ON THE GALOIS COHOMOLOGY OF SPIN(14)

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Preliminary Notes

Note from May/June 2006

I am very grateful to Skip Garibaldi for comments. They led to several corrections and additions.

In the version from 1999 I had claimed without proof \( ed(\text{Spin}_{13}) = 6 \). I have now added a new section (Section 10) containing a proof.

Abstract

Let \( k \) be a field with \( \text{char} \, k \neq 2 \). For \( i = 6, 7 \) we define invariants

\[
h_i : H^1(k, \text{Spin}(14)) \to H^i(k, \mathbb{Z}/2)/(−1)H^{i−1}(k, \mathbb{Z}/2).
\]

Further we show that the natural map

\[
H^1(k, (G_2 \times G_2) \rtimes \mu_8) \to H^1(k, \text{Spin}(14))
\]

is surjective.

One concludes that the essential dimension of \( \text{Spin}(14) \) is equal to 7.

Similar considerations are done for \( \text{Spin}(12) \). We also present the list of essential dimensions of the split groups \( \text{Spin}(n) \) for \( n \leq 14 \).

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1. The Arason invariant

1.1. The invariants $e_i$, $i \leq 3$. Let
$$e_i : \Lambda^i(k)/\Lambda^{i+1}(k) \to H^i(k; \mathbb{Z}/2), \quad i = 0, \ldots, 3$$
be the first invariants on the graded Witt ring given by dimension, discriminant, the Hasse-Witt invariant, and Arason’s invariant, cf. [1, 26].

1.2. The split groups of type $D_n$. We denote by $SO(n, n)$ the automorphism group of the quadratic form
$$n \sum_{1}^{n} (x_i^2 - y_i^2).$$
Furthermore, $Spin(n, n)$ denotes the universal cover of $SO(n, n)$ and $PSO(n, n) = SO(n, n)/\{\pm 1\} = Spin(n, n)/\mu$ denotes the corresponding adjoint group. Here $\mu$ is the center of $Spin(n, n)$. One has $\mu = \mu_2 \times \mu_2$ if $n$ is even and $\mu = \mu_4$ if $n$ is odd.

If $n$ is odd, every split group of type $D_n$ is isomorphic to one of $Spin(n, n)$, $SO(n, n)$, $PSO(n, n)$.

1.3. Galois cohomology of $SO(n, n)$. The set $H^1(k, SO(n, n))$ consists of the isomorphism classes of $2n$-dimensional quadratic forms with trivial discriminant. We consider $H^1(k, SO(n, n))$ as a subset of $\Lambda^2(k) \subset W(k)$.

The image of $H^1(k, SO(n, n)) \to H^1(k, PSO(n, n))$ consists of the similarity classes of the quadratic forms in $H^1(k, SO(n, n))$. For $u \in H^1(k, Spin(n, n))$ let $q_u$ be the corresponding quadratic form.

The image of $H^1(k, Spin(n, n)) \to H^1(k, SO(n, n))$ consists of those classes in $H^1(k, SO(n, n))$ with trivial Hasse-Witt invariant.

1.4. The invariant $\tilde{e}_3$ in $K^M_3/2$. Let $K^M_3 k$ be Milnor’s $K$-group [18].

By Merkurjev’s theorem [2, 16, 31] the invariant $e_2$ is bijective. Furthermore, Milnor’s homomorphism
$$s_3 : K^M_3 k/2 \to \Lambda^3(k)/\Lambda^4(k)$$
is bijective (cf. [11, 17, 18, 25]).

Putting things together yields natural maps
$$\tilde{e}_3 : H^1(k, Spin(n, n)) \to K^M_3 k/2.$$
For $u \in H^1(Spin(n, n))$ the class $\tilde{e}_3(u)$ depends alone on $q_u$. For $u \in H^1(Spin(8, 8))$ the corresponding quadratic form $q_u$ is a 3-fold Pfister form (cf. [5, 15, 20, 26]); if $q_u = \langle a, b, c \rangle$, then $\tilde{e}_3(u) = \{a, b, c\}$. Furthermore, the maps $\tilde{e}_3$ behave additively with respect to the natural inclusions
$$Spin(n, n) \times Spin(m, m) \to Spin(n + m, n + m).$$
These properties determine the family of maps $\tilde{e}_3$ uniquely.
2. Reduced squares

It has been observed by Serre that for any \( n \geq 2 \) there is a natural map
\[
P: K^M_n k/2 \to K^M_{2n} k/(2K^M_{2n} k + \{-1\}^{n-1} K^M_{n+1} k)
\]
characterized by
\[
P(\sum x_i) = \sum_{i<j} x_i x_j \mod (2K^M_{2n} k + \{-1\}^{n-1} K^M_{n+1} k)
\]
where \( x_i \) are symbols. (An element \( x \in K^M_n k/2 \) is called a symbol if it is of the form \( x = \{a_1, \ldots, a_n\} \) for some \( a_i \in k^* \).)

To define the operation \( P \) one checks that the right hand side of this formula does not depend on the presentation of an element as a sum of symbols. This follows easily from the definition of Milnor’s \( K \)-theory and the identity \( \{a, a\} = \{a, -1\} \), cf. [18].

Let \( \alpha_n: K^M_n k/2 \to H^n(F, \mathbb{Z}/2) \) be the norm residue homomorphism [18]. Milnor’s conjecture (cf. [30]) asserts that \( \alpha_n \) is bijective. With Milnor’s conjecture, the operations \( P \) give rise to corresponding maps
\[
H^n(k, \mathbb{Z}/2) \to H^{2n}(k, \mathbb{Z}/2)/(-1)^{n-1} H^{n+1}(k, \mathbb{Z}/2).
\]
Combining this with the fact that \((-1)H^{2n-1}(k, \mathbb{Z}/2)\) is in the kernel of the natural maps \( H^{2n}(k, \mathbb{Z}/2) \) to \( H^{2n}(k, \mathbb{Z}/4) \), one obtains operations
\[
H^n(k, \mathbb{Z}/2) \to H^{2n}(k, \mathbb{Z}/4).
\]
In the case \( n = 2 \) this operation is nothing else than the Pontryagin square, cf. [3, 4, 32, 33]. For \( n > 2 \) I don’t know any explanation of the operations \( P \) by an operation defined on the cohomology of topological spaces.

