On the spinor norm and $A_0(X, K_1)$ for quadrics
by Markus Rost

This is a TeXed version (Sept. 1996) of the original preprint Sept. 1988.

I. Introduction

1) We denote by $K_n F$ the $n$-th Milnor $K$-group of field (for convenience).

For $X/F$ projective there is a complex

$$\bigoplus_{v \in X(1)} K_n K(v) \xrightarrow{d} \bigoplus_{v \in X(0)} K_n K(v) \xrightarrow{N} K_n F$$

where $d$ is given by the tame symbol and $N$ is given by the norm map in Milnor $K$-Theory. We denote coker $d$ by $A_0(X, K_n)$ and by $N_X : A_0(X, K_n) \to K_n F$ the induced norm map.

2) In this note all fields have characteristic different from 2.

Let $\varphi : V \to F$ be a quadratic module. We denote by $X_\varphi \subset \mathbb{P}V$ the associated projective quadric hypersurface and we put $N_\varphi = N_{X_\varphi}$. Note that $X_{c\varphi} = X_\varphi$.

Let $C(\varphi)$, $C_0(\varphi)$ be the (even) Clifford algebra of $\varphi$ and consider $V$ as a subspace of $C(\varphi)$ in the usual way.

The special Clifford group of $\varphi$ is defined as

$$\mathrm{ST}(\varphi) = \{ \alpha \in C_0(\varphi) \mid \alpha V \alpha^{-1} = V \text{ in } C(\varphi) \}.$$ 

One has a commutative diagram

$$\begin{array}{ccc}
\mathrm{ST} & \longrightarrow & F^* \\
\downarrow & & \downarrow\text{sn} \\
\text{ST}(\varphi) & \longrightarrow & \text{SO}(\varphi) \\
\downarrow & & \downarrow\text{sn} \\
F^* & \xrightarrow{2} & F^*/(F^*)^2 \\
\end{array}$$

where $F^* \subset \mathrm{ST}(\varphi)$ is central and $\mathrm{ST}(\varphi)$ acts on $(V, \varphi)$ by $\alpha(v) = \alpha v \alpha^{-1}$; the spinor norm $\text{sn}$ is given by $\text{sn}(\alpha) = \alpha t \alpha$.

If $\dim \varphi = 2$, the $\mathrm{ST}(\varphi) = C_0(\varphi)^*$, $C_0(\varphi)$ is the quadratic extension of $F$ defined by the discriminant of $\varphi$ and $\text{sn}$ is given by the norm for $C_0(\varphi)/F$.

If $\dim \varphi = 3$, then $\mathrm{ST}(\varphi) = C_0(\varphi)^*$, $C_0(\varphi)$ is a quaternion algebra and $\text{sn}$ is induced by the reduced norm for the algebra $C_0(\varphi) | F$.

If $\varphi_0$ is a subform of $\varphi$ (i.e. $\varphi_0 = \varphi | V_0$ for some subspace $V_0$ of $V$), then $\mathrm{ST}(\varphi_0) \subset \mathrm{ST}(\varphi)$.

3) In this note we construct a natural homomorphism

$$\tilde{\omega}_{\varphi} : \mathrm{ST}(\varphi) \longrightarrow A_0(X_\varphi, K_1)$$
such that \( N_\phi \circ \tilde{\omega}_\phi = \text{sn} \).

\( \tilde{\omega}_\phi \) is surjective (at least if \( F \) has no odd extensions). Therefore, if one investigates the injectivity of \( N_\phi \), one is let to consider the kernel of the spinor norm.

4) An element \( \alpha \in S\Gamma(\varphi) \) is called plane, if \( \alpha \in S\Gamma(\varphi_0) \) for a 2-dimensional subform \( \varphi_0 \) of \( \varphi \). It is known that \( S\Gamma(\varphi) \) is generated by plane elements (Dieudonné). We denote by \( S\Gamma(\varphi) \) the quotient of \( S\Gamma(\varphi) \) by its commutator subgroup and elements of the form \( \alpha \beta^{-1} \), where \( \alpha, \beta \in S\Gamma(\varphi) \) are plane such that \( \text{sn}(\alpha) = \text{sn}(\beta) \). Let \( \text{sn} : S\Gamma(\varphi) \to F^* \) be the homomorphism induced by \( \text{sn} \). An element of \( S\Gamma(\varphi) \) is called plane, if it equals \( \bar{\alpha} \) for some plane \( \alpha \in S\Gamma(\varphi) \). It turns out that \( \tilde{\omega}_\phi \) factors through a homomorphism \( \omega_\phi : S\Gamma(\varphi) \to A_0(X_\phi, K_1) \).

