UNRAMIFIED COHOMOLOGY OF QUADRICS, I

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Abstract. Given a quadric X over a field F of characteristic $\neq 2$, we compute the kernel and cokernel of the natural map in degree 4 from the mod 2 Galois cohomology of F to the unramified mod 2 cohomology of $F(X)$, when $\dim X > 10$ and in several smaller-dimensional cases. Applications of these results to real quadrics and to the unramified Witt ring are given.

Contents

Introduction

Let F be a field, let X be a smooth, proper, and irreducible variety over F and let n be prime to $\operatorname{char} F$. Consider the restriction map

$$H^i(F, \mu_n^{\otimes(i-1)}) \to H^i(F(X), \mu_n^{\otimes(i-1)})$$

in Galois cohomology.

The kernel of this map has been studied in various cases, for unirational and notably for homogeneous varieties. In the case of the Brauer group ($i = 2$) this goes

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back to Witt [56] for conics and Amitsur [2] for Severi-Brauer varieties. For $i = 3$ the kernel has been determined by Arason [1] for quadrics. Other computations for $i = 3$ and homogeneous varieties (Suslin [49], Peyre [39, 40], Merkurjev [29], Esnault et al. [14]) rely on the Merkurjev-Suslin theorem (see below). The Merkurjev-Suslin/Rost theorem makes it possible to study also the case $i = 4$ and $n = 2$. Here certain quadrics have been considered by Jacob-Rost [16], Szyjewski [51], Rost and Merkurjev [27].

Unramified cohomology. The image of this map sits inside the *unramified cohomology group*

$$H^i_{nr}(F(X)/F, \mu_n^{\otimes(i-1)}) := \operatorname{Ker}\big((H^i F(X), \mu_n^{\otimes(i-1)}) \xrightarrow{(\partial_x)} \bigoplus_{x\in X^{(1)}} H^{i-1}(F_x, \mu_n^{\otimes(i-2)})\big).$$

Here x runs through the codimension 1 points of X, and

$$\partial_x\colon H^i(F(X), \mu_n^{\otimes(i-1)}) \to H^{i-1}(F_x, \mu_n^{\otimes(i-2)})$$

is the residue map associated to the discrete valuation ring $\mathcal{O}_{X,x}$.

We study the cokernel of the corresponding map

$$\eta^i_n\colon H^i(F, \mu_n^{\otimes(i-1)}) \to H^i_{nr}(F(X)/F, \mu_n^{\otimes(i-1)}).$$

The group $H^i_{nr}(F(X)/F, \mu_n^{\otimes(i-1)})$ is a birational invariant and does not change when X is replaced by $X \times \mathbf{P}^1$. Moreover for the base field extension from F to the rational function field $K = F(t)$ one has an exact sequence

$$(*)\quad 0 \to H^i_{nr}(F(X)/F, \mu_n^{\otimes(i-1)}) \to H^i_{nr}(K(X)/K, \mu_n^{\otimes(i-1)}) \to \\ \to \bigoplus_{y\in\mathbf{A}^1_{(0)}} H^{i-1}_{nr}(F_y(X)/F_y, \mu_n^{\otimes(i-2)}) \to 0.$$

(*cf.* [7, §3.3, th. 4.1.1, 4.1.5], [42, (7.3),(12.10)].)

Saltman [43, 44], Bogomolov [3], Colliot-Thélène-Ojanguren [8], and Peyre [38], inter alia, have studied the unramified cohomology of certain unirational varieties over an algebraically closed F (there are many more references available on this case, for which the reader can consult for instance [7, § 4]). The unramified cohomology of real varieties has been studied in high cohomological degree by Colliot-Thélène-Parimala [9] and Scheiderer [46]. For quadrics X and $i \leq 3$, the cokernel of η^i_2 was studied by Colliot-Thélène-Sujatha [10] ($F = \mathbf{R}$), Peyre [39] (F arbitrary, X a conic), Sujatha [48] (F arbitrary, X defined by a neighbour of a 3-fold Pfister form) and Kahn [17] (F arbitrary, $\dim X > 2$). It follows from [56, 1, 17] that η^2_2 (*resp.* η^3_2) is bijective for $\dim X > 2$ (*resp.* $\dim X > 6$) and has kernel and cokernel of order at most 2 for all X (*resp.* for $\dim X > 2$).

In this paper, we show that η^4_2 is bijective if X is a quadric of dimension > 14, and that $\left|\operatorname{Ker}\eta^4_2\right| \leq 2$, $\left|\operatorname{Coker}\eta^4_2\right| \leq 4$ for $\dim X > 4$. To our knowledge, this is the first non-trivial instance where unramified cohomology of degree 4 is determined over an arbitrary field.

The Milnor conjecture. Before a more detailed discussion we recall the Bloch-Kato conjecture, the Milnor conjecture, and the related conjecture about existence of cohomological invariants for the graded Witt ring.

The Bloch-Kato conjecture predicts that if n is invertible in F, then the norm residue homomorphism h_i of degree i,

$$h_i\colon K_i^{\mathrm{M}}F/n \to H^i(F,\mu_n^{\otimes i})$$

is an isomorphism for all $i \geq 0$. Here $K_i^{\mathrm{M}}F$ denotes the i-th Milnor K-group of F [34]. If n is a power of 2, this is called the Milnor conjecture. The bijectivity of h_i holds trivially for $i = 0$ and by Kummer theory for $i = 1$. The Merkurjev-Suslin theorem states that h_2 is an isomorphism [31] and Merkurjev-Suslin/Rost [32] proved the Milnor conjecture for $i = 3$. The second author announced a proof of the Milnor conjecture for $i = 4$. Voevodsky [54] recently released a proof of the Milnor conjecture for all i. In the present paper we use only the Milnor conjecture for $i \leq 3$, except in appendix C for an application to the unramified Witt group which relies on the Milnor conjecture for $i = 4$.

Let $W(F)$ denote the Witt ring of F and let $(I^iF)_{i\geq 0}$ be the filtration on $W(F)$ given by the higher powers of the fundamental ideal $I(F)$ ([26, 45]). For a_1, a_2, $\dots$, $a_i \in F^*$ let $\langle\langle a_1,\dots,a_i\rangle\rangle$ denote the *i-fold Pfister form* $\langle 1,-a_1\rangle \otimes\cdots\otimes \langle 1,-a_i\rangle$ (*ibid.*). A quadratic form q is a *neighbour* of an i-fold Pfister form φ if $\dim q > 2^{i-1}$ and q is similar to a subform of φ. For simplicity, write

$$H^iF = H^i(F,\mathbf{Z}/2).$$

We define a *symbol* to be an element of H^iF of the form $(a_1,\dots,a_i)$, where $a_j \in F^*$, (a_j) is the class of a_j in H^1F by Kummer theory and $(a_1,\dots,a_i) := (a_1)\cdot\ldots\cdot(a_i)$ is the cup-product. Let $P_i(F)$ denote the set of classes of i-fold Pfister forms in $W(F)$. There are maps

$$e^i\colon P_i(F) \to H^iF$$

which send the class of an i-fold Pfister form $\langle\langle a_1,\dots,a_i\rangle\rangle$ to the corresponding cohomological symbol $(a_1,\dots,a_i)$. It is conjectured that these maps extend to well-defined isomorphisms

$$\overline{e}^i\colon I^iF/I^{i+1}F \to H^iF$$

for all $i \geq 0$. For $i \leq 2$, it is easy to see that these maps are well-defined, being the rank mod 2, signed discriminant, and the Clifford invariant, respectively, for $i = 0$, 1, 2. Arason [1] proved that $\overline{e}^3$ is well-defined and Jacob-Rost/Szyjewski [16, 51] proved that $\overline{e}^4$ is well-defined. Granting the existence of $\overline{e}^i$, the Milnor conjecture in degree i implies the bijectivity of $\overline{e}^i$. Voevodsky has recently announced a proof of the existence and bijectivity of the homomorphisms $\overline{e}^i$ for all i, jointly with Orlov and Vishik [35].

We now describe our results. Let q be a (regular) quadratic form over a field F with $\operatorname{char} F \neq 2$ and let X be the associated projective quadric. Unless mentioned otherwise, we assume $\dim q \geq 3$.

Note that if q is isotropic, then the extension $F(X)/F$ is purely transcendental and therefore η_2^i is bijective for all i (*cf.* proposition 2.5).

Kernel of η_2^i. Here the major result is

Theorem 1 (*cf.* theorem 6.3)**.** *Let $i \leq 4$. If $\dim q = 2^{i-2}+1$, then every element in $\operatorname{Ker}\eta_2^i$ is a symbol.*

For $i = 3$ and, in case of a Pfister neighbour, for $i = 4$, this theorem follows easily from the computations of $\operatorname{Ker}\eta_2^i$ by Arason [1], Jacob-Rost [16], and Szyjewski [51, 6.1].