3. Lambda operations

Let \( \widetilde{W}(k) \) be the Grothendieck (\(-\)-Witt) ring of quadratic forms over \( k \). One defines \( \lambda \)-operations
\[
\lambda^i: \widetilde{W}(k) \to \widetilde{W}(k)
\]
in the usual fashion (see for instance [13]):

For a quadratic form \( \varphi: V \to k \) let \( \lambda^i \varphi: \bigwedge^i V \to k \) be its \( i \)-th exterior power. One has \( \lambda^0 \varphi = (1) \) and \( \lambda^1 \varphi = \varphi \). The form \( \lambda^2 \) is also given by the Killing form on the Lie algebra \( \text{so}(\varphi) \) (at least if \( Q \subset k \)).

One forms the formal power series
\[
\lambda_t \varphi = \sum_{i \geq 0} t^i \lambda^i \varphi.
\]
Then
\[
\lambda_t (\varphi \downarrow \psi) = \lambda_t \varphi \otimes \lambda_t \psi.
\]

The series \( \lambda_t \) extends to \( \widetilde{W}(k) \) by
\[
\lambda_t (\varphi - \psi) = \lambda_t \varphi \otimes (\lambda_t \psi)^{-1}
\]
and the operations $\lambda^i$ on $\hat{W}(k)$ are defined by

$$\lambda^i(x) = \sum_{i \geq 0} t^i \lambda^i(x)$$

for $x \in \hat{W}(k)$.

We are mainly interested in $\lambda^2$. Note that

$$\lambda^0(x) = 1,$$

$$\lambda^1(x) = x,$$

$$\gamma^2 = \dim y + 2\lambda^2(y),$$

$$\lambda^2(x + y) = \lambda^2(x) + xy + \lambda^2(y),$$

$$\lambda^2(x - y) = \lambda^2(x) - y(x - y) - \lambda^2(y),$$

$$\lambda^2(x - y) = \lambda^2(x) - xy + \dim y + \lambda^2(y),$$

$$\lambda^2(a) = \lambda^2(x)$$

for $x, y \in \hat{W}(k)$ and $a \in k^*$.

Let $I(k) \subset \hat{W}(k)$ be the fundamental ideal of zero dimensional virtual quadratic forms. The projection $\hat{W}(k) \to W(k)$ induces identifications $\hat{I}^n(k) = I^n(k)$ for $n > 0$. $\hat{I}^n(k)$ is additively generated by elements of the form

$$\langle a_1, \ldots, a_n \rangle - \langle 1 \rangle^n = \langle a_1, \ldots, a_{n-1} \rangle - \langle a_n \rangle \langle a_1, \ldots, a_{n-1} \rangle.$$

**Lemma 3.1.** Let $\varphi$ be an $n$-fold Pfister form and $x = \varphi - \langle 1 \rangle^n$. Then

$$\lambda^2(x) = \langle -1 \rangle^{n-1} x.$$

**Proof.** Write $\varphi = \psi \langle a \rangle$ where $\psi$ is an $(n - 1)$-fold Pfister form and where $a \in k^*$. Then $x = \psi - \langle a \rangle \psi$ and one finds

$$\lambda^2(x) = \lambda^2(\psi - \langle a \rangle \psi)$$

$$= \lambda^2(\psi) - \langle a \rangle \psi x - \lambda^2(\psi)$$

$$= -\langle a \rangle \langle -1 \rangle^{n-1} x$$

$$= \langle -1 \rangle^{n-1} \langle -a \rangle x = \langle -1 \rangle^{n-1} x.$$

Here one uses $\psi^2 = \langle -1 \rangle^{n-1} \psi$, $\langle -a \rangle x = \langle a \rangle x$ if $\dim x = 0$, and $\langle -a \rangle \langle a \rangle = \langle a \rangle$. \qed

**Corollary 3.2.** Let $\varphi$ be an $n$-fold Pfister form. Then

$$\lambda^2(\varphi) \simeq \varphi' \langle -1 \rangle^{n-1},$$

$$\lambda^2(\varphi') \simeq \varphi(\langle -1 \rangle^{n-1})'.$$

We define operations

$$P': I^n(k) \to I^{2n}(k),$$

$$P'(x) = \lambda^2(x) - \langle -1 \rangle^{n-1} x.$$

It follows from Lemma 3.1 and $\lambda^2(x + y) = \lambda^2(x) + xy + \lambda^2(y)$ that indeed $P'(x) \in I^{2n}(k)$.

These operations lift the operations $P$ to the Witt ring.
4. Multiplicative transfer

Let \( L/F \) be separable field extension. In addition to the restriction map
\[ r_{L/F}: W(F) \to W(L), \quad \varphi \mapsto [\varphi_L] \]
and the corestriction map
\[ c_{L/F}: W(L) \to W(F), \quad \psi \mapsto [\text{trace}_{L/F} \psi] \]
one may define a multiplicative transfer map
\[ N_{L/F}: W(L) \to W(F). \]
This map is analogous to the multiplicative transfer in cohomology, cf. [6, 12, 29].

We are interested in the case \( [L:F] = 2 \). Let \( \sigma \) denote the generator of the
Galois group. Then for a quadratic form \( \psi: W \to L \) the form \( N_{L/F}(\psi) \) is given by
the restriction of \( \psi \otimes \sigma^*: W \otimes \sigma W \to L \) to the subspace of invariants \((W \otimes \sigma W)^\sigma\).

Suppose \( L = F(\sqrt{a}) \). One has the following rules
\[
\dim_F(N_{L/F}(\psi)) = (\dim_F \psi)^2,
\]
\[
N_{L/F}(\langle \alpha \rangle) = \langle N_{L/F}(\alpha) \rangle,
\]
\[
N_{L/F}(x + y) = N_{L/F}(x) + c_{L/F}(x\sigma(y)) + N_{L/F}(y),
\]
\[
N_{L/F}(x - y) = N_{L/F}(x) - c_{L/F}(x\sigma(y)) + N_{L/F}(y),
\]
\[
\lambda^2(c_{L/F}(x)) = c_{L/F}(\lambda^2(x)) + aN_{L/F}(x),
\]
\[
N_{L/F}(\langle \alpha \rangle) = \langle a \rangle + \begin{cases} \langle \langle \text{trace} \alpha, -aN_{L/F}(\alpha) \rangle \rangle & \text{if trace } \alpha \neq 0, \\ 0 & \text{if trace } \alpha = 0, \end{cases}
\]
\[
N_{L/F}(\langle \alpha_1, \ldots, \alpha_n \rangle) = \langle \alpha \rangle^n + \prod \langle \langle \text{trace} \alpha_i, -aN_{L/F}(\alpha_i) \rangle \rangle \quad \text{if trace } \alpha_i \neq 0, \quad 0 \quad \text{else.}
\]
In particular, if \(-1\) is a square in \( F \), then
\[
N_{L/F}(I^n(L)) \subset I^{2n}(F)
\]
for \( n \geq 2 \).