5) Since \( N \circ \omega_\phi = \text{sn} \) and \( \omega_\phi \) is surjective, we have

**Theorem 1.**

If every element of \( S\Gamma(\varphi) \) can be written as a product of two plane elements, then \( N_\phi \) is injective in degree 1.

All forms \( \varphi \), for which I can prove the injectivity of \( N_\phi \) satisfy the hypothesis of Theorem 1. We have

**Proposition 2.**

\( \varphi \) satisfies the hypothesis of Theorem 1 in the following cases

i) \( \dim \varphi \leq 5 \)

ii) \( \varphi = \rho \otimes (\psi \oplus (c)) \), where \( \rho \) is a Pfister form, \( \psi \) is a Pfister neighbor and \( c \in F^* \)

iii) \( \varphi = \psi \oplus c(1,1) \), where \( \psi \) is a Pfister neighbor and \( c \in F^* \).

Recall that a Pfister neighbor is a form of type \( \psi_0 \oplus b\psi_1 \) where \( \psi_0 \) is a Pfister form and \( \psi_1 \) is a subform of \( \psi_0 \). Note that every Pfister neighbor (hence every Pfister form) is included in case ii).

6) The perhaps simplest type of quadratic forms not covered by i), ii) or iii) are 6-dimensional forms \( \varphi \) such that \( C_0(\varphi) \) has (maximal) index 4 over its center. We have

**Proposition 3.** There exists a field \( F \) and 6-dimensional quadratic form \( \varphi \) over \( F \), such that \( N_\varphi \) is not injective. \( \varphi \) can be chosen to have discriminant 1.

This result is based on a relation between \( \text{Ker}N_\varphi \) and \( SK_1(C_0(\varphi)) \) which is obtained by Swan’s computation of \( K_1(X_\varphi) \).

To give an explicit example, let \( A = D(a, b) \otimes D(\bar{a}, \bar{b}) \) a tensor-product of two quaternion algebras such that \( |SK_1A| > 2 \). Let

\[ \varphi = \langle -a, -b, ab \rangle \oplus -\langle -\bar{a}, -\bar{b}, \bar{a}\bar{b} \rangle \]
be the associated form (\(X_\varphi\) is the Grassmanian for submodules of \(A\) of rank 8).

By Swan we have \(K_1(X) = (K_1F)^4 \oplus K_1A \oplus K_1A\). Consider
\[
  j : A_0(X_\varphi, K_1) = H^4(X; K_5) \longrightarrow K_1(X) \longrightarrow K_1A
\]
where the last map is given by projection to one of the factors. One may check that \(\text{Nrd} \circ j = 2N_{\varphi}\). Hence \(j(\ker N_{\varphi}) \subset SK_1A\). One can show that \(SK_1(A)/j(\ker N_{\varphi})\) is of order at most 2; it is generated by \(j(u)\) for any \(u\) such that \(N_{\varphi}(u) = -1\).

II. The special Clifford group

\textit{Remark added in July 1996:} In the following I seem to care only on anisotropic forms. The much simpler case of isotropic forms should considered in the very beginning.

Let \(\varphi : V \rightarrow F\) be a quadratic module. For \(\alpha \in \Gamma(\varphi)\) let
\[
  \text{supp} \alpha = \{ v \in V ; \alpha(v) = v \}^\perp.
\]
Note that \(\alpha \in \Gamma(\varphi | \text{supp}(\alpha))\). Clearly
\[
  \text{supp} \alpha = 0 \iff \alpha \in F^* \\
  \dim \text{supp} \alpha \leq 2 \iff \alpha \text{ is plane}.
\]
Since the product of any two reflections in \(O(\varphi)\) is a plane rotation, we have the following consequence of the theorem of Cartan-Dieudonné.

\textbf{Proposition 4.} 
Any element of \(\Gamma(\varphi)\) can be written as product of \(\left[ \frac{\dim \varphi}{2} \right] \) plane elements.

Consequently \(\dim(\text{supp} \alpha) \neq 1, 3\) for \(\alpha \in \Gamma(\varphi)\). Two plane elements \(\alpha, \beta\) are called to be linked if \(\dim(\text{supp} \alpha + \text{supp} \beta) \leq 3\). In this case \(\alpha\beta\) is again plane, because \(\text{supp} \alpha \beta \subset \text{supp} \alpha + \text{supp} \beta\) and \(\dim(\text{supp} \alpha \beta) = 3\) is impossible.