The proof of theorem 1 for $i = 4$ in general is an extension of the discussions in [16, 27] to arbitrary 5-dimensional quadratic forms. The method used there provides a very general description of $\operatorname{Ker}\eta_2^4$ for quadratic forms of dimension ≥ 5 (see [27]) as follows:

$$(**) \qquad \operatorname{Ker}\eta_2^4 = \operatorname{Ker}(H^2(X,\mathcal{K}_3) \to H^2(\overline{X},\mathcal{K}_3)).$$

For an Albert form with corresponding biquaternion algebra A (see below) this yields an isomorphism $\mathrm{SK}_1(\mathrm{A}) = \operatorname{Ker}\eta_2^4$ (*ibid.*).

For a five dimensional form q, let $\widehat{q} = q \perp \langle -d\rangle$, $d = \det(q)$ be the attached Albert form. In this case $(**)$ yields a surjective homomorphism

$$\mathrm{O}(\widehat{q})/[\mathrm{O}(\widehat{q}),\mathrm{O}(\widehat{q})]\mathrm{O}(q) \twoheadrightarrow \operatorname{Ker}\eta_2^4.$$

Theorem 1 follows from a careful computation of this map.

Using this theorem and some quadratic form theory, we show in section 3:

Corollary 2. *Let $i \leq 4$. Then*

(1) *A symbol $(a_1,\dots,a_i) \neq 0$ lies in $\operatorname{Ker}\eta_2^i$ if and only if q is similar to a subform of $\langle\langle a_1,\dots,a_i\rangle\rangle$.*
(2) *$\operatorname{Ker}\eta_2^i$ consists of symbols for $\dim q > 2^{i-2}$.*
(3) *$\left|\operatorname{Ker}\eta_2^i\right| \leq 2$ for $\dim q > 2^{i-1}$.*
(4) *$\operatorname{Ker}\eta_2^i = 0$ for $\dim q > 2^i$.*

Note that the bound $\dim q > 2^{i-2}$ in (2) is optimal: if q is an $(i-2)$-fold Pfister form $\langle\langle a_1,\dots,a_{i-2}\rangle\rangle$, then $\operatorname{Ker}\eta_2^i$ contains $H^2F \cdot (a_1,\dots,a_{i-2})$, which obviously contains non-symbolic elements in general.

One may expect that theorem 1 also holds for higher degree i. Then corollary 2 would hold accordingly (using the existence and bijectivity of the $\overline{e}^i$).

Cokernel of η_2^i. Results on $\operatorname{Coker}\eta_2^i$ are best understood in terms of $\operatorname{Coker}\eta^i$, where η^i is the restriction map

$$\eta^i\colon H^i(F,\mathbf{Q}/\mathbf{Z}(i-1)) \to H^i_{nr}(K/F,\mathbf{Q}/\mathbf{Z}(i-1)).$$

Here $\mathbf{Q}/\mathbf{Z}(i-1) := \varinjlim \mu_m^{\otimes(i-1)}$, where m runs through all integers prime to $\operatorname{char} F$. A consequence of Milnor's conjecture in degree $i-1$ is that $H^iF \to H^i(F,\mathbf{Q}/\mathbf{Z}(i-1))$ is injective; hence $\operatorname{Ker}\eta_2^i = \operatorname{Ker}\eta^i$ since the latter group is killed by 2 (*cf.* lemma 7.2 a)). To relate $\operatorname{Coker}\eta_2^i$ and $\operatorname{Coker}\eta^i$, define

$$(\operatorname{Ker}\eta_2^i)_0 = \{\, \alpha \in \operatorname{Ker}\eta_2^i \mid (-1)\cdot\alpha = 0 \in H^{i+1}F \,\}.$$

We prove in section 7:

Proposition 3 (*cf.* proposition 7.4)**.** *For any quadric X and $i \leq 4$, there is an exact sequence*

$$0 \to (\operatorname{Ker}\eta_2^i)_0 \to \operatorname{Coker}\eta_2^i \to \operatorname{Coker}\eta^i.$$

This way, information on $\operatorname{Ker}\eta_2^i$ and $\operatorname{Coker}\eta^i$ immediately yields information on $\operatorname{Coker}\eta_2^i$.

To describe our picture about $\operatorname{Coker}\eta^i$ we formulate:

Conjecture.

(1) $\operatorname{Coker}\eta^i = 0$ *if* q *is Pfister neighbour.*
(2) $\operatorname{Coker}\eta^i = 0$ *for* $\dim q \leq 5$.
(3) $\operatorname{Coker}\eta^i$ *is finite for* $\dim q > 6 \cdot 2^{i-4}$.
(4) $\operatorname{Coker}\eta^i = 0$ *for* $\dim q > 6 \cdot 2^{i-3}$.

In appendix A we show $\operatorname{Coker}\eta^i = 0$ for $\dim q = 3$ and—after an appropriate modification of the target group—also for $\dim q = 2$. Here the arguments rely on the Milnor conjecture in degree $i-2$ (*resp.* $i-1$).

For $i \leq 2$ one has (*cf.* section 9):

Theorem 4. $\operatorname{Coker}\eta^i = 0$ *for* $i \leq 2$.

For $i = 3$ there appears a peak in the story at $\dim q = 6$. For the reader's convenience, we recall the various types of 6-dimensional quadratic forms. Let $d_\pm q$ be the signed discriminant and let $C(q)$ be the Clifford algebra of q [26]. The *pure subform* $\langle\langle a_1, \dots, a_i\rangle\rangle'$ of the Pfister form $\langle\langle a_1, \dots, a_i\rangle\rangle$ is defined by

$$\langle\langle a_1, \dots, a_i\rangle\rangle = \langle 1\rangle \perp \langle\langle a_1, \dots, a_i\rangle\rangle'.$$

There are four types of anisotropic quadratic forms q of dimension 6:

- $d = d_\pm q \neq 1$, $q_{F(\sqrt{d})}$ is anisotropic. For convenience, we shall call such forms *virtual Albert forms.*
- $d_\pm q = 1$, *i.e.*, $q \in I^2F$. Such forms are called (anisotropic) *Albert forms.* An Albert form is similar to a form of type
$$\langle\langle a, b\rangle\rangle' \perp -\langle\langle e, f\rangle\rangle' \qquad a,\, b,\, e,\, f \in F^*.$$
To an Albert form there is attached the biquaternion algebra $A = (a, b) \otimes (e, f)$ which is the central simple algebra with $C(q) \simeq M_2(A)$. The form q is isotropic if and only if A is not a skewfield.
- $d = d_\pm q \neq 1$, $q_{F(\sqrt{d})}$ is hyperbolic. These forms are the 6-dimensional Pfister neighbours (*cf.* [24, p. 10]).
- $d = d_\pm q \neq 1$, $q_{F(\sqrt{d})}$ is isotropic without being hyperbolic. Such forms are of type $a\tau \perp b\langle 1, -d\rangle$, where $a,\, b \in F^*$ and τ is a 2-fold Pfister form.

Now for $i = 3$ our result is (*cf.* section 10):

Theorem 5. $\operatorname{Coker}\eta^3 = 0$, *except if* q *is an anisotropic Albert form, for which* $\operatorname{Coker}\eta^3 \simeq \mathbf{Z}/2$.

This result completes those of [17].

The main results about $\operatorname{Coker}\eta^4$ are:

Theorem 6 (*cf.* section 12)**.**

(1) *For* $\dim q > 6$ *there is an exact sequence*

$$0 \to \operatorname{Coker}\eta^4 \to (CH^3X)_{\text{torsion}} \to \big(CH^1X \otimes H^3(F, \mathbf{Q}/\mathbf{Z}(2))\big) \oplus H^5(F, \mathbf{Q}/\mathbf{Z}(3)).$$

Here CH^pX *denotes the* p*-th Chow group of* X*.*

(2) $\left|\operatorname{Coker}\eta^4\right| \leq 2$ *if* $\dim q \neq 4,\, 5,\, 6$.
(3) $\operatorname{Coker}\eta^4 = 0$ *if*
  - q *is a Pfister neighbour*
  - $\dim q = 8$ *and* $d_\pm q = 1$
  - $\dim q > 12$.

In the proof of theorem 6 we use Karpenko's results on CH^3 of a quadric. The main computational tool is, as in [17], to mix the use of the Hochschild-Serre and Bloch-Ogus [4] spectral sequences, here with divisible coefficients. We use in particular the fact that these two spectral sequences are modules over the Galois cohomology of F. Divisible coefficients have been used also in [23, 6, 39, 40]. It is clear that these methods can also be applied to other homogeneous varieties.

Theorem 7. *If q is an anisotropic Albert form or a virtual Albert form over F, then $\operatorname{Coker}\eta^4$ is infinite for $q_{F(t)}$.*

This theorem is an easy consequence of theorem 5 and the exact sequence $(*)$.

From corollary 2, proposition 3, and theorem 6 one easily deduces:

Corollary 8.