5. The invariants \( h_6 \) and \( h_7 \)

For this section it is assumed for simplicity that \( \sqrt{-1} \in k \).

We define
\[
h_6: H^1(k, \text{Spin}(7,7)) \to H^6(k, \mathbb{Z}/2),
\]
\[
h_6(u) = \alpha_6 \circ P \circ \epsilon_3(u).
\]

The invariant \( h_6(u) \) depends only on \( q_u \).

By the remarks of Section 3 one can lift this invariant to \( I^6(k) \).

In some cases the invariant \( h_6 \) can be described explicitly. For a Pfister form \( \varphi \)
one denotes by \( \varphi' \) its pure subform (one has \( \varphi = (1) \bot \varphi' \)). Let \( a_i, b_i \in k^*, \ i = 1, \ 2, \ 3 \), and put
\[
(1) \quad q = c(\langle a_1, a_2, a_3 \rangle') \bot -\langle b_1, b_2, b_3 \rangle')
\]
Then \( q = q_u \) for some \( u \in H^1(k, \text{Spin}(7,7)) \) and for any such \( u \) one finds
\[
(2) \quad h_6(u) = (a_1, a_2, a_3, b_1, b_2, b_3).
\]

**Lemma 5.1.** Let \( u \in H^1(k, \text{Spin}(7,7)) \). If \( q_u \) is isotropic, then \( h_6(u) = 0 \).
Proof. If $q_u$ is isotropic, the $q_u$ has a representation (1) with $a_1 = b_1$, see [19, Satz 14, Zusatz] or [24]. The claim follows from (2). □

Proposition 5.2. Let $u \in H^1(k, \text{Spin}(7,7))$ and let $c$ be a nonzero value of $q_u$. The element

$$h_6(u) \cup (c) \in H^7(k, \mathbb{Z}/2)$$

does not depend on the choice of $c$.

Proof (Variant 1). Write $q = q_u$. If $q$ is isotropic, then $h_6(u) = 0$ by Lemma 5.1. We may therefore assume that $q$ is anisotropic. Let $c = q(v)$ and $c' = q(v')$ be two values of $q$ with $v, v'$ linearly independent. Then $c/c'$ is a norm from the quadratic extension $L$ splitting the 2-dimensional subform $q|(vk + v'k)$. Say $c/c' = N_{L/k}(\lambda)$. Then

$$h_6(u) \cup (c) - h_6(u) \cup (c') = h_6(u) \cup (c/c')$$
$$= h_6(u) \cup N_{L/k}(\lambda)$$
$$= N_{L/k}(h_6(u_L) \cup (\lambda))$$
$$= N_{L/k}(0 \cup (\lambda)) = 0$$

since $q_u$ is isotropic and by Lemma 5.1. □

Proof (Variant 2). Write $q = q_u$ as $q: V \rightarrow k$. Then any $x = [v] \in PV$ determines an element

$$q(x) \in k(x)/k(x)^\times$$

Let $\xi \in PV$ be the generic point and consider

$$\omega = h_6(u) \cup (q(\xi)) \in H^7(k(PV), \mathbb{Z}/2).$$

The element $\omega$ is unramified on $PV$, except possibly at the divisor

$$Z = \{q = 0\} \subset PV$$

Here the residue is a multiple of (in fact, equal to)

$$h_6(u)_{k(Z)} \in H^6(k(Z), \mathbb{Z}/2)$$

But the quadratic form $q_{k(Z)}$ is isotropic, whence $h_6(u)_{k(Z)} = 0$ by Lemma 5.1. Hence $\omega$ is unramified everywhere on $PV$ and therefore $\omega = (\omega_0)_{k(PV)}$ for some $\omega_0 \in H^7(k, \mathbb{Z}/2)$. The claim follows by specialization. □

Proposition 5.2 gives rise to an invariant

$$h_7: H^1(k, \text{Spin}(7,7)) \rightarrow H^7(k, \mathbb{Z}/2),$$

$$h_7(u) = h_6(u) \cup (q_u(v))$$

where $q_u(v)$ is any nonzero value of $q_u$.

As for $h_6$, the invariant $h_7(u)$ depends only on $q_u$. If $q_u = q$ with $q$ as in (1), then

$$h_7(u) = (a_1, a_2, a_3, b_1, b_2, b_3, c).$$

This computation shows that the invariant $h_7$ is non-trivial.

In the next two statements (Proposition 5.3, Lemma 5.4) we assume that $k$ contains the algebraic closure of $\mathbb{Q}$. This assumption is made to be sure that we can neglect some universal constants arising in decompositions of Killing forms and of $\lambda^2(q)$. I have not tried to figure out the best possible conditions.
Proposition 5.3. Assume $\overline{Q} \subset k$. Any value of $h_6$ and of $h_7$ is a symbol.

Proof. It suffices to consider $h_6$. Let $u \in H^1(k, \text{Spin}(7, 7))$ and write $q = q_u$. Then $h_6(u)$ is represented by 92-dimensional form

$$\lambda^2 q \perp \langle 1 \rangle.$$  

The form $\lambda^2 q$ is also given by the Killing form on $so(q)$.

We may assume that $u$ is induced from an element $x \in H^1(k, (G_2 \times G_2) \rtimes \mu_8)$, see Corollary 7.3. Let $g \subset so(q)$ be the Lie algebra of type $G_2 + G_2$ corresponding to $x$. Its Killing form is the trace of the Killing form of a Lie algebra of type $G_2$ over some quadratic extension. In view of the next Lemma, this form is hyperbolic.

Therefore the 92-dimensional form $\lambda^2 q \perp \langle 1 \rangle$ contains a 28-dimensional hyperbolic subform. Thus $h_6(u)$ is represented by a $92 - 28 = 64$-dimensional quadratic form, which therefore must be a multiple of a 6-fold Pfister form. □

This proof indicates that one may represent $h_6(u)$ by a form on the spinor representation $S$, cf. below. In fact there is a natural way to represent $h_6(u)$ as $N_{L/k}(\psi)$ on $S$, where $\psi/L$ is the 3-fold Pfister form corresponding to a reduction $x \in H^1(k, (G_2 \times G_2) \rtimes \mu_8)$ of $u$, cf. [24].