Note that \(\alpha\) and \(\beta\) commute, if \(\text{supp} \alpha \perp \text{supp} \beta\).

\textbf{Theorem 5.} 
Let \(G\) be the free group on the set of all plane elements of \(\Gamma(\varphi)\). Denote by \(g_\alpha\) the generator corresponding to \(\alpha\). Then
\[
  G \longrightarrow \Gamma(\varphi), \quad g_\alpha \longrightarrow \alpha
\]
is surjective and its kernel is the normal subgroup generated by elements of the form

\begin{align*}
R_1 \quad & g_\alpha g_\beta g_\alpha^{-1} \quad \text{if } \alpha \text{ and } \beta \text{ are linked.} \\
R_2 \quad & [g_\alpha, g_\beta] \quad \text{if } \text{supp}(\alpha) \perp \text{supp}(\beta).
\end{align*}
For \( v_1, v_2 \in V \) such that \( \varphi(v_1) = \varphi(v_2) \) we denote by \( \varepsilon(v_2, v_1) \in \Sigma(\varphi) \) the trivial element if \( v_1 = v_2 \), otherwise any plane element such that \( \varepsilon(v_2, v_1)(v_1) = v_2 \). Note that a plane \( \alpha \in \Sigma(\varphi) \) is linked with \( \varepsilon(\alpha(v), v) \) for any \( v \in V \).

Remark added in Jan 1998: The plane element \( \varepsilon(v_2, v_1) \) is assumed to have support in the subspace generated by \( v_1, v_2 \).

Let \( \hat{G} \) be the quotient of \( G \) by the relations \( R_1, R_2 \) and denote by \( \hat{g} \) the image of \( g_\alpha \) in \( \hat{G} \). Theorem 5 states that \( \hat{G} \to \Sigma(\varphi) \), \( \hat{g} \to \alpha \) is bijective. Surjectivity follows from Proposition 4. In the following we proof the injectivity (I don’t know, whether and in how far this question has been considered in the literature).

**Lemma 6.**

\[ \hat{g}_\alpha \hat{g}_\beta \hat{g}_\alpha^{-1} = \hat{g}_{\alpha \beta \alpha}^{-1} \quad \text{for plane} \quad \alpha, \beta \in \Sigma(\varphi). \]

**Proof.** Let \( W = (\text{supp} \alpha)^\perp \cap \text{supp} \beta \).

If \( \dim W = 2 \), then \( \text{supp} \beta \subset (\text{supp} \alpha)^\perp \). Therefore \( \alpha \beta \alpha^{-1} = \beta \) and the lemma follows from \( R_2 \).

If \( \dim W < 2 \), then there exists a nonzero \( v \in W^\perp \cap \text{supp} \beta \). Then \( \alpha, \beta \) are linked with \( \gamma = \varepsilon(\alpha(v), v) \), hence by \( R_2 \):

\[ \hat{g}_\alpha \hat{g}_\beta \hat{g}_\alpha^{-1} \hat{g}_{\alpha \beta \alpha}^{-1} = \hat{g}_{\alpha \beta \alpha}^{-1} \hat{g}_{\gamma} \hat{g}_{\beta} \hat{g}_{\alpha} \hat{g}_{\alpha \beta \alpha}^{-1} \]

with \( \alpha' = \alpha \gamma^{-1} \), \( \beta' = \gamma \beta \gamma^{-1} \).

Note that \( \gamma \) fixes \( W \), because \( v \in W^\perp \) and \( \alpha \) fixes \( W \). This shows \( W \subset W' = (\supp \alpha')^\perp \cap \text{supp} \beta' \). Moreover \( \alpha(v) = \gamma(v) \in W' \setminus W \), hence \( \dim W' > W \) and we are left with the case \( \dim W = 2 \) after eventually repeating this argument. \( \square \)

**End of the proof of Theorem 5:** Let \( \alpha_1, \ldots, \alpha_N \in \Sigma(\varphi) \) be plane such that \( \alpha_1 \ldots \alpha_N = 1 \). We show \( \hat{g}_{\alpha_1} \ldots \hat{g}_{\alpha_N} = 1 \) by induction on \( \dim \varphi \).