(1) $\operatorname{Coker}\eta_2^2 = 0$, *unless q is a neighbour of a 2-fold Pfister form $\langle\langle a,b\rangle\rangle$. In this case, it is 0 or $\mathbf{Z}/2$ according as $(-1,a,b)\in H^3F$ is nonzero or not.*

(2) *If q is an anisotropic Albert form,* $(\operatorname{Ker}\eta_2^3)_0 = 0$ *and* $\operatorname{Coker}\eta_2^3 \simeq \mathbf{Z}/2$. *In all other cases,* $\operatorname{Coker}\eta_2^3 \simeq (\operatorname{Ker}\eta_2^3)_0$. *In particular:*
   a) *If q is a neighbour of a 3-fold Pfister form $\langle\langle a,b,c\rangle\rangle$, then* $\operatorname{Coker}\eta_2^3 = 0$ *or $\mathbf{Z}/2$ according as $(-1,a,b,c)\in H^4F$ is nonzero or not.*
   b) *If q is not an Albert form and does not embed into a 3-fold Pfister form (up to similarity), then* $\operatorname{Coker}\eta_2^3 = 0$.

(3) a) *If $\dim q > 12$ and q is not a neighbour of a 4-fold Pfister form, then* $\operatorname{Coker}\eta_2^4 = 0$.
   b) *If q is a neighbour of a 4-fold Pfister form $\langle\langle a,b,c,d\rangle\rangle$, then* $\operatorname{Coker}\eta_2^4 = 0$ *or $\mathbf{Z}/2$ according as $(-1,a,b,c,d)\in H^5F$ is nonzero or not.*
   c) *For $\dim q > 8$,* $\left|\operatorname{Coker}\eta_2^4\right| \le 4$.

In the case $F = \mathbf{R}$, these results immediately imply that $\operatorname{Coker}\eta_2^i = 0$ for all quadrics when $i \le 4$. In appendix B, we also get partial results on the $\mathcal{H}$-cohomology of such quadrics.

The unramified Witt ring. As in [17, corollary], we wish to apply the above results to the *unramified Witt ring.* Denote by $W_{nr}(K/F)$ the unramified Witt ring of a finitely generated extension K of F (*cf.* [8]) and, for $i \ge 0$, by $I^i_{nr}(K/F)$ the intersection $I^iK \cap W_{nr}(K/F)$. Let

$$\chi^i\colon I^iF/I^{i+1}F \to I^i_{nr}(K/F)/I^{i+1}_{nr}(K/F)$$

be the natural map analogous to η_2^i and η^i. The homomorphism $\bar{e}^i$ induces a homomorphism

$$I^i_{nr}K/I^{i+1}_{nr}K \to H^i_{nr}K$$

and hence a homomorphism

$$\operatorname{Coker}\chi^i \to \operatorname{Coker}\eta_2^i.$$

Using the Milnor conjecture in degree $i = 4$, we find:

Theorem 9. *Let $i \le 4$. For any quadric X, the composition*

$$\operatorname{Coker}\chi^i \to \operatorname{Coker}\eta_2^i \to \operatorname{Coker}\eta^i$$

is injective.

From corollary 2, theorem 6, and theorem 9 we deduce:

Corollary 10.

(1) *The map* $W(F)/I^3F \to W_{nr}(F(X)/F)/I^3_{nr}(F(X)/F)$ *is surjective. It is bijective for* $\dim q > 4$.
(2) *The map* $W(F)/I^4F \to W_{nr}(F(X)/F)/I^4_{nr}(F(X)/F)$ *is surjective, except if* q *is an anisotropic Albert form. In this case, its cokernel is* $\simeq \mathbf{Z}/2$ *(by* [17, remark 3, p. 249]*). It is bijective for* $\dim q > 8$.
(3) *The map* $W(F)/I^5F \to W_{nr}(F(X)/F)/I^5_{nr}(F(X)/F)$ *is surjective in the following cases*
   - q *is a Pfister neighbour.*
   - $\dim q = 8$ *and* $d_\pm q = 1$.
   - $\dim q > 12$.

   It is bijective for $\dim q > 16$. □

Remark. Voevodsky's work yields also the computation of $\operatorname{Ker}\eta_2^i$ for i-fold Pfister forms. A consequence of [54] and [35] is the validity of proposition 3 and of theorem 9 for all i.

However these announcements do not imply a generalization of theorem 1 to higher degree or a proof of the conjecture in an obvious way.

In a sequel to this paper, we plan to complete the present results by determining $\operatorname{Ker}\eta^4$ and $\operatorname{Coker}\eta^4$ in the cases pending here, and by refining results on real quadrics. In particular, we shall show that $\operatorname{Ker}\eta_2^4$ is always generated by its symbols. For this, we shall use a method different from that of this paper, based on the following result:

Theorem ([19])**.** *Let* X *be a quadric and let* C *be a conic such that there exist a morphism* $C \to X$*. Let* D *be the quaternion algebra associated to* C*. Then there is a canonical isomorphism*

$$\operatorname{Ker}\eta_2^4 \xrightarrow{\sim} \frac{H^0(X, K_2^D)}{K_2 D}.$$

Here $H^0(X, K_2^D)$ is the intersection of the kernels of the residue maps

$$K_2(D \otimes_F F(X)) \to K_1(D \otimes_F F(x)),$$

where x runs through the points of codimension 1 in X.

This paper is organized as follows. In part I we recall some known results on Chow groups, in particular of Karpenko, and of $\mathcal{K}$- and $\mathcal{H}$-cohomology of quadrics, supplying proofs when they don't appear in the literature. Part II is devoted to the proof of theorem 1 and corollary 2, and part III to the proof of proposition 3, theorem 5, and theorem 6. In appendix A we consider $\operatorname{Coker}\eta^i$ for quadrics of dimension 0 and 1. In appendix B we consider the case $F = \mathbf{R}$. In appendix C we prove theorem 9.

Notation and conventions. For any scheme X, H^iX denotes the i-th étale cohomology group of X with coefficients $\mathbf{Z}/2$. We use the abbreviation $H^i(X, j)$ for $H^i_{\text{ét}}(X, \mathbf{Q}/\mathbf{Z}(j))$ and similarly $H^p(X, \mathcal{H}^q(j))$ for the E_2-term of the coniveau spectral sequence with divisible coefficients. If X is smooth over a field F, we denote by CH^iX its i-th Chow group, the group of cycles of codimension i over X modulo rational equivalence. We denote by cl^i_X (*resp.* cl'^i_X) the cycle map with $\mathbf{Z}/2$ coefficients $CH^iX/2 \to H^{2i}X$ (*resp.* with divisible coefficients $CH^iX \otimes \mathbf{Q}/\mathbf{Z} \to$

$H^{2i}(X, \mathbf{Q}/\mathbf{Z}(i)))$. We set $\overline{X} = X \times_F F_s$, where F_s is a separable closure of F. The word "quadric" means smooth, projective and absolutely irreducible quadric. If X is an even-dimensional quadric over F (assumed of characteristic $\neq 2$), we denote by $d(X) \in F^*/F^{*2}$ the (signed) discriminant $d_\pm q$ of any quadratic form q defining X. If $\varphi\colon A \to B$ is a homomorphism between two abelian groups, we sometimes allow ourselves to write B/A or $\dfrac{B}{A}$ for $\operatorname{Coker}\varphi$, even if φ is not injective.

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Part I. Collected results on quadrics

In this part, we state general results on Chow groups, $\mathcal{H}$- and $\mathcal{K}$-cohomology of quadrics, most of which are "classical". The reader is advised to skip it and to use it only for reference purposes. We merely give references when published proofs are available, and provide proofs otherwise.

We say that a quadric X is *anisotropic* (*resp. isotropic*, *split*) if a quadratic form defining X is anisotropic (*resp.* is isotropic, has maximal index).

1. Chow groups

The following proposition is mainly a reformulation of results of N. Karpenko [20].

1.1. Proposition. *Let X be a quadric over a field F of characteristic $\neq 2$. Let h denote the class of a hyperplane section in CH^1X.*

a) *For any $i \geq 0$, the natural map $CH^iX \to CH^i\overline{X}$ has kernel the torsion subgroup of CH^iX (which is 2-primary).*
b) *For $i < \dim X/2$, $CH^iX/\operatorname{torsion}$ is generated by h^i; $CH^iX \to CH^i\overline{X}$ is surjective.*
c) *For $i > \dim X/2$, $CH^iX/\operatorname{torsion}$ is generated by h^i if $i \leq \dim X - n$ and by the class of a rational linear subspace of codimension i if $i > \dim X - n$. Here n is the Witt index of a quadratic form defining X. $\operatorname{Coker}(CH^iX \to CH^i\overline{X}) \simeq \mathbf{Z}/2$ if $i \leq \dim X - n$ and $CH^iX \to CH^i\overline{X}$ is surjective if $i > \dim X - n$.*
d) *For $i = \dim X/2$ and $d(X) = 1$ one has*
$$CH^iX/\operatorname{torsion} \simeq \mathbf{Z} \oplus \mathbf{Z},$$
with basis $(h^i, 2^t l)$, where $t \geq 0$ and l is the class of some maximal linear subspace of $\overline{X}$. Moreover $\operatorname{Coker}(CH^iX \to CH^i\overline{X}) \simeq \mathbf{Z}/2^t$. We have $t = 1$ if $i = 1$ and X is anisotropic, and $t = 2$ if $i = 2$ and X is anisotropic.
e) *For $i = \dim X/2$ and $d(X) \neq 1$ one has*
$$CH^iX/\operatorname{torsion} \simeq \mathbf{Z}$$
generated by h^i. The map $CH^iX \to (CH^i\overline{X})^{G_F}$ is surjective, where G_F is the absolute Galois group of F.