Lemma 5.4. Assume $\overline{Q} \subset k$. Let $g$ be a Lie algebra of type $G_2$ and let $\varphi$ be the associated 3-fold Pfister form. Then the Killing form on $g$ is hyperbolic.

Proof. Let $V$ be the 7-dimensional representation of $g$. Then

$$g \perp V = \bigwedge^2 V.$$  

Let further $\psi$ denote the Killing form on $g$ and let $\varphi$ be the associated 3-fold Pfister form. Then

$$\psi \perp \varphi' = \lambda^2(\varphi') = \varphi'((-1, -1))'$$  

by Corollary 3.2. The claim follows. □

Our considerations in the construction of the invariants $h_6, h_7$ may be also applied to the group $SO(6)$. This leads to invariants

$$H^1(k, SO(6)) \to H^1(k, \mathbb{Z}/2)$$

for $i = 4, 5$, given by

$$c(\langle a_1, a_2 \rangle', \langle b_1, b_2 \rangle') \mapsto (a_1, a_2, b_1, b_2),$$  

$$c(\langle a_1, a_2 \rangle', \langle b_1, b_2 \rangle') \mapsto (a_1, a_2, b_1, b_2, c).$$

The latter coincides with the invariant

$$\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle \mapsto (a_1, a_2, a_3, a_4, a_5)$$  

defined by Serre.

6. A Reduction Lemma

Let $G$ be an algebraic group over $k$ and let $i: H \subset G$ be a subgroup. For $x \in H^1(k, G)$ we denote by $P_x$ a corresponding $G$-torsor.

Lemma 6.1. Let $x \in H^1(k, G)$. Then $x$ is in the image of

$$i_*: H^1(k, H) \to H^1(k, G)$$

if and only if the variety $P_x/H$ has a $k$-rational point.
Proof. Indeed, if \( x = i_s(y) \), then \( P_x \simeq P_y \times_H G \) and \( P_x/H \) has the \( k \)-rational point given by \([P_y, 1] \mod H\).

Conversely, if \( z \in P_x/H \) is \( k \)-rational, then the fiber of \( z \) under \( P \to P_x/H \) is an \( H \)-torsor \( Q \) with \( Q \times_H G \simeq P_x \). □

This simple lemma is the basis of many structure theorems on quadratic forms and algebras. It applies usually when there is a “small” representation of \( G \), i.e., a representation \( G \to \text{GL}(V) \) with \( \dim V < \dim G \).

A fairly simple example is given by \( G = \text{O}(n) \) and \( H = \text{O}(n - 1) \times \mu_2 \). Let \( x \in H^1(k, \text{O}(n)) \); if \( q_x : V \to k \) is the corresponding quadratic form, then \( P_x/H \) is naturally isomorphic to \( U = PV \setminus \{q_x = 0\} \). Since \( U \) has a rational point, it follows that \( x \) has a reduction to \( H \).

Her majesty \( E_8 \) does not have a small representation.

7. 14-DIMENSIONAL SPINORS

Let \( \text{Spin}(7, 7) \to \text{GL}(S) \) be one of the spinor representations (\( \dim S = 64 \)) and let \( \text{PSO}(7, 7) \to \text{PGL}(S) \) be the induced homomorphism. We denote \( G = \text{PSO}(7, 7) \) and define \( H \subset G \) as the image of \((G_2 \times G_2) \rtimes \mathbb{Z}/2 \to \text{PSO}(7, 7)\)
given by
\[
(g, h)^n \mapsto \begin{pmatrix}
\rho(g) & 0 \\
0 & \rho(h)
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}^n
\]
where \( \rho : G_2 \to \text{Spin}(7) \) is the standard representation.

We need the following fact, see [7, 9, 21, 24].

**Proposition 7.1.** The action of \( G \) on \( \text{PS} \) has an open and dense orbit \( U \). If \( k \) is algebraically closed, then the isotropy group \( H_u \) of \( u \in U \) is conjugate to \( H \). In particular, \( U = G/H \).

Now let \( x \in H^1(k, G) \). Then \( X_x = P_x \times_G \text{PS} \) is a Brauer-Severi variety whose Brauer class coincides with the Tits class \( t(x) \in H^2(k, \mu_4) \) of \( x \). Further, the variety \( U_x = P_x \times_G U = P_x/H \) is a dense open subscheme of \( X_x \). It follows that \( P_x/H \) has \( k \)-rational points if and only if \( t(x) = 0 \) (to be sure, let us assume that \( k \) is infinite).

Lemma 6.1 shows

**Corollary 7.2.** An element \( x \in H^1(k, G) \) has an \( H \)-reduction if and only if \( t(x) = 0 \). □

Let \( \tilde{H} \) be the preimage of \( H \) under \( \text{Spin}(7, 7) \to \text{PSO}(7, 7) \). One finds (see [24])

\[
\tilde{H} = (G_2 \times G_2) \rtimes \mu_8
\]
where \( \mu_8 \subset \text{Spin}(7, 7) \) is the normalizer of \( G_2 \times G_2 \).

**Corollary 7.3.** The homomorphism
\[
H^1(k, (G_2 \times G_2) \rtimes \mu_8) \to H^1(k, \text{Spin}(7, 7))
\]
is surjective.
Proof. This follows from a diagram chase in

\[
\begin{array}{cccccc}
H^1(k, \mu_4) & \rightarrow & H^1(k, (G_2 \times G_2) \rtimes \mu_8) & \rightarrow & H^1(k, (G_2 \times G_2, Z/2)) & \rightarrow & H^2(k, \mu_4) \\
\| & & \downarrow & & \downarrow & & \| \\
H^1(k, \mu_4) & \rightarrow & H^1(k, \text{Spin}(7, 7)) & \rightarrow & H^1(k, \text{PSO}(7, 7)) & \rightarrow & H^2(k, \mu_4)
\end{array}
\]

It can be shown that there exist a field \(k\) and \(x \in H^1(k, \text{Spin}(7, 7))\) such that \(x\) has no reduction to the subgroup \((G_2 \times G_2) \rtimes \mu_4\). This means that the appearing forms of \(G_2 \times G_2\) are necessarily of type \(R_{\ell/k}(G_2)\) with \(\ell/k\) a quadratic field extension. Examples have been provided in [8] using residue arguments and in [10] using computations of the \(K\)-theory of certain homogeneous varieties.