Let \( v \in V \) be any anisotropic vector, let \( v_i = \alpha_i \ldots \alpha_n(v), v_{N+1} = v = v_1 \), let \( \gamma_i = \varepsilon(v_i, v_{i+1}) \) and \( \beta_i = \alpha_i \gamma_i^{-1} \). \( \alpha_i \) and \( \gamma_i \) are linked, because \( \alpha_i(v_{i+1}) = v \), thus \( \hat{g}_{\alpha_i} = \hat{g}_{\beta_i} \hat{g}_{\gamma_i} \).

Put \( \delta_i = \gamma_1 \ldots \gamma_i \), \( (\delta_0 = 1) \); \( \delta_i \) is plane and is one choice of \( \varepsilon(v, v_{i+1}) \). Hence \( \gamma_i \) and \( \delta_{i-1} \) are linked and therefore \( \hat{g}_{\delta_i} = \hat{g}_{\gamma_i} \ldots \hat{g}_{\gamma_1} \).

Let \( \rho_i = \delta_{i-1} \beta_i \delta_{i-1}^{-1} \). Then \( \hat{g}_{\delta_{i-1}} \hat{g}_{\beta_i} \hat{g}_{\delta_{i-1}}^{-1} = \hat{g}_{\rho_i} \) by the lemma. Taking things together we find

\[ g_{\alpha_1} \ldots g_{\alpha_N} = g_{\beta_1} g_{\gamma_1} g_{\beta_2} \ldots g_{\beta_N} g_{\gamma_N} = g_{\rho_1} \ldots g_{\rho_N} g_{\rho_N}. \]

Now, we are done, because \( \rho_1 \ldots \rho_N \) and \( \delta_N \) fix \( v \). \( \square \)

We have the following consequence of Theorem 5.
Corollary 7. Let \( \bar{G} \) be the free abelian group generated by all plane elements of \( S\Gamma(\varphi) \). Denote by \( \bar{g}_\alpha \) the generator corresponding to \( \alpha \). Then

\[
\bar{G} \longrightarrow S\Gamma(\varphi), \quad \bar{g}_\alpha \longrightarrow \bar{\alpha}
\]

is surjective and its kernel is generated by elements of the form

\[
\bar{R}_0) \quad \bar{g}_\alpha \bar{g}_\beta^{-1} \quad \text{if } \text{sn}(\alpha) = \text{sn}(\beta)
\]

\[
\bar{R}_1) \quad \bar{g}_\alpha \bar{g}_\beta \bar{g}_\alpha^{-1} \quad \text{if } \alpha \text{ and } \beta \text{ are linked}.
\]

The corollary is the basis of our construction of an epimorphism \( \omega_\varphi : S\Gamma(\varphi) \rightarrow A_0(X, K_1) \) described in the next section.

The rest of this section is devoted to the proof of Proposition 2. Clearly Proposition 2 i) follows from Proposition 4.

Let

\[
D(\varphi) = \{ \varphi(v); \ v \in V \text{ anisotropic} \} \subset F^*
\]

and

\[
N(\varphi) = \{ \text{sn}(\alpha); \ \alpha \in S\Gamma(\varphi) \text{ plane} \} \subset F^*.
\]

\( N(\varphi) \) consists just of all norms from quadratic extensions \( K/F \) such that \( \varphi_K \) is isotropic. Define

\[
\Sigma_\varphi : N(\varphi) \longrightarrow \bar{S}\Gamma(\varphi), \quad \Sigma_\varphi(\text{sn}(\alpha)) = \bar{\alpha}
\]

\( \Sigma_\varphi \) is welldefined by the very definition of \( \bar{S}\Gamma(\varphi) \). \( \Sigma_\varphi \) is injective, since \( \text{sn} \) is a left inverse. \( \Sigma_\varphi(N(\varphi)) \) generates \( \bar{S}\Gamma(\varphi) \) by Proposition 4.

For a subform \( \varphi_0 \) of \( \varphi \), we denote by \( i_* : \bar{S}\Gamma(\varphi_0) \rightarrow \bar{S}\Gamma(\varphi) \) the homomorphism induced by the inclusion \( \bar{S}\Gamma(\varphi_0) \subset \bar{S}\Gamma(\varphi) \).