Proof. a), b), and c) follow easily from [20, (2.1), (2.2), and (2.7)]; d) and e) follow from the same references plus [20, (2.4) and (5.5)]. ∎

1.2. Corollary. *Let X be a quadric over F, D a divisible group and $i \geq 0$. If $i \neq \dim X/2$ or $i = \dim X/2$ and $d(X) = 1$, there is an exact sequence*

$$0 \to \operatorname{Tor}(D, \operatorname{Coker}(CH^i X \to CH^i \overline{X})) \to D \otimes CH^i X \to D \otimes CH^i \overline{X} \to 0.$$

This is clear, since $CH^i\overline{X}$ is torsion-free and the map $CH^i X \to CH^i \overline{X}$ has torsion kernel and cokernel.

1.3. Corollary. *Let X be a quadric of dimension $2i$ over F such that $d = d(X) \neq 1$, $E = F(\sqrt{d})$ and $\Gamma = \operatorname{Gal}(E/F)$. Then the Γ-module $CH^i X_E/\operatorname{torsion}$ is isomorphic to $\operatorname{Ind}_1^\Gamma \mathbf{Z}$ if X_E is split, and to $\mathbf{Z} \oplus \mathbf{Z}(\varepsilon)$ otherwise, where $\mathbf{Z}(\varepsilon)$ is the Γ-module of support $\mathbf{Z}$ with the nontrivial action of Γ. On the other hand, the G_F-module $CH^i\overline{X}$ is always isomorphic to $\operatorname{Ind}_{G_E}^{G_F} \mathbf{Z}$. The maps*

$$CH^i X/\operatorname{torsion} \to (CH^i \overline{X})^{G_F}$$

and

$$CH^i X \otimes \mathbf{Q}/\mathbf{Z} \to H^0(F, CH^i \overline{X} \otimes \mathbf{Q}/\mathbf{Z})$$

are isomorphisms.

Proof. By prop. 1.1 d), $CH^i X_E/\operatorname{torsion}$ is a free $\mathbf{Z}$-module with basis $(h^i, 2^t l)$ for some $t \geq 0$. It is clear that $t = 0$ if and only if l is rational over E, *i.e.*, iff X_E is split. In this case, another basis of $CH^i X_E/\operatorname{torsion}$ is (l, l') where $l' = h^i - l$ is the class of another linear section, and this basis is permuted by Γ, hence the first claim. If $t > 0$, another basis of $CH^i X_E/\operatorname{torsion}$ is given by $(h^i, 2^{t-1}h^i - 2^t l)$, and $2^{t-1}h^i - 2^t l$ is a -1-eigenvector for Γ. This proves the second claim. The isomorphism $CH^i X/\operatorname{torsion} \xrightarrow{\sim} (CH^i \overline{X})^{G_F}$ has been seen in prop. 1.1 a) and e). To get the other, we consider the short exact sequence of G_F-modules

$$0 \to CH^i \overline{X} \to CH^i \overline{X} \otimes \mathbf{Q} \to CH^i \overline{X} \otimes \mathbf{Q}/\mathbf{Z} \to 0$$

obtained by tensoring the exact sequence $0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$ by the flat $\mathbf{Z}$-module $CH^i \overline{X}$. Since $CH^i \overline{X} \simeq \operatorname{Ind}_{G_E}^{G_F} \mathbf{Z}$, we have $H^1(F, CH^i \overline{X}) = 0$ by Shapiro lemma, hence we get an isomorphism

$$CH^i X \otimes \mathbf{Q}/\mathbf{Z} \xrightarrow{\sim} (CH^i \overline{X})^{G_F} \otimes \mathbf{Q}/\mathbf{Z} \xrightarrow{\sim} H^0(F, CH^i \overline{X} \otimes \mathbf{Q}/\mathbf{Z}). \quad \square$$

1.4. Corollary. *Let X be an anisotropic quadric over F, $K = F(X)$ and $i_1(X)$ the Witt index of q_K, where q is a quadratic form defining X.*

a) *For $i < \dim X/2$, $CH^i X/\operatorname{torsion} \to CH^i X_K/\operatorname{torsion}$ is bijective.*
b) *For $i > \dim X/2$, $CH^i X/\operatorname{torsion} \to CH^i X_K/\operatorname{torsion}$ is bijective if $i \leq \dim X - i_1(X)$ and $\operatorname{Coker}(CH^i X/\operatorname{torsion} \to CH^i X_K/\operatorname{torsion}) \simeq \mathbf{Z}/2$ if $i > \dim X - i_1(X)$.*
c) *For $i = \dim X/2$, $CH^i X/\operatorname{torsion} \to CH^i X_K/\operatorname{torsion}$ is bijective if $d(X) \neq 1$. If $d(X) = 1$, $\operatorname{Coker}(CH^i X/\operatorname{torsion} \to CH^i X_K/\operatorname{torsion})$ is cyclic of order 2^s for some $s \leq t$, where t is as in prop. 1.1 d). For $i = 1$ or 2 we have $s = 1$.*
d) *For $i = \dim X/2$ and $d = d(X) \neq 1$, let $E = F(\sqrt{d})$. Then*

$$CH^i X_E/\operatorname{torsion} \to CH^i X_{KE}/\operatorname{torsion}$$

is bijective if X_E is isotropic and has cokernel as in c) otherwise.

Proof. Everything directly follows from proposition 1.1, except for the last sentence of c) and for d). Assume $\dim X = 2$. Then X is defined by an anisotropic quaternion form, (= a 2-fold Pfister form). hence $i_1(X) = 2$, which implies $s = 1$ in this case by proposition 1.1 d). Assume $\dim X = 4$. Then X is defined by an anisotropic Albert form φ, hence $i_1(X) = 1$. Let Y be the quadric over K given by the anisotropic part of φ_K ($\dim Y = 2$). By [20, (2.2)] (*cf.* lemma 2.1 below), there is an isomorphism $CH^1Y \xrightarrow{\sim} CH^2X_K$. This isomorphism is natural under change of base field, as can be seen from the proof of lemma 2.1. In particular, the diagram

$$\begin{array}{ccc} CH^1Y & \xrightarrow{\sim} & CH^2X_K \\ \downarrow & & \downarrow \\ CH^1\overline{Y} & \xrightarrow{\sim} & CH^2\overline{X_K} \end{array}$$

commutes, which shows that $\operatorname{Coker}(CH^2X_K \to CH^2\overline{X_K}) \simeq \mathbf{Z}/2$. Moreover, since $\operatorname{Coker}(CH^2X \to CH^2\overline{X}) \simeq \mathbf{Z}/4$ and $CH^2\overline{X} \xrightarrow{\sim} CH^2\overline{X_K}$, this indeed shows that $s = 1$. Finally, d) is obvious since, if X_E is isotropic, the extension KE/E is purely transcendental (compare [20, proof of (3.12)]). □

1.5. Proposition. *Let X be a quadric over F, where F is separably closed (of characteristic $\neq 2$). Then $H^jX = 0$ for j odd and the cycle maps*

$$cl^i_X \colon CH^iX/n \to H^{2i}(X, \mu_n^{\otimes i})$$

are isomorphisms for any n prime to $\operatorname{char} F$ *and $i \geq 0$.*

Proof. This follows from proposition 2.2 a) and the computation of $H^*_{\text{ét}}(X, \mathbf{Z}_l)$ [11, th. 3.3]. □

1.6. Proposition. *Let X be a quadric over F.*

a) (classical) *CH^1X is torsion-free.*
b) ([20, (6.1)]) *$(CH^2X)_{\text{torsion}} \simeq \mathbf{Z}/2$ if X is defined by a neighbour of an anisotropic 3-fold Pfister form, and is 0 otherwise.*
c) ([20, (2.6)],[21],[22]) *$(CH^3X)_{\text{torsion}} \simeq 0$ or $\mathbf{Z}/2$. If $\dim X = 3$ or $\dim X > 10$, $(CH^3X)_{\text{torsion}} = 0$.*

2. $\mathcal{K}$- and $\mathcal{H}$-cohomology

We denote by $H^i(X, \mathcal{K}_j)$ and $H^i(X, \mathcal{H}^j(\mu_m^{\otimes k}))$ the cohomology groups of the localization complexes with terms

$$\bigoplus_{x \in X^{(i)}} K_{j-i}F(x) \qquad \textit{resp.} \qquad \bigoplus_{x \in X^{(i)}} H^{j-i}(F(x), \mu_m^{\otimes(k-i)}).$$

These cohomology groups are defined for any equidimensional, not necessarily smooth variety. For the long exact excision sequences and for homotopy invariance we refer to [42]. The interpretation for smooth X as Zariski cohomology groups is not needed here.