8. THE ESSENTIAL DIMENSION OF \(\text{Spin}(14)\)

We denote by \(\text{ed}(G)\) the essential dimension of \(G\), see [22].

**Proposition 8.1.** \(\text{ed}(\text{Spin}(14)) = 7\).

**Proof.** \(\text{ed}(\text{Spin}(14)) \geq 7\) follows from the non-triviality of the invariant \(h_7\).

It remains to show \(\text{ed}(\text{Spin}(14)) \leq 7\). By Corollary 7.3 it suffices show \(\text{ed}(\tilde{H}) \leq 7\). To describe any \(\tilde{H}\)-torsor one needs one parameter to describe a class \((a) \in H^1(k, \mu_8) = k^*/(k^*)^8\) and 3 \(\cdot\) 2 parameters to describe an octonion algebra

\[
O(a_1 + \sqrt{a} b_1, a_2 + \sqrt{a} b_2, a_3 + \sqrt{a} b_3)
\]

over \(k(\sqrt{a})\). \(\square\)

9. ON THE COHOMOLOGY OF \(\text{Spin}(12)\)

We briefly sketch a proof of \(\text{ed}(\text{Spin}(6, 6)) = 6\).

We define \(H \subset \text{SO}(6, 6)\) as the image of

\[
\text{SL}(6) \rtimes \mathbf{Z}/2 \rightarrow \text{SO}(6, 6)
\]

given by

\[
ge^a \mapsto \left(\begin{array}{cc}
g & 0 \\ 0 & (g^t)^{-1}
\end{array}\right)\left(\begin{array}{cc}
0 & 1 \\ 1 & 0
\end{array}\right)^a.
\]

Here we understand coordinates \((x, y)\) with respect to the quadratic form \(\sum_i x_i y_i\). The preimage \(\tilde{H}\) of \(H\) in \(\text{Spin}(6, 6)\) is

\[
\tilde{H} = \text{SL}(6) \rtimes \mu_4.
\]

By the mentioned theorem of Pfister ([19, Satz 14, Zusatz] or [24]), any \(\text{Spin}(6, 6)\)-torsor admits an \(\tilde{H}\)-reduction. Since any hermitian form can be diagonalized, the map

\[
H^1(k, \text{SO}(6, 6) \rtimes \mu_4) \rightarrow H^1(k, \tilde{H})
\]

is surjective. Hence

**Corollary 9.1.** \(\text{ed}(\text{Spin}(6, 6)) \leq 6\).

We define invariants in \(H^3(\mathbf{Z}/2), H^6(\mathbf{Z}/2)\) by a variant of the previous method. It is based on the following facts:
Lemma 9.2. Let $a \in k^*$. Then the kernel of

$$W(k) \to W(k), \quad x \mapsto \langle a \rangle x$$

is generated by 2-dimensional forms of the form $\langle N_{\ell/k}(a) \rangle$ with $a \in \ell^*$, $\ell = k(\sqrt{a})$.

Proof. Well known. \hfill \Box

Lemma 9.3. Let $a, b \in k^*$ and let $x, y \in W(k)$. If

$$\langle a \rangle x = \langle b \rangle y,$$

then there exist $z \in W(k)$ with

$$\langle a \rangle x = \langle a \rangle z = \langle b \rangle z = \langle b \rangle y.$$

Moreover, any such $z$ may be written as a sum of 2-dimensional forms of the form $\langle N_{\ell/k}(a) \rangle$ with $a \in \ell^*$, $\ell = k(\sqrt{ab})$.

Proof. Let $\varphi$ be a quadratic form representing $x$, let $K = k(\sqrt{b})$, and suppose that $\langle a \rangle \varphi_K$ is split.

As $\langle a \rangle \varphi_K$ is isotropic, one has $\langle a \rangle \varphi = \langle a \rangle (\langle c \langle d \rangle \rangle + \varphi')$ such that $\langle a, d \rangle_K$ is isotropic. To see this, let $\varphi = \langle a_1, \ldots, a_n \rangle$ and let $q: V = L^n \to k$

$$q(\lambda_1, \ldots, \lambda_n) = \sum_i a_i N_{L/k}(\lambda_i)$$

with $L = k(\sqrt{a})$. Note that $q = \langle a \rangle \varphi$ and that $q(\lambda v) = N_{L/k}(\lambda)q(v)$ for $\lambda \in L$. If $q_K$ is isotropic, there exists a 2-dimensional $L$-submodule $W$ of $V$ such that $q|W$ is isotropic over $K$. Next note that $q|W = c\langle a, d \rangle$ for some $c, d$.

We may assume $\varphi = c\langle d \rangle \perp \varphi'$. There exists $e$ such that

$$\langle a, d \rangle = \langle a, e \rangle = \langle b, e \rangle$$

Then $\langle a \rangle \varphi = c\langle a, e \rangle + \langle a \rangle \varphi'$. The claim follows by induction on $\dim \varphi$ and Lemma 9.2. \hfill \Box

Let $I_2(k) \subset I(k)$ be the subset of elements which are split over some quadratic extension. One defines an operation

$$Q: I_2(k) \to I_2(k), \quad Q(\langle a \rangle x) = \langle a \rangle \varphi^2(x).$$

This map is well defined by Lemma 9.2 and Lemma 9.3.

We assume that $-1$ is a square. Let $u \in H^1(k, \Spin(6, 6))$. Then

$$q_u = a\langle b \rangle (\langle c, d \rangle' - \langle e, f \rangle')$$

and

$$Q(q_u) = \langle b, c, d, e, f \rangle$$

Hence we an invariant

$$k_5: H^1(k, \Spin(6, 6)) \to H^5(k, \mathbb{Z}/2).$$

If $q_u$ is isotropic, then $q_u = a\langle b, c', d'' \rangle \perp (1, -1)$. This shows $Q(q_u) = 0$. By the same argument as in the proof of Proposition 5.2 we get an invariant

$$k_6: H^1(k, \Spin(6, 6)) \to H^6(k, \mathbb{Z}/2),$$

$$k_6(u) = k_5(u) \cup (q_u(v))$$
where \( q_a(v) \) is any nonzero value of \( q_a \).