Lemma 8.
Let \( \varphi \) be a quadratic form. Then

i) \( D(\varphi) \cdot D(\varphi) = N(\varphi) \).

ii) If \( \varphi \) represents 1, then \( D(\varphi) \subset N(\varphi) \).

iii) Let \( v \in V \) and \( \alpha \in S\Gamma(\varphi) \) plane, such that \( \varphi(v) = 1 \) and \( v \in \text{supp} \alpha \). Then \( \text{sn}(\alpha) \in D(\varphi) \).

iv) Let \( v \in V \) and \( \alpha \in S\Gamma(\varphi) \). Then \( \alpha = \alpha_1 \ldots \alpha_n \) with \( \alpha_i \) plane and \( v \in \text{sup} \alpha_i \).

v) If \( \varphi \) represents 1, then

\[
\Sigma_\varphi(\varphi(v_1) \varphi(v_2)) = \Sigma_\varphi(\varphi(v_1)) \Sigma_\varphi(\varphi(v_2))
\]

for any anisotropic \( v_1, v_2 \in V \). (The left hand side is defined by i)).
vi) Suppose $\psi$ represents 1 and let $\varphi = \psi \oplus (b)$. Then

$$\overline{ST(\varphi)} = i_*(\overline{ST(\psi)}) \cdot \sum_{\varphi}(D(\varphi)).$$

**Proof.**

It is easy to check i) - iii) for dim $\varphi = 2$ and iv) - vi) for dim $\varphi = 3$. One may however reduce to these cases by restriction to appropriate subforms. This is obvious except for iv) and vi). For iv) one has to consider only plane elements $\alpha$ by Proposition 4; hence one may replace $\varphi$ by $\varphi|_{(\text{supp } \alpha + uf)}$, which is of dimension $\leq 3$.

For vi) the reduction can be done as follows. Write $(V, \varphi) = (W, \psi) \oplus (F, \langle b \rangle)$ and let $v_0 = (0, 1) \in W \times F = V$. For a given $\alpha \in \text{ST}(\varphi)$ put $\beta = \varepsilon(\alpha(v_0), v_0)$. Then $\beta^{-1} \cdot \alpha$ fixes $v_0$ and is therefore contained in $\text{ST}(\psi)$. Hence it remains to show $\bar{\beta} \in i_* (\text{ST}(\psi)) \cdot \sum_{\varphi}(D(\varphi))$ for which one may restrict to $\varphi|_{V'}$ with $V' = \text{supp } \beta + Fv_1$, where $v_1 \in V$ is such that $\varphi(v_1) = 1$. \hfill $\square$

**Lemma 9.**

Let $\psi = (1) \oplus \psi'$ be a Pfister form, let $\bar{\psi}'$ be a subform of $\psi'$ and let $\bar{\psi} = (1) \oplus \bar{\psi}'$. Moreover let $\zeta$ be an arbitrary form representing 1 and let $b \in F^*$. Put $\varphi = (\psi \otimes \zeta) \oplus (b)$ and $\hat{\varphi} = \varphi \oplus b\bar{\psi}' = (\psi \otimes \zeta) \oplus b\bar{\psi}$. Then

$$i_* : \overline{ST(\varphi)} \longrightarrow \overline{ST(\hat{\varphi})}$$

is surjective.

**Proof.**

Since $\hat{\varphi}$ represents 1 we know that $\sum_{\varphi}(D(\hat{\varphi}))$ generates $\overline{ST(\hat{\varphi})}$ by Lemma 8 i). Note that

$$D(\hat{\varphi}) \subset D(\varphi) \cdot D(\bar{\psi}) \subset D(\varphi) \cdot D(\varphi) = N(\varphi)$$

since $\psi$ is multiplicative and $\zeta$ represents 1. Hence

$$\sum_{\varphi}(D(\hat{\varphi})) \subset \sum_{\varphi}(N(\varphi)) \subset i_*(\overline{ST(\varphi)})$$

by Lemma 8 v). \hfill $\square$

**Lemma 10.**

Let $\varphi$ be a Pfister neighbor, i.e. $\varphi = \psi \oplus b\bar{\psi}$ where $\psi$ is a Pfister form, $\bar{\psi}$ is a subform of $\psi$ and $b \in F^*$. Then $\overline{\text{sn}} : \overline{ST(\varphi)} \rightarrow F^*$ is injective and has image $N(\varphi) = D(\psi \otimes \ll \rightangle{b} \rangle)$.  

6
Proof.
Let \( \rho = \psi \otimes \ll -b \gg \). Then \( D(\rho) \) is a group as for every Pfister form and therefore \( D(\rho) = N(\rho) \) by Lemma 8 i). Since \( \varphi_K \) is isotropic if and only if \( \rho_K \) is isotropic, we have \( N(\varphi) = D(\rho) \). Therefore \( N(\varphi) \) is a group, hence \( \text{Im } sn = N(\varphi) = D(\rho) \).