2.1. Lemma. *Let X be an isotropic quadric of dimension $n \geq 2$ over a field F of characteristic $\neq 2$; write $q \simeq q' \perp x_nx_{n+1}$ for a quadratic form defining X and*

$Z \subseteq X$ for the codimension 2 subquadric with equation $q' = 0$. Then, for $0 < i < n$, $j \geq i$ and $k \in \mathbf{Z}$, there are isomorphisms:

$$\begin{aligned} K_jF &\xrightarrow{\sim} H^0(X, \mathcal{K}_j),\\ H^{i-1}(Z, \mathcal{K}_{j-1}) &\xrightarrow{\sim} H^i(X, \mathcal{K}_j),\\ K_{j-n}F &\xrightarrow{\sim} H^n(X, \mathcal{K}_j), \end{aligned}$$

$$\begin{aligned} H^j(F, \mu_m^{\otimes k}) &\xrightarrow{\sim} H^0(X, \mathcal{H}^j(\mu_m^{\otimes k})),\\ H^{i-1}(Z, \mathcal{H}^{j-1}(\mu_m^{\otimes (k-1)})) &\xrightarrow{\sim} H^i(X, \mathcal{H}^j(\mu_m^{\otimes k})),\\ H^{j-n}(F, \mu_m^{\otimes (k-n)}) &\xrightarrow{\sim} H^n(X, \mathcal{H}^j(\mu_m^{\otimes k})). \end{aligned}$$

These isomorphisms commute with change of base field, transfer and cup-product by K_*F (resp. $H^*(F, \mu_m^{\otimes l})$).

Remark. If $n = 2$, the "subquadric" Z is $\operatorname{Spec} E$, with $E = F[t]/t^2 - d$, where $d = d(X)$ is the (signed) discriminant of the 2-dimensional form q'. E is a quadratic extension of F if $d \neq 1$ and $F \times F$ otherwise.

Proof. (compare [16, lemma 3.3], [51, prop. 3.2.1] for $\dim X = 3$, and [20, (2.2)] for $i = j$). We do it only for $\mathcal{K}$-cohomology, the case of $\mathcal{H}$-cohomology being identical. Let Y be the hyperplane section $x_{n+1} = 0$ and $P = (0 : \cdots : 0 : 1 : 0)$. We have excision exact sequences:

$$\cdots \to H^{i-1}(Y, \mathcal{K}_{j-1}) \to H^i(X, \mathcal{K}_j) \to H^i(X \setminus Y, \mathcal{K}_j) \to H^i(Y, \mathcal{K}_{j-1}) \to \cdots$$

$$\begin{aligned} \cdots \to H^{i-n}(P, \mathcal{K}_{j-n}) \to H^{i-1}(Y, \mathcal{K}_{j-1}) \to H^{i-1}(Y \setminus P, \mathcal{K}_{j-1}) \to\\ \to H^{i-n+1}(P, \mathcal{K}_{j-n}) \to \cdots \end{aligned}$$

On the other hand, $X \setminus Y$ is isomorphic to the affine space and P makes the projection $Y \setminus P \to Z$ a vector bundle with affine line fibres. Homotopy invariance of $\mathcal{K}$-cohomology yields new exact sequences and isomorphisms:

$$H^{i-1}(Y, \mathcal{K}_{j-1}) \xrightarrow{\sim} H^i(X, \mathcal{K}_j) \quad \text{if } i > 1, \tag{2.1}$$

$$0 \to H^0(X, \mathcal{K}_j) \to K_jF \to H^0(Y, K_{j-1}) \to H^1(X, \mathcal{K}_j) \to 0 \tag{2.2}$$

and

$$H^{i-1}(Z, \mathcal{K}_{j-1}) \xrightarrow{\sim} H^{i-1}(Y \setminus P, \mathcal{K}_{j-1}) \quad \text{if } i < n-2, \tag{2.3}$$

$$0 \to H^{n-2}(Y, \mathcal{K}_{j-1}) \to H^{n-2}(Z, \mathcal{K}_{j-1}) \to K_{j-n}F \to H^{n-1}(Y, \mathcal{K}_{j-1}) \to 0. \tag{2.4}$$

The composition $K_jF \to H^0(X, \mathcal{K}_j) \to K_jF$ is the identity, so that the exact sequence (2.2) splits as two isomorphisms. Similarly, for $n > 0$, $K_{j-n}F \to H^{n-1}(Y, \mathcal{K}_{j-1}) \xrightarrow{\sim} H^n(X, \mathcal{K}_j)$ is induced by the inclusion of the closed point P in X and composition with the norm $H^n(X, \mathcal{K}_j) \xrightarrow{N} K_{j-n}F$ is the identity, so the exact sequence (2.4) also splits as two isomorphisms. Putting all this together, we get the isomorphisms of lemma 2.1. The naturality statements are clear from the construction of these isomorphisms. □

2.2. Proposition. *Let X be a split quadric over a field F of characteristic $\neq 2$, and $h \in CH^1X$ the class of a hyperplane section.*

a) *$CH^iX \simeq \mathbf{Z}$ for $i \neq \dim X/2$, and $CH^iX \simeq \mathbf{Z}\oplus\mathbf{Z}$ if $i = \dim X/2$; h^i is a generator of CH^iX for $i < \dim X/2$ and twice a generator for $i > \dim X/2$. For $i = \dim X/2$, a basis of CH^iX is given by the classes l, l' of two maximal linear subspaces of X such that $h^i = l + l'$.*

b) *The natural maps*
$$K_{j-i}F \otimes CH^iX \to H^i(X, \mathcal{K}_j)$$
given by the Brown-Gersten-Quillen spectral sequence and cup-product are isomorphisms.

c) *The natural maps*
$$H^{j-i}(F, \mu_m^{\otimes k}) \otimes CH^iX \to H^i(X, \mathcal{H}^j(\mu_m^{\otimes(i+k)}))$$
given by the coniveau spectral sequence and cup-product are isomorphisms for all $m \geq 1$.

Proof. a) follows from proposition 1.1. Let us prove b) (as above, the proof of c) is identical and omitted). (See [29, prop. 1 and prop. 3] for another approach.) The quadric X is isomorphic to

$$X_n : \begin{cases} x_0x_1 + \dots + x_nx_{n+1} = 0 & \text{if } n \text{ is even} \\ x_0^2 + x_1x_2 + \dots + x_nx_{n+1} = 0 & \text{if } n \text{ is odd} \end{cases}$$

where $n = \dim X$. As a consequence of lemma 2.1, the diagram

$$\begin{array}{ccc} H^{i-1}(X_{n-2}, \mathcal{K}_{j-1}) & \xrightarrow{\sim} & H^i(X_n, \mathcal{K}_j) \\ \cup\uparrow & & \cup\uparrow \\ K_{j-i}F \otimes CH^{i-1}X_{n-2} & \xrightarrow{\sim} & K_{j-i}F \otimes CH^iX_n \end{array} \quad \text{for } 0 < i < n$$

commutes for $i \leq j$. Inductively we get the commutative diagrams

$$\begin{array}{ccc} H^0(X_{n-2i}, \mathcal{K}_{j-i}) & \xrightarrow{\sim} & H^i(X_n, \mathcal{K}_j) \\ \cup\uparrow & & \cup\uparrow \\ K_{j-i}F \otimes CH^0X_{n-2i} & \xrightarrow{\sim} & K_{j-i}F \otimes CH^iX_n \end{array} \quad \text{for } 2i \leq n,$$

$$\begin{array}{ccc} H^{2i-n}(X_{2i-n}, \mathcal{K}_{j+i-n}) & \xrightarrow{\sim} & H^i(X_n, \mathcal{K}_j) \\ \cup\uparrow & & \cup\uparrow \\ K_{j-i}F \otimes CH^{2i-n}X_{2i-n} & \xrightarrow{\sim} & K_{j-i}F \otimes CH^iX_n \end{array} \quad \text{for } 2i \geq n.$$