If \( q_a = a\langle b \rangle (\langle c, d \rangle - \langle e, f \rangle) \), then

\[
k_6(u) = (a, b, c, d, e, f).
\]

This shows that \( k_6 \) is nontrivial.

**Corollary 9.4.** \( \text{ed}(\text{Spin}(6, 6)) \geq 6 \).

10. **On the cohomology of Spin(13)**

Let

\[
q : V = k^{13} \to k
\]

\[
q(x_1, \ldots, x_{13}) = (x_1^2 + x_2^2 + x_3^2) - (x_4^2 + x_5^2 + x_6^2) - (x_7^2 + x_8^2 + x_9^2) + (x_{10}^2 + x_{11}^2 + x_{12}^2) - x_{13}^2
\]

An element of \( H^1(k, \text{SO}(q)) \) is given by a 13-dimensional quadratic form \( q' \) with

\[
q' \perp (1) \in H^1(k, \text{SO}(7, 7)) \subset I^2 \subset W(k)
\]

Let \( G \) be the subgroup of \( \text{SO}(q) \) generated by (matrix notation with respect to \( k^{13} = k^3 \times k^3 \times k^3 \times k^3 \times k \))

\[
U(g, h) = \begin{pmatrix}
g & 0 & 0 & 0 & 0 \\
0 & g & 0 & 0 & 0 \\
0 & 0 & h & 0 & 0 \\
0 & 0 & 0 & h & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and

\[
V(\alpha, \beta) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \beta & 0 \\
0 & 0 & 0 & 0 & \alpha \beta
\end{pmatrix}, \quad W(\eta) = \begin{pmatrix}
0 & 0 & \eta & 0 & 0 \\
0 & 0 & 0 & \eta & 0 \\
\eta & 0 & 0 & 0 & 0 \\
0 & \eta & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

with \( g, h \in \text{SO}(3), \alpha, \beta \in \mu_2 \) and \( \eta \in \mu_4 \). One has

\[
G = (\text{SO}(3) \times \mu_2)^2 \times \mu_4 \subset \text{SO}(q)
\]

with \( \mu_4 \) acting via the projection \( \mu_4 \to \mu_2 = \mathbb{Z}/2 \) by permutation of the factors.

We consider the commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & \mu_2 \\
\| & & \downarrow j \\
\mu_2 \longrightarrow & \stackrel{\pi}{\longrightarrow} & G \\
\| & \downarrow j & \| \\
1 & \longrightarrow & \text{Spin}(q) \\
& & \downarrow j & \text{SO}(q) \\
& & \longrightarrow 1
\end{array}
\]

where \( \tilde{G} \subset \text{Spin}(q) \) is the preimage of \( G \) under the projection \( \pi : \text{Spin}(q) \to \text{SO}(q) \) and \( j, j \) are the inclusions.

We describe the image

\[
J = j_* (H^1(k, G)) \subset H^1(k, \text{SO}(q))
\]
**Lemma 10.1.** The set $J$ consists exactly of the (isomorphism classes of) quadratic forms $q'$ of the type

\[(4) \quad q' = \bar{q} \perp (-\det(\bar{q}))\]

with

\[(5) \quad \bar{q} = (T_{K/k})_* \langle (s, 1, -\lambda, -\mu_1, -\mu_2, \mu_1 \mu_2) \rangle\]

with $K = k[s]/(s^2 - b)$ for some $b \in k^\times$ and $\lambda, \mu_1, \mu_2 \in K^\times$.

**Proof.** Note that $G \subset SO(q)$ leaves the subspace $V' = k^{12} \times \{0\} \subset V$ invariant. Let

$$\ell : G \to O(q|V')$$

$$\ell(g) = j(g)|_{V'}$$

Then

$$j(g) = (\ell(g), \det(\ell(g))) \in O(q|V') \times O(1) \subset O(q)$$

This yields the decomposition (4).

It remains to show that $\ell_* (H^1(k, G)) \subset H^1(O(q|V'))$ consists of the forms $\bar{q}$ as in (5).

Elements of $H^1(k, G)$ are the isomorphism classes of triples $(K', \varphi, \varphi_1)$, where $K' = k[t]/(t^4 - b)$ is a Galois $\mu_4$-algebra and where $\varphi, \varphi_1$ are quadratic forms over the quadratic subextension $K = k[s] \subset K'$, $s = t^2$ with $\varphi$ of rank 3 and determinant 1 and with $\varphi_1$ of rank 1. Let

$$H = (O(1) \times O(1) \times O(1)) \cap SO(3) \simeq \mu_2 \times \mu_2$$

and

$$G' = (H \times \mu_2)^2 \times \mu_4 \subset G$$

Since quadratic forms (over $K$) can be diagonalized, it follows that $H^1(k, G') \to H^1(k, G)$ is surjective.

The claim follows from Corollary 10.3 below. \[\square\]

**Lemma 10.2.** Let

$$G'' = (\mu_2)^2 \times \mu_4$$

generated by $\mu_4$ and elements $\alpha, \beta$ with the relations

$$\alpha^2 = \beta^2 = (\alpha\beta)^2 = 1, \quad \zeta\alpha\zeta^{-1} = \beta, \quad \zeta\beta\zeta^{-1} = \alpha$$

for a generator $\zeta$ of $\mu_4$.

Let

$$q_0 : k^2 \to k$$

$$q_0(x, y) = x^2 - y^2$$

and let

$$\varphi : G'' \to O(q_0)$$

be the homomorphism with

$$\varphi(\alpha) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varphi(\eta) = \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}$$

with $\alpha, \beta \in \mu_2$ and $\eta \in \mu_4$.

Let $\xi \in H^1(k, G'')$ and write the corresponding Galois $G''$-algebra as

$$E_\xi = k[t, s, x, y]/(t^4 - b, s - t^2, x^2 - u - sv, y^2 - u + sv)$$
with $b, u, v \in k, b \neq 0, u^2 - bv^2 \neq 0$. Here the action of $G''$ is given by

\[
\begin{align*}
\zeta(t) &= \zeta t, \quad \zeta(s) = -s, \quad \zeta(x) = y, \quad \zeta(y) = x \\
\alpha(t) &= t, \quad \alpha(s) = s, \quad \alpha(x) = -x, \quad \alpha(y) = y \\
\beta(t) &= t, \quad \beta(s) = s, \quad \beta(x) = x, \quad \beta(y) = -y
\end{align*}
\]

Then the associated quadratic form $q_\xi = \varphi_*(\xi) \in H^1(k, O(q_0))$ is given by

\[
q_\xi = (T_{K/k}_*)(\langle s \rangle (u + sv))
\]

with $K = k[s] \subset E_\xi$.