Injectivity of \( \text{sn} \): If \( \varphi \) is a Pfister form (i.e. \( \bar{\psi} = \psi \)), then \( \sum_{\varphi} \) is a homomorphism in view of Lemma 8 v) and is a left inverse to \( \text{sn} \).

In the general case one may apply Lemma 9 with \( \zeta = \langle 1 \rangle \) to reduce to case \( \bar{\psi} = \langle 1 \rangle \). In this case we find by Lemma 8 vi) and the above remarks for Pfister forms:

\[
\overline{\text{ST}(\varphi)} = \sum_{\varphi}(N(\psi)) \cdot \sum_{\varphi}(D(\varphi)).
\]

Hence every element in \( \overline{\text{ST}(\varphi)} \) can be written as product of two plane elements and we are done. \( \square \)

Proof of Proposition 2 ii).
By Lemma 9 we may replace \( \varphi = \rho \otimes (\psi \oplus \langle c \rangle) \) by \( \bar{\varphi} = (\rho \otimes \psi) \oplus \langle c \rangle \). Since \( \rho \otimes \psi \) is itself a Pfister neighbor, we may assume \( \varphi = \psi \oplus \langle c \rangle \). By Lemma 8 vi) and Lemma 10 we have

\[
\overline{\text{ST}(\varphi)} = \sum_{\varphi}(N(\psi)) \cdot \sum_{\varphi}(D(\varphi)).
\]

\( \square \)

Proof of Proposition 2 iii).
Write \( (V, \varphi) = (W, \psi) + (F \times F, \langle c, c \rangle) \) and let \( v_0 = (0, 0, 1) \), \( v_1 = (0, 1, 0) \) \( \in V = W \times F \times F \).

For given \( \alpha \in \text{ST}(\varphi) \) let \( \beta = \varepsilon(\alpha(v_0), v_0) \) and \( \gamma = \varepsilon(\beta^{-1}\alpha(v_1), v_1) \). Then \( \delta = \gamma^{-1}\beta^{-1}\alpha \) fixes \( v_0 \) and \( v_1 \), hence \( \delta \in \text{ST}(\psi) \) and \( \bar{\delta} \in \sum_{\varphi}(N(\psi)) \) by Lemma 10. By Lemma 8 iii) we have \( \text{sn}(\beta), \text{sn}(\gamma) \in D(c\varphi) \) and therefore \( \bar{\beta} \cdot \bar{\gamma} \in \sum_{\varphi}(D(c\varphi)) \cdot \sum_{\varphi}(D(c\varphi)) = \sum_{\varphi}(N(\varphi)). \)

Hence \( \bar{\alpha} = \bar{\beta} \cdot \bar{\gamma} \cdot \bar{\delta} \) is in \( \sum_{\varphi}(N(\psi)) \cdot \sum_{\varphi}(N(\varphi)). \) \( \square \)
III. The map $\overline{\text{ST}(\varphi)} \longrightarrow A_0(X_\varphi, K_1)$

We will use the following

**Theorem 11.**

i) $A_0(X_\varphi, K_0) \hookrightarrow K_0 F$ for arbitrary $\varphi$

ii) $A_0(X_\varphi, K_n) \hookrightarrow K_n F$ for isotropic $\varphi$

iii) $A_0(X_\varphi, K_1) \hookrightarrow K_1 F$ for $\dim \varphi = 3$.

i) is proved in [Merkuriev, Suslin; On the norm homomorphism in degree 3].

ii) follows from i) by the norm principle.

iii) is one of the main points in the proof of Hilbert Satz 90 for $K_2$ for quadratic extensions. It stands at the heart of our construction. It shows that for a 3-dimensional form $\varphi$ the groups $\overline{\text{ST}(\varphi)} = \text{ST}(\varphi)/[\text{ST}(\varphi), \text{ST}(\varphi)]$ and $A_0(X_\varphi, K_1)$ are naturally isomorphic, because $\text{sn}$ and $N_\varphi$ are injective and have the same image in $F^* = K_1 F$.