Still by lemma 2.1, $K_{j-i}F \otimes CH^0X_{n-2i} \xrightarrow{\cup} H^0(X_{n-2i}, \mathcal{K}_{j-i})$ and
$$K_{j-i}F \otimes CH^{2i-n}X_{2i-n} \xrightarrow{\cup} H^{2i-n}(X_{2i-n}, \mathcal{K}_{j+i-n})$$
are isomorphisms (note that for $2i = n$, X_0 is a disjoint union of two points so $CH^0X_0 \simeq \mathbf{Z}\oplus\mathbf{Z}$). The claim follows. □

2.3. Lemma. *Let $i > 0$ and X be a quadric over F, with $\dim X \neq 2i$ or $\dim X = 2i$ and $d(X) = 1$. Then, for all $j \geq i$, the map $K_{j-i}F \otimes CH^iX \to H^i(X, \mathcal{K}_j)$ has 2-primary torsion kernel and cokernel. If $\dim X = 2i$ and $d(X) = d \neq 1$, the composition*
$$(K_{j-i}E \otimes CH^iX_E)_\Gamma \to H^i(X_E, \mathcal{K}_j)_\Gamma \xrightarrow{N} H^i(X, \mathcal{K}_j)$$

has 2-primary torsion kernel and cokernel, where $E = F(\sqrt{d})$*,* $\Gamma = \mathrm{Gal}(E/F)$ *and* N *is the transfer in* $\mathcal{K}$*-cohomology.*

Proof. Assume first that $\dim X \neq 2i$ or $\dim X = 2i$ and $d(X) = 1$. Let K be a splitting field for X that is finite and Galois over F, with $[K:F]$ a power of 2 (some multiquadratic extension will do) and $G = \mathrm{Gal}(K/F)$. In the commutative diagram

$$\begin{array}{ccc} K_{j-i}F \otimes CH^iX & \longrightarrow & H^i(X,\mathcal{K}_j) \\ {\scriptstyle\mathrm{Res}}\downarrow & & {\scriptstyle\mathrm{Res}}\downarrow \\ (K_{j-i}K \otimes CH^iX_K)^G & \longrightarrow & H^i(X_K,\mathcal{K}_j)^G \end{array}$$

the bottom horizontal map is an isomorphism by proposition 2.2 and the right vertical map has 2-primary torsion kernel by the usual transfer argument. Therefore, to prove the claim, it is enough to show that the left vertical map has 2-primary torsion kernel and cokernel. But this map decomposes as

$$K_{j-i}F \otimes CH^iX \to K_{j-i}F \otimes CH^iX_K \to \\ \to (K_{j-i}K)^G \otimes CH^iX_K = (K_{j-i}K \otimes CH^iX_K)^G$$

since $CH^iX_K \simeq \mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z}$ with trivial Galois action by propositions 1.1 and 2.2. The same proposition shows that the left map has 2-primary torsion kernel and cokernel, while the transfer argument shows the same for the right map. This implies the same for their composition.

Assume now that $\dim X = 2i$ and $d(X) = d \neq 1$. By the above, the left map in the composition of lemma 2.3 has 2-primary torsion kernel and cokernel, and the same holds for the right one still by a transfer argument. This concludes the proof of lemma 2.3. □

2.4. Proposition. *Let* $i > 0$ *and* X *be a quadric over* F*, with* $\dim X \neq 2i-2$ *or* $\dim X = 2i-2$ *and* $d(X) = 1$*. Then there is an exact sequence:*

$$H^1(F,1) \otimes CH^{i-1}X \to H^{i-1}(X,\mathcal{H}^i(i)) \to (CH^iX)_{\mathrm{torsion}} \to 0.$$

If $\dim X = 2i-2$ *and* $d(X) = d \neq 1$*, there is an exact sequence:*

$$(H^1(E,1) \otimes CH^{i-1}X_E)_\Gamma \to H^{i-1}(X,\mathcal{H}^i(i)) \to (CH^iX)_{\mathrm{torsion}} \to 0,$$

where $E = F(\sqrt{d})$*,* $\Gamma = \mathrm{Gal}(E/F)$ *and the left map is cup-product followed by transfer.*

Proof. Tensoring the maps of lemma 2.3 with $\mathbf{Q}/\mathbf{Z}$ we get surjections:

$$K_1F \otimes CH^{i-1}X \otimes \mathbf{Q}/\mathbf{Z} \twoheadrightarrow H^{i-1}(X,\mathcal{K}_i) \otimes \mathbf{Q}/\mathbf{Z}$$

if $\dim X \neq 2i-2$ or $\dim X = 2i-2$ and $d(X) = 1$,

$$(K_1E \otimes CH^{i-1}X_E \otimes \mathbf{Q}/\mathbf{Z})_\Gamma = (K_1E \otimes CH^{i-1}X_E)_\Gamma \otimes \mathbf{Q}/\mathbf{Z} \\ \twoheadrightarrow H^{i-1}(X,\mathcal{K}_i) \otimes \mathbf{Q}/\mathbf{Z}$$

if $\dim X = 2i-2$ and $d(X) = d \neq 1$.

By Kummer theory, $K_1F \otimes \mathbf{Q}/\mathbf{Z} \simeq H^1(F,1)$. The claim now follows from the short exact sequence [5, 3.6]

$$0 \to H^{i-1}(X,\mathcal{K}_i) \otimes \mathbf{Q}/\mathbf{Z} \to H^{i-1}(X,\mathcal{H}^i(i)) \to (CH^iX)_{\mathrm{torsion}} \to 0. \qquad \square$$

2.5. Proposition.

a) ([8, prop. 1.2]) *If the quadric X is isotropic over F, then η_2^i and η^i are isomorphisms for all $i \geq 0$.*

b) *In general,* $\operatorname{Ker}\eta^i$ *and* $\operatorname{Coker}\eta^i$ *have exponent ≤ 2.*

c) (*cf.* [10, lemma 1.3]) *Suppose X and X' are two quadrics such that X' becomes isotropic over $F(X)$. Then there is a natural commutative diagram for all i, k:*

$$\begin{tikzcd} & H^i_{nr}(F(X)/F,k) \\ H^i(F,k) \arrow[ur, "\eta^i_X"] \arrow[dr, "\eta^i_{X'}"'] & \\ & H^i_{nr}(F(X')/F,k) \arrow[uu] \end{tikzcd}$$

so that $\operatorname{Ker}\eta^i_{X'} \subseteq \operatorname{Ker}\eta^i_X$ *and* $\operatorname{Coker}\eta^i_{X'}$ *maps to* $\operatorname{Coker}\eta^i_X$. *If X also becomes isotropic over $F(X')$, then* $\operatorname{Ker}\eta^i_X = \operatorname{Ker}\eta^i_{X'}$ *and* $\operatorname{Coker}\eta^i_X \simeq \operatorname{Coker}\eta^i_{X'}$.

Proof. a) comes from the fact that the extension $F(X)/F$ is purely transcendental if X is isotropic (compare [7, th. 4.1.5]). b) follows from a) and a transfer argument, using a quadratic extension of F which makes X isotropic. We give a proof of c) which is more elementary than [10, lemma 1.3]. Consider the diagram of fields

$$\begin{tikzcd} & F(X) \arrow[dr] & \\ F \arrow[ur] \arrow[dr] & & F(X \times_F X') \\ & F(X') \arrow[ur] & \end{tikzcd}$$

There are natural maps

$$H^i_{nr}(F(X)/F,k) \to H^i_{nr}(F(X \times_F X')/F,k)$$

and

$$H^i_{nr}(F(X')/F,k) \to H^i_{nr}(F(X \times_F X')/F,k).$$

If X' becomes isotropic over $F(X)$, then $F(X \times_F X')/F(X)$ is a purely transcendental extension. Therefore, by [8, prop. 1.2] or [7, th. 4.1.5], the first quoted map is an isomorphism, which proves the first claim. If X is also purely transcendental over $F(X')$, then $H^i_{nr}(F(X')/F,k) \to H^i_{nr}(F(X)/F,k)$ is clearly the inverse of $H^i_{nr}(F(X)/F,k) \to H^i_{nr}(F(X')/F,k)$, hence the second claim. □

Part II. Ker $\boldsymbol{\eta_2^4}$ for higher-dimensional quadrics

3. Generalities on Ker $\boldsymbol{\eta_2^i}$

We fix a number i and assume that the invariant $\overline{e}^r_F$ is is well-defined for $r \leq i-1$ and an isomorphism for $r \leq i-2$ (for all fields F).

3.1. Lemma. *A symbol $(a_1,\dots,a_i) \neq 0$ lies in* $\operatorname{Ker}\eta_2^i$ *if and only if q is similar to a subform of $\langle\langle a_1,\dots,a_i\rangle\rangle$.*

Proof. Let $\varphi = \langle\langle a_1, \dots, a_i \rangle\rangle$. Since $(a_1, \dots, a_i)_{F(X)} = 0$, [16, remark p. 555] implies $\varphi_{F(X)} \sim 0$, and the claim follows by the Cassels-Pfister theorem [26, th. IX.1.3]. $\square$

As observed in the introduction, for $\dim q \le 2^{i-2}$ the group $\operatorname{Ker} \eta_2^i$ does not consist of symbols in general. The following proposition and corollary indicate that this bound on $\dim q$ might be sharp. Recall that two n-fold Pfister forms φ, ψ are *linked* if there exist $a_1, \dots, a_{n-1}, a_n, b_n \in F^*$ such that $\varphi \simeq \langle\langle a_1, \dots, a_{n-1}, a_n \rangle\rangle$ and $\psi \simeq \langle\langle a_1, \dots, a_{n-1}, b_n \rangle\rangle$. We define the notion of linked symbols (in Galois cohomology) similarly.

3.2. Proposition. *Let* $\dim q \ge 2^{i-2} + 1$, *let* X *be the projective quadric with equation* $q = 0$ *and let* $\varphi, \psi \in \operatorname{Ker}\big(W(F) \to W(F(X))\big)$ *be two* i*-fold Pfister forms. Then* φ *and* ψ *are linked, hence* $\varphi \perp -\psi$ *is Witt-equivalent to a third* i*-fold Pfister form* (*up to similarity*).

Proof. (*cf.* Fitzgerald [15, proof of th. 2.1]) The forms φ and ψ are multiplicative and split over $F(X)$. Therefore q is isomorphic to subforms of φ and of ψ (assuming q represents 1). Hence the split form $q \perp -q$ is a subform of $\varphi \perp -\psi$ and the Witt index $n = i(\varphi \perp -\psi)$ is $\ge 2^{i-2} + 1$. By [13, th. 4.5], n is a power of 2, hence $n \ge 2^{i-1}$. Therefore φ and ψ are linked, again by *loc. cit.*. $\square$

3.3. Lemma. *For* $\dim q \ge 2^{i-2} + 1$ *the following are equivalent:*

a) $\operatorname{Ker} \eta_2^i$ *consists of symbols;*
b) $\operatorname{Ker} \eta_2^i$ *is additively generated by symbols.*

Proof. Let $\alpha, \beta \in \operatorname{Ker} \eta_2^i$ be two symbols and $\tilde\alpha$, $\tilde\beta$ Pfister forms such that $e^i(\tilde\alpha) = \alpha$, $e^i(\tilde\beta) = \beta$. By [16, remark p. 555], $\tilde\alpha, \tilde\beta \in \operatorname{Ker}(W(F) \to W(F(X)))$. By proposition 3.2, $\tilde\alpha$ and $\tilde\beta$ are linked, so α and β are linked and $\alpha + \beta$ is a symbol. $\square$

3.4. Proposition. *Assume that for* $\dim q = 2^{i-2} + 1$ *the group* $\operatorname{Ker} \eta_2^i$ *consists of symbols. Then*

(1) $\operatorname{Ker} \eta_2^i$ *consists of symbols for* $\dim q > 2^{i-2}$.
(2) $\left|\operatorname{Ker} \eta_2^i\right| \le 2$ *for* $\dim q > 2^{i-1}$.
(3) $\operatorname{Ker} \eta_2^i = 0$ *for* $\dim q > 2^i$.

Remark. The hypothesis of proposition 3.4 is satisfied for $i \le 3$ by [1]. For $i = 4$ it will be proved in the next three sections (theorem 6.3).

Proof. (1) Let $\dim q' \ge 2^{i-2} + 1$ and let q be a subform of q' of dimension $2^{i-2} + 1$. The claim follows now from proposition 2.5 c) applied to the associated quadrics X' *resp.* X.

(2) + (3) If $\dim q > 2^{i-1}$, for any nonzero symbol $(a_1, \dots, a_i) \in \operatorname{Ker} \eta_2^i$, q is a neighbour of the Pfister form $\langle\langle a_1, \dots, a_i \rangle\rangle$ by lemma 3.1. But q can be a neighbour of at most one Pfister form, up to isomorphism, *cf.* [24, 7.4]. Moreover if $\langle\langle a_1, \dots, a_i \rangle\rangle \simeq \langle\langle b_1, \dots, b_i \rangle\rangle$, then $(a_1, \dots, a_i) = (b_1, \dots, b_i)$ [1, Satz 1.6]. Hence $\left|\operatorname{Ker} \eta_2^i\right| \le 2$, and if it is nontrivial then $\dim q \le 2^i$. $\square$

4. 6-dimensional spinors

For generalities on Clifford algebras and the special Clifford group see [45].

Let $q \colon V \to F$ be 6-dimensional, let $C_0(q)$ be the even Clifford algebra of q, and let $Z(q)$ be the center of $C_0(q)$. Then $Z(q)$ is a quadratic extension of F and $C_0(q)$

is an Azumaya algebra of degree 4 over $Z(q)$. The standard involution $x \mapsto x^t$ is an involution of second kind. Denote by

$$\mathrm{Nrd}\colon C_0(q) \to Z(q)$$

the reduced norm.

Let

$$\mathrm{S\Gamma}(q) = \{\, x \in C_0(q)^* \mid xVx^{-1} = V \,\}$$

be the special Clifford group and let

$$\mathrm{G\Gamma}(q) = \{\, x \in C_0(q)^* \mid xx^t \in F^* \,\}.$$

Then $\mathrm{S\Gamma}(q) \subset \mathrm{G\Gamma}(q)$ and there is an exact sequence

$$1 \to \mathrm{Spin}(q) \to \mathrm{S\Gamma}(q) \xrightarrow{\mathrm{sn}} F^*$$

where $\mathrm{sn}(x) = xx^t$ is the spinor norm.

Suppose $d_\pm(q) = 1$. Then $Z(q) = F \times F$ and $C_0(q) = A \times A'$ where A and A' are algebras of degree 4 over F. The standard involution identifies A' with A^{op}, so that there is an isomorphism

$$C_0(q) = A \times A^{\mathrm{op}} \tag{4.1}$$

under which the standard involution reads as $(a,b)^t = (b,a)$. The reduced norm $C_0(q) \to Z(q)$ is then componentwise the reduced norm on A. The group $\mathrm{G\Gamma}(q)$ becomes the group of pairs

$$\{\, (a, a^{-1}f) \in A \times A^{\mathrm{op}} \mid a \in A^*,\ f \in F^* \,\}.$$

Let $\pi\colon C_0(q) \to A$ be the projection onto the first factor. Then

$$\mathrm{G\Gamma}(q) = A^* \times F^*$$

via $x \mapsto (\pi(x), xx^t)$.

We call an element $x \in \mathrm{S\Gamma}(q)$ *plane* if $x \in \mathrm{S\Gamma}(\tilde q)$ where $\tilde q$ is a regular 2-dimensional subform of q. If q is anisotropic, then x is plane if and only if x is the product vw of two vectors v, $w \in V \setminus 0$.

4.1. Lemma. *$\mathrm{Nrd}(x) = (xx^t)^2$ for $x \in \mathrm{S\Gamma}(q)$.*

Proof. The action of $\mathrm{S\Gamma}(q)$ on V induces an exact sequence

$$1 \to F^* \to \mathrm{S\Gamma}(q) \to \mathrm{SO}(q) \to 1. \tag{4.2}$$

Since $\mathrm{O}(q)$ is generated by reflections, it follows that $\mathrm{S\Gamma}(q)$ is generated by products $x = vw$ where v, $w \in V$ are anisotropic vectors. For generic v, w, the subspace of V generated by v, w is regular of dimension 2. Therefore it suffices to check the claim for plane elements.

So suppose $x \in C_0(\tilde q)$ for a regular 2-dimensional subform $\tilde q$. Write $L = C_0(\tilde q)$. The standard involution leaves L invariant and restricts to the canonical involution of L/F. Hence $x \mapsto xx^t$ restricts on L to the norm of L/F. The claim follows now from the fact that $C_0(q)$ is of degree 4 and $L \otimes Z(q) \subset C_0(q)$ is of degree 2 over $Z(q)$. □

4.2. Proposition.

a) $$\mathrm{S\Gamma}(q) = \{\, x \in C_0(q)^* \mid xx^t \in F^*,\ \mathrm{Nrd}(x) = (xx^t)^2 \,\},$$

b) $$\mathrm{Spin}(q) = \{\, x \in C_0(q)^* \mid xx^t = 1,\ \mathrm{Nrd}(x) = 1 \,\}.$$

Proof. The inclusions "$\subset$" are clear from lemma 4.1.

We may assume that F is algebraically closed. In this case it easy to see that $\mathrm{S\Gamma}(q) = \mathrm{Spin}(q) \cdot F^*$ and similarly for the righthand sides. Hence b) $\Rightarrow$ a).

The righthand side of b) becomes under (4.1) the group of pairs

$$\{ (a, a^{-1}) \in A \times A^{\mathrm{op}} \mid a \in A^*, \, \mathrm{Nrd}(a) = 1 \} = \mathrm{SL}(1, A).$$

This way b) reads as the classical isomorphism $\mathrm{Spin}(6) = \mathrm{SL}(4)$ [55, 12, 25]. $\square$

4.3. Corollary. *If $d_\pm(q) = 1$ and A is as above, there is an exact sequence*

$$1 \to \mathrm{S\Gamma}(q) \to A^* \times F^* \xrightarrow{\omega} F^*$$

where $\omega(a, f) = \mathrm{Nrd}(a)/f^2$.

4.4. Corollary. *Let A be a biquaternion skewfield and let $a \in A^*$ and $f \in F^*$ with*

$$\mathrm{Nrd}(a) = f^2.$$

Then there exist a_1, a_2, $a_3 \in A^*$ with the following properties:

- *$a = a_1 a_2 a_3$,*
- *Every a_i lies in a quadratic subfield $L_i \subset A$,*
- *$N_{L_1/F}(a_1) N_{L_2/F}(a_2) N_{L_3/F}(a_3) = f$.*

Proof. If $A = (a, b) \otimes (c, d)$, let $q = \langle -a, -b, ab, c, d, -cd \rangle$. We are then in the situation of corollary 4.3. The form q is anisotropic. Since every element in $\mathrm{SO}(q)$ is a product of 6 reflections, it follows that every element in $\mathrm{S\Gamma}(q)$ is a product of 3 plane elements x_i. Their images $a_i = \pi(x_i)$ in A^* do the job. $\square$

We will have to make use of the following variant of the Cartan-Dieudonné theorem for anisotropic forms.

4.5. Lemma. *Let $q \colon V \to F$ be anisotropic of dimension n and let $V_1 \subset V_2 \subset \cdots \subset V_n = V$ be a sequence of subspaces with $\dim V_i = i$. Then every $\alpha \in \mathrm{O}(q)$ is a product*

$$\alpha = \tau_{i_k} \cdots \tau_{i_1}$$

where $1 \le i_1 < \cdots < i_k \le n$ and τ_j is a reflection in $\mathrm{O}(q|V_j)$.

Proof. By induction on n. Let $v \neq 0$ be orthogonal to V_{n-1}. If $\alpha \notin \mathrm{O}(q|V_{n-1})$, let τ be the reflection at the hyperplane orthogonal to $\alpha(v) - v$. Then $\tau\alpha \in \mathrm{O}(q|V_{n-1})$. $\square$

5. $\operatorname{Ker} \eta_2^4$ and $H^2(X, \mathcal{K}_3)$

The following theorem reduces the computation of $\operatorname{Ker} \eta_2^4$ for $\dim q \ge 5$ to a $\mathcal{K}$-cohomology computation. The statement is essentially a reformulation of arguments in [27], where the result has been applied to Albert forms. The principal idea has already been used in [16] for 5-dimensional Pfister neighbours.

5.1. Theorem. *Let X be a smooth quadric of dimension ≥ 3. Then there exist isomorphisms*

$$\rho_3 \colon \operatorname{Ker}(H^2(X, \mathcal{K}_2) \to H^2(\overline{X}, \mathcal{K}_2)) \xrightarrow{\sim} \operatorname{Ker}(H^3 F \to H^3 F(X)),$$

$$\rho_4 \colon \operatorname{Ker}(H^2(X, \mathcal{K}_3) \to H^2(\overline{X}, \mathcal{K}_3)) \xrightarrow{\sim} \operatorname{Ker}(H^4 F \to H^4 F(X)).$$

These are functorial with respect to base change $F \to F'$ and with respect to transfer for finite extensions F'/F.

Moreover they commute with multiplication with K_1 in the following sense:

$$\rho_4(\{a\}\cdot x) = (a)\cdot \rho_3(x)$$

for $a \in F^*$ and $x \in \operatorname{Ker}(H^2(X,\mathcal{K}_2) \to H^2(\overline{X},\mathcal{K}_2))$.

Proof. The (inverse of the) map ρ_4 has been described in [27, p. 71] in terms of a diagram chase. It is also proven that the map is an isomorphism (it should be noted that condition (7) in [27, p. 71] is superfluous). From this definition the compatibility with respect to base change and transfer is easy to see.

One proceeds in a similar way for ρ_3 (see also [16, diagrams (3.5), (4.2)]), just by repeating the diagram chase in one degree less. The product formula is then obvious. □

Remark. It can be shown that the maps ρ_i are functorial with respect to linear morphisms $j\colon X \subset X'$ of quadrics. Namely then $\operatorname{Ker}\eta^i_{X'} \subset \operatorname{Ker}\eta^i_X$ (proposition 2.5 c)) and the maps ρ_i commute with these inclusions and with the pull back maps j^* on $H^2(-,\mathcal{K}_{i-1})$.

Our aim is to compute the map ρ_4 for $\dim X = 3$.

Let q be a form defining X and let $A = C_0(q)$ be its even Clifford algebra (a biquaternion algebra). We denote by $h \in CH^1(X)$ the class of a hyperplane section. If A is split, then X is split. In this case $CH^2(X) = \mathbf{Z}$, generated by the class l of a line and one has $h^2 = 2l$.

5.2. Theorem. *For $\dim X = 3$ there exist surjections*

$$\theta_3\colon K_0A \oplus K_0F \twoheadrightarrow H^2(X,\mathcal{K}_2),$$
$$\theta_4\colon K_1A \oplus K_1F \twoheadrightarrow H^2(X,\mathcal{K}_3),$$

functorial with respect to base change and transfer, compatible with multiplication with K_1, and such that θ_3 has the following properties:

a) *$\theta_3(0,1) = -h^2$.*

b) *If A is split, let $y \in K_0A$ be the element with $\operatorname{Nrd}(y) = 1$. Then $\theta_3(y,0) = l$.*

Proof. All the essential arguments are in [27, p. 74–76]. In fact, in the 5-dimensional case the situation is somewhat simpler. By [27, prop. 2],

$$H^2(X,\mathcal{K}_3) = K_1(X)^{(2)}/K_1(X)^{(3)}$$

where the superscripts refer to the filtration coming from the Brown-Gersten-Quillen (BGQ) spectral sequence [41]. In our case one finds $K_1(X)^{(2)} = K_1A \oplus K_1F$. (We don't have to care about a computation of $K_1(X)^{(3)}$). □

Let

$$\operatorname{Res}_{\overline{F}/F}\colon H^2(X,\mathcal{K}_i) \to H^2(\overline{X},\mathcal{K}_i)$$

be the restriction to a separable closure. Theorem 5.2 a), b) shows

$$\operatorname{Res}_{\overline{F}/F}\circ\theta_3(y,m) = (\operatorname{Nrd}(y) - 2m)\cdot l \in CH^2(\overline{X}) = \mathbf{Z}\cdot l.$$

The compatibility of θ_3, θ_4 with multiplication with K_1 yields

$$\operatorname{Res}_{\overline{F}/F}\circ\theta_4(z,\{f\}) = (\operatorname{Nrd}(z) - 2\{f\})\cdot l \in H^2(\overline{X},\mathcal{K}_3) = K_1\overline{F}\cdot l.$$

This together with theorem 5.1 and theorem 5.2 yields

5.3. Corollary. *There are surjections*

$$\omega_3\colon \operatorname{Ker}(K_0A \oplus K_0F \xrightarrow{(\mathrm{Nrd},-2)} K_0F) \twoheadrightarrow \operatorname{Ker}\eta_2^3,$$
$$\omega_4\colon \operatorname{Ker}(K_1A \oplus K_1F \xrightarrow{(\mathrm{Nrd},-2)} K_1F) \twoheadrightarrow \operatorname{Ker}\eta_2^4,$$

compatible with base change, transfer and multiplication with K_1. □

6. Computations for dim $X = 3$

Let $q\colon W \to F$ be 5-dimensional and $A = C_0(q)$. We assume that the determinant of q is -1. Put

$$\widehat{q}\colon F \oplus W \to F$$
$$\widehat{q}(t, w) = t^2 + q(w).$$

Then $C_0(\widehat{q}) = A \times A^{\mathrm{op}}$ as in (4.1).

We first compute ω_3, ω_4 in the case when $\widehat{q}$ is isotropic. In this case q represents -1 and therefore

$$q \simeq \langle -1 \rangle \perp c\langle\langle a, b\rangle\rangle$$

for some a, b, $c \in F^*$. Moreover $A = M_2(F) \otimes D$ where D is the quaternion algebra associated to a, b.

6.1. Proposition. *Let* $y \in K_0A = K_0D$ *be the element represented by the* D*-module* D *(i.e.,* $\mathrm{Nrd}(y) = 2$*) and let* $\varepsilon = (y, 1)$.

(1) *The sequences*

$$K_0F \xrightarrow{\cdot\,\varepsilon} K_0A \oplus K_0F \xrightarrow{(\mathrm{Nrd},-2)} K_0F,$$
$$K_1F \xrightarrow