**Proof.** One has (more or less by definition)

\[
q_u = (q_0 \otimes_k E)(k^2 \otimes_k E)^{G''}
\]

with $G''$ acting on $k^2$ via $O(q_0)$ and on $E$ as Galois algebra, respectively.

The claim follows from the following explicit computation (for a related consideration see Garibaldi’s Lens notes from May 2006, Example 16.5):

One finds that $(k^2 \otimes_k E)^{G''}$ is the free $k$-module with basis

\[
X = (xt, -yt), \quad Y = (xt^3, yt^3) = (xts, yts)
\]

For $c, d \in k$ one has with $\lambda = x^2 = u + sv$ and $\bar{\lambda} = y^2 = u - sv$

\[
q_0(cX + dY) = (xt(c + ds))^2 - (yt(-c + ds))^2 = \lambda s(c + ds)^2 + \bar{\lambda}(s)(c - ds)^2 = T_{K/k}(\lambda s(c + ds)^2)
\]

\[\square\]

**Corollary 10.3.** Let $n, m \geq 0$, let $U = (\mu_2)^n$ and let

\[\Phi: U \rightarrow O(1)^m \subset O(m)\]

be some homomorphism. Let

\[G'' = U^2 \rtimes \mu_4\]

with $\mu_4$ acting via the projection $\mu_4 \rightarrow \mu_2 = \mathbb{Z}/2$ by permutation of the factors and let

\[\varphi: G'' \rightarrow O(m, m)\]

\[\varphi(u_1, u_2) = \begin{pmatrix} \Phi(u_1) & 0 \\ 0 & \Phi(u_2) \end{pmatrix}\]

\[\varphi(\zeta) = \begin{pmatrix} 0 \\ \zeta \\ 0 \end{pmatrix}\]

for $(u_1, u_2) \in U^2$ and a generator $\zeta$ of $\mu_4$.

Let $\xi \in H^1(k, G'')$ and write the corresponding Galois $G''$-algebra as

\[E_\xi = k[t, s, x_i, y_i; i = 1, \ldots, n]/(t^4 - b, s - t^2, x_i^2 - u_i - sv_i, y_i^2 - u_i + sv_i)\]

with $b, u_i, v_i \in k, b \neq 0, u_i^2 - bv_i^2 \neq 0$ (with obvious $G''$ action, see Lemma 10.2).

Then the associated quadratic form $q_\xi = \varphi_*(\xi) \in H^1(k, O(m, m))$ is given by

\[q_\xi = (T_{K/k}_*)(\langle s \rangle (\mu_1, \ldots, \mu_m))\]
with $K = k[s] \subset E_\xi$ and with

$$\mu_j = \prod_{i=1}^{n} \lambda_i^{\Phi_{ij}} \in K, \quad j = 1, \ldots, m$$

where

$$\lambda_i = u_i + sv_i$$

and where $\Phi_{ij} = 0, 1$ is defined by

$$\Phi(\alpha_1, \ldots, \alpha_n) = \left( \prod_{i=1}^{n} \alpha_i^{\Phi_{ij}} \right)_{j=1, \ldots, m}$$

Proof. One easily reduces to the case $m = 1$, $n = 1$ and $\Phi = \text{id}$, which is treated in Lemma 10.2.

Proposition 10.4. The natural map $\tilde{j}_*: H^1(\tilde{G}) \to H^1(\text{Spin}(q))$ is surjective.

Proof. Let $u \in H^1(k, \text{Spin}(q))$ and let $q_u \in H^1(k, \text{SO}(q))$ be the associated quadratic form. Then

$$q_u \perp (1) \in I^3$$

By the results on 14-dimensional forms in $I^3$ one has

$$q_u \perp (1) = (T_{K/k})_*((s)\varphi')$$

with $K = k[s]/(s^2 - b)$ for some $b \in k^\times$ and with $\varphi$ a 3-fold Pfister form over $K$ (and with $\varphi = (1) \perp \varphi'$). Since the left hand side reduces to 1, there exists a value $-\lambda$ of $\varphi'$ with $T_{K/k}(-s\lambda) = 1$. As for any (invertible) value $-\lambda$ of $\varphi'$, one has $\varphi = \langle \lambda, \mu_1, \mu_2 \rangle$ for some $\mu_1, \mu_2 \in K^\times$. Note that

$$(T_{K/k})_*((-s\lambda)) = (1, -N_{K/k}(\lambda))$$

Thus

$$q_u = (T_{K/k})_*((s)\langle \lambda \rangle \langle \mu_1, \mu_2 \rangle') \perp (-N_{K/k}(\lambda))$$

By Lemma 10.1 it follows that $q_u \in J$. A diagram chase (see diagram (3)) involving the coboundary maps $H^1(k, G), H^1(k, \text{SO}(q)) \to H^2(k, \mu_2)$ shows that there exists $\tilde{u} \in H^1(k, \tilde{G})$ such that $\tilde{j}(\tilde{u}), u \in H^1(k, \text{Spin}(q))$ have the same image in $H^1(k, \text{SO}(q))$. Another diagram chase shows that we can arrange $\tilde{j}(\tilde{u}) = u$.

We next compute $\tilde{G} \subset \text{Spin}(q) \subset C(q)$ inside the Clifford algebra. Let $e_1, \ldots, e_{12}$ be the standard base of $V$.

Let $\zeta$ be a primitive 4-th root of unity.

For $v, w \in V$ with $q(v) = 1$, $q(w) = -1$ and $v \perp w$ let

$$\omega(v, w) = \frac{1 + \zeta vw}{\sqrt{2}}$$

Then $\omega(v, w)\omega(v, w) = 1$ and therefore $\omega(v, w) \in \text{Spin}(q)$. Moreover $\omega(v, w)^2 = \zeta vw$ and $\omega(v, w)^4 = -1$. Furthermore $\omega(v, w)v = v\omega(v, w)^{-1}$ and $\omega(v, w)w = w\omega(v, w)^{-1}$. Also $\omega(v, w)\omega(v, w)^{-1} = \zeta w$ and $\omega(v, w)w\omega(v, w)^{-1} = \zeta v$.