**Theorem 12.**

For quadratic forms $\varphi$ over $F$ there exists unique homomorphisms

$$\omega_\varphi : \overline{\text{ST}(\varphi)} \longrightarrow A_0(X_\varphi, K_1)$$

such that

i) If $\dim \varphi = 3$, then $\omega_\varphi = N_\varphi^{-1} \circ \text{sn}$.

ii) If $\varphi_0$ is a subform of $\varphi$, then

$$\begin{array}{ccc}
\overline{\text{ST}(\varphi_0)} & \xrightarrow{\omega_{\varphi_0}} & A_0(X_{\varphi_0}, K_1) \\
\downarrow i_* & & \downarrow i_* \\
\overline{\text{ST}(\varphi)} & \xrightarrow{\omega_\varphi} & A_0(X_\varphi, K_1)
\end{array}$$

is commutative.

**Proof.**

Let $\bar{G}$ be as in Corollary 7 and define

$$\hat{\omega}_\varphi : \bar{G} \longrightarrow A_0(X_\varphi, K_1)$$

as follows. For $\alpha \in \text{ST}(\varphi)$ plane choose a subform $\varphi_0$ of $\varphi$ of dimension 2 such that $\alpha \in \text{ST}(\varphi_0)$. ($\varphi_0$ is unique if $\alpha \not\in F^*$). Then $\alpha \in \text{ST}(\varphi_0) = K_1 F(X_{\varphi_0}) = A_0(X_{\varphi_0}, K_1)$ and we define $\hat{\omega}_\varphi(\bar{g}_\alpha)$ to be the image of $\alpha$ under $A(X_{\varphi_0}, K_1) \rightarrow A(X_\varphi, K_1)$. This definition does not depend on the choice of $\varphi_0$ because of Theorem 11 i). In order to prove Theorem 12, it suffices to show that $\hat{\omega}_\varphi$ vanishes on the relations $\bar{R}_0, \bar{R}_1$ in Corollary 7. This is clear for $\bar{R}_1$ because of Theorem 11 iii) and follows for $\bar{R}_0$ from
Proposition 12.

Let $v_1, v_2 \in X(0)$ be two points of degree 2, let $F_i = K(v_i)$ and let $\alpha_i \in F_i^*$ such that $N_{F_i|F}(\alpha_1) = N_{F_2|F}(\alpha_2)$. Then $[\{\alpha_1\}, v_1] - [\{\alpha_2\}, v_2]$ is in the image of
\[
\bigoplus_{v \in X(1)} K_2 K(v) \overset{d}{\to} \bigoplus_{v \in X(0)} K_1 K(v).
\]

Proof.

Let $F_0 = F_1 \otimes_F F_2$. We have
\[\alpha_i = \{\beta\} + \{N_{F_0|F_i}(\gamma)\}\]
for some $\beta \in F_i^*$ and $\gamma \in F_0^*$ (take $\beta = (\text{tr} \alpha_1 + \text{tr} \alpha_2)^{-1}$, $\gamma = \alpha_1 + \alpha_2$ in the generic case).

Hence
\[[\{\alpha_1\}, v_1] - [\{\alpha_2\}, v_2] = \{\beta\} \cdot (v_1 - v_2) + \text{cor}_{F_1|F}(u_1 - u_2)\]
where $u_1, u_2 \in \bigoplus_{v \in (X_{F_1})^0} K_1 K(v)$ are given by
\[u_1 = [N_{F_0|F_1}\{\gamma\}, \tilde{v}_1],\]
with $\tilde{v}_1$ a rational point of $X_{F_1}$ and
\[u_2 = [\gamma, \tilde{V}_2]\]
with $\tilde{v}_2$ the point over $v_2$.

Now $\{\beta\} \cdot (v_1 - v_2) \in \text{Im} d$, because $A_0(X_\varphi, K_0) \hookrightarrow K_1 F$.

Furthermore, over $F_1$ we have $N_\varphi(u_1 - u_2) = 0$, hence $u_1 - u_2 \in \text{Im} d$, because $X_{F_1}$ has a rational point. \qed

Proposition 13.

If $F$ has no extension of odd degree, then $\omega_\varphi : ST(\varphi) \to A_0(X_\varphi, K_2)$ is surjective.

It follows from Knebusch’s norm principle that $\text{Im } (N_\varphi \circ \omega_\varphi) = \text{Im } N_\varphi$. One may use the proof of Knebusch’s norm principle to show that $\omega_\varphi$ is surjective in general. Since this is a bit tedious I omit a proof here.

Proof of Theorem 1.

