(X_L) \cong \mathrm{Pic}(Y_L)$ as groups by the previous proposition. Therefore the claim follows with use of Proposition 1.2.7. □

8.3. **Curves of Genus** 0 **and Arbitrary Brauer-Severi Varieties.** The following criterion for motivic equivalence of Brauer-Severi varieties in Chow motives was given by Nikita A. Karpenko in [Kar00].

Theorem 8.3.1 (Karpenko)**.** *Let X and Y be two Brauer-Severi varieties over an arbitrary field k with associated Azumaya algebras A_X and A_Y. Then*

$$X \cong Y \text{ in } \mathrm{CH}(k) \iff A_X \cong A_Y \text{ or } A_X \cong (A_Y)^{op}$$

Remark 8.3.2. Note that Karpenko's description of the associated Azumaya algebra looks different but it yields the same Azumaya algebra up to isomorphism as our approach (cf. [Ker90, Ch.30]).

The next theorem which establishes a connection between motivic homotopy theory and Chow motives, needs a hypothesis on k:

Hypothesis 8.3.3. There is a fully faithful embedding

$$\mathrm{CH}(k) \hookrightarrow \mathrm{DM}^{\mathrm{eff}}_{-}(k)$$

which acts on objects by $X \mapsto M(X)$ where $M(X)$ was the motive of X.

Remark 8.3.4. The hypothesis holds for all fields of characteristic 0 by Proposition 7.4.11. Note that it should be true for all perfect fields (cf. Remark 7.4.12).

Theorem 8.3.5. *Let k be a field fulfilling Hypothesis 8.3.3 and X, Y in $SmProj/k$. Then*

$$h(X) \cong h(Y) \text{ in } \mathcal{H}(k) \Longrightarrow X \cong Y \text{ in } \mathrm{CH}(k)$$

Proof. Recall from section 7.5 that we have a commutative diagram

$$\begin{tikzcd} Sm/k \arrow[r, "h"] \arrow[dr, "M"'] & \mathcal{H}(k) \arrow[d, "H"] \\ & \mathrm{DM}^{\mathrm{eff}}_{-}(k) \end{tikzcd}$$

up to functor isomorphism. Hence, if we take two disjoint unions of smooth projective varieties X, Y with $h(X) \cong h(Y)$, we get that their motives $M(X)$ and $M(Y)$ are isomorphic. But by Hypothesis 8.3.3 $\mathrm{CH}(k)$ embeds fully faithfully in $\mathrm{DM}^{\mathrm{eff}}_{-}(k)$ via $X \mapsto M(X)$ and therefore we get $X \cong Y$ in $\mathrm{CH}(k)$. □

Together with Karpenko's Theorem this connection yields a necessary condition for motivic homotopy equivalent Brauer-Severi varieties.

Corollary 8.3.6. *Let k be a field fulfilling Hypothesis 8.3.3 and X and Y two Brauer-Severi varieties with associated Azumaya algebras A_X and A_Y. Then*

$$h(X) \cong h(Y) \text{ in } \mathcal{H}(k) \Longrightarrow A_X \cong A_Y \text{ or } A_X \cong (A_Y)^{op}$$

Remark 8.3.7. Of course, this immediately implies that the dimension of Brauer-Severi varieties is homotopy invariant for a field k fulfilling the Hypothesis 8.3.3 since $\dim_k(A^{op}) = \dim_k(A)$ for an algebra A over k.

This necessary condition provides the answer to our question for the curves of genus 0.

Corollary 8.3.8. *Let k be a field fulfilling Hypothesis 8.3.3 and C_1, C_2 two curves of genus 0. Then*

$$h(C_1) \cong h(C_2) \text{ in } \mathcal{H}(k) \Longleftrightarrow C_1 \cong C_2 \text{ in } Sm/k$$

Proof. Let Q_1 and Q_2 be the associated quaternion algebras. Then $Q_1 \cong Q_2$ or $Q_1 \cong Q_2^{op}$ if $h(C_1) \cong h(C_2)$ by the previous corollary. Using Proposition 1.5.10 we get the claim. □

8.4. **Summary.** Summarizing all results of the previous subsections about the homotopy types of smooth projective curves we get the following theorem which gives the answer to the question and proves the conjecture of the introduction for all algebraically closed fields and all fields fulfilling Hypothesis 8.3.3.

Theorem 8.4.1. *Let k be an algebraically closed field or a field fulfilling Hypothesis 8.3.3 and let C_1, C_2 be two smooth projective curves over k. Then*

$$h(C_1) \cong h(C_2) \text{ in } \mathcal{H}(k) \Longleftrightarrow C_1 \cong C_2 \text{ in } Sm/k$$

Proof. Since the genus of a curve is homotopy invariant by Proposition 8.2.3, we can assume that $g(C_1) = g(C_2)$. Then we have two cases: $g = 0$ and $g > 0$. The claim follows for the curves of genus > 0 by Theorem 8.1.2. If k is algebraically closed, there is only one curve of genus 0 up to isomorphism, namely $\mathbb{P}^1$ (cf. Proposition 1.5.1), and the claim follows. If k is a field fulfilling the hypothesis, then the claim for the curves of genus 0 follows by Corollary 8.3.8. □

Remark 8.4.2. Since Hypothesis 8.3.3 is valid for all fields of characteristic 0, the previous theorem implies Theorem B of the introduction. Furthermore the only point where the hypothesis is needed is the case of genus 0. The rest is valid for arbitray fields. That is, we have also proved Theorem A of the introduction. Since the hypothesis should be true for all perfect fields, the previous theorem and hence the conjecture of the introduction should hold for all perfect fields.

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