Consider the element

$$\omega = \omega(e_1, e_7)\omega(e_2, e_8)\omega(e_3, e_9)\omega(e_{10}, e_4)\omega(e_{11}, e_5)\omega(e_{12}, e_6) \in \text{Spin}(q)$$

Its image in $\text{SO}(q)$ is $\pi(\omega) = W(\zeta)$. Moreover

$$\omega^4 = 1$$
Next let \( \tilde{\alpha} = e_4e_5e_6e_{13}, \quad \tilde{\beta} = -\zeta e_{10}e_{11}e_{12}e_{13} \)

Both elements are in \( \text{Spin}(q) \) and \( \pi(\tilde{\alpha}) = V(-1, 1) \) and \( \pi(\tilde{\beta}) = V(1, -1) \). Moreover

\[
\begin{align*}
\tilde{\alpha}^2 &= 1 \\
\tilde{\beta}^2 &= 1 \\
\tilde{\alpha}\tilde{\beta} &= -\tilde{\beta}\tilde{\alpha} \\
(\tilde{a}\tilde{b})^2 &= -1 \\
\omega\tilde{a}\omega^{-1} &= \tilde{\beta} \\
\omega\tilde{b}\omega^{-1} &= -\tilde{a} \\
\omega\tilde{a}\tilde{b}\omega^{-1} &= \tilde{a}\tilde{b} \\
\omega\tilde{a}\omega^{-1} &= \tilde{a}\tilde{b}\omega \\
\tilde{a}\omega\tilde{b}^{-1} &= -\omega^2
\end{align*}
\]

Let \( H \) be the subgroup generated by \( \omega \) and \( \tilde{\alpha} \). Then \( \tilde{\beta} \in H \) and

\[
H = (\mu_4 \times \mu_4) \rtimes \mu_2
\]

with the \( \mu_2 \) generated by \( \tilde{\alpha} \) and \( \mu_4 \times \mu_4 \) generated by \( \omega \) and \( \tilde{\alpha}\omega\tilde{a}^{-1} \).

Note further that the diagonal embedding \( \text{SO}(3) \to \text{SO}(3, 3) \) lifts to \( \text{Spin}(3, 3) \). Thus the connected component of \( G \) lifts (uniquely) to \( \text{Spin}(q) \). This yields:

**Lemma 10.5.** One has

\[
\tilde{G} \simeq (\text{SO}(3))^2 \rtimes_{\varphi} H
\]

where \( H \) acts by permutation of the factors via \( \varphi: H \to \mathbb{Z}/2, \varphi(\tilde{\alpha}) = 0, \varphi(\omega) = 1 \).

(I was surprised about the simple structure of \( H \). There ought to be a better approach to the subgroup \( \tilde{G} \) of \( \text{Spin}(13) \) than just by a computation starting from \( G \).)

**Proposition 10.6.** \( \text{ed}(\tilde{G}) \leq 6 \)

**Proof.** Elements of \( H^1(k, H) \) are given by Galois \( H \)-algebras which can be written as

\[
L = k[z, x, y]/(z^2 - a, x^4 - u + sv, y^4 - u + sv)
\]

with \( a, u, v \in k, a \neq 0, u^2 - av^2 \neq 0 \). For the generic case we may assume \( v \neq 0 \) and replace \( s \) by \( sv \) and \( a \) by \( av^2 \). Then \( v = 1 \). Therefore \( H \)-torsors are parameterized by \( a \) and \( u \) and we have \( \text{ed}(H) \leq 2 \).

Thus an element of \( H^1(k, \tilde{G}) \) is given by a Galois \( H \)-algebra

\[
L = k[z, x, y]/(z^2 - a, x^4 - u - s, y^4 - u + s)
\]

and a quadratic form of rank 3 and determinant 1 over \( K = k[t] \subset L \) with \( t = (xy)^2 \)

and \( t^2 = u^2 - a \). Thus \( \text{ed}(\tilde{G}) \leq \text{ed}(H) + 2 \cdot 2 \).

**Corollary 10.7.** \( \text{ed}(\text{Spin}(q)) \leq 6 \)

**Proof.** This is clear from Proposition 10.4 and Proposition 10.6. 

\[\square\]
11. The essential dimension of split Spin(n) for $n \leq 14$

Let $\text{Spin}_n$ denote a split form of $\text{Spin}(n)$.

**Theorem.**

\[
\begin{align*}
\text{ed}(\text{Spin}_n) &= 0 \quad \text{for } n \leq 6, \\
\text{ed}(\text{Spin}_7) &= 4, \\
\text{ed}(\text{Spin}_8) &= 5, \\
\text{ed}(\text{Spin}_9) &= 5, \\
\text{ed}(\text{Spin}_{10}) &= 4, \\
\text{ed}(\text{Spin}_{11}) &= 5, \\
\text{ed}(\text{Spin}_{12}) &= 6, \\
\text{ed}(\text{Spin}_{13}) &= 6, \\
\text{ed}(\text{Spin}_{14}) &= 7.
\end{align*}
\]

**Proof.** (Sketch) The cases $n = 12, 14$ have been just considered. It is not difficult to extend our considerations to the case $n = 11$.

As for $n = 13$: By corollary 10.7 one has $\text{ed}(\text{Spin}_{13}) \leq 6$. The invariant $h_6$ restricted to $\text{Spin}_{13}$ is nontrivial, for example for $q \perp \langle 1 \rangle = b_1$ ($\langle \langle a_1, a_2, a_3 \rangle \rangle' - \langle \langle b_1, b_2, b_3 \rangle \rangle'$)

Hence $\text{ed}(\text{Spin}_{13}) \geq 6$.

For $n = 7, 10$ one uses that any $\text{Spin}_n$-torsor admits a reduction to $G_2 \times \mu_2$ resp. to $G_2 \times \mu_4$. For $n = 8, 9$ one may use the fact that

$\text{Spin}_8 \to \text{Spin}_9 \to F_4$

induce surjections on $H^1$ at the prime 2 and Serre’s $H^5(\mathbb{Z}/2)$-invariant for $F_4$, cf. [27, III. Annexe, § 3.4] or [28, III. Appendix 2, 3.4] and [14, § 40], [23]. For $n \leq 6$ note that any $n$-dimensional quadratic form with trivial $e_1$-, $e_2$-invariants is split. □

**References**


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