Since $A_0(X_\varphi, K_1) \hookrightarrow K_1 F$ for isotropic $\varphi$, we have $2\text{Ker } N_\varphi = 0$ by a transfer argument. Hence we may assume that $F$ has no odd extension, again using transfers. But then by Proposition 13 and the very definition of $ST(\varphi)$:
\[\text{Ker } N_\varphi = \omega_\varphi(\text{Ker } \overline{\mathfrak{n}}) = 0.\] \qed

For a quadratic form put $D_1(\varphi) = \text{Im } \text{sn} = \text{Im } N_\varphi$. For the proof of Proposition 13 we need the following lemma which can be deduced also from the arguments in [Merkuriev, Suslin; On the norm homomorphism in degree 3].
Lemma 14.
Let \( \text{dim} \varphi = 4 \) and let \( H/F \) be a quadratic extension. Then for every \( u \in D_1(\varphi_H) \) there exists (over \( F \)) two 3-dimensional subforms \( \varphi', \varphi'' \) of \( \varphi \) such that \( u \in D_1(\varphi'_H) \cdot D_1(\varphi''_H) \).

Proof. (sketch)
Write \( \varphi = \langle -a, -b, ab, c \rangle \). It is easy to check that \( D_1(\varphi) = \text{Nrd} (D(a,b) \otimes F(\sqrt{c})) \cap F^* \subset F(\sqrt{c})^* \).

Using this for \( \varphi_H \) one finds \( u = \text{Nrd} (d) \cdot (1 + c \text{Nrd} (d')) \) for some \( d, d' \in D(a,b) \otimes F_H \) with \( d' + \overline{d'} = 0 \). Now put \( \varphi' = \langle -a, -b, ab \rangle \), then \( \text{Nrd} (d) \in D_1(\varphi'_H) \).

It is not hard to find \( \overline{a}, \overline{b} \in F^* \) such that \( D(\overline{a}, \overline{b}) \simeq D(\overline{ac}, \overline{bc}, \overline{a}, \overline{b}, \overline{ac}, \overline{bc}, \overline{a}, \overline{b}) \) and \( 1 + c \text{Nrd} (d') \in \text{Nrd} (D(\overline{ac}, \overline{bc}, \overline{a}, \overline{b})) \).

Since \( \varphi' = (\overline{a}, \overline{b}, \overline{ac}, \overline{bc}, \overline{a}, \overline{b}) \) is similar to \( \varphi \) by quadratic form theory. Hence \( \varphi'' \) is similar to a subform \( \varphi'' \) of \( \varphi \).

Consequence.
Let \( H/F \) be a quadratic extension. Then \( \text{cor}_{H/F}(\text{Im} \omega_{\varphi_H}) \subset \text{Im} \omega_{\varphi} \).

Proof. We may assume \( \text{dim} \varphi = 4 \).

For \( \alpha \in \text{ST} (\varphi_H) \) plane there exists by Lemma 14 subforms \( \varphi', \varphi'' \) of \( \varphi \) of dimension 3 and \( \alpha' \in \text{ST}(\varphi'_H), \alpha'' \in \text{ST}(\varphi''_H) \) such that \( \text{sn}(\alpha) = \text{sn}(\alpha') \text{sn}(\alpha'') \). We know that \( \overline{\text{sn}} \) is injective, hence \( \text{cor}_{H/F}(\omega_{\varphi_H}(\alpha)) = \text{cor}_{H/F}(\omega_{\varphi'_H}(\overline{\alpha'})) + \omega_{\varphi''_H}(\overline{\alpha''}) \).

Now \( \text{cor}_{H/F}(\omega_{\varphi_H}(\overline{\alpha})) \) is in the image of \( A_0(X_\varphi, K_1) \rightarrow A_0(X_\varphi, K_1) \). But we know that \( \omega_{\varphi'} \) is surjective, hence \( \text{cor}_{H/F}(\omega_{\varphi_H}(\overline{\alpha})) \in \text{Im} \omega_{\varphi} \). Similarly \( \text{cor}_{H/F}(\omega_{\varphi_H}(\overline{\alpha'})) \in \text{Im} \omega_{\varphi} \).

Proof of Proposition 13.
By the consequence we have the norm principle for \( \text{Im} \omega_{\varphi} \) if \( F \) has no odd extensions. Since \( A_0(X_\varphi, K_1) \) is generated by corestrictions from splitting fields \( K \) of \( \varphi \) and since \( A_0(X_{\varphi_K}, K_1) = \text{Im} \omega_{\varphi_K} \) we conclude \( A_0(X_\varphi, K_n) = \text{Im} \omega_{\varphi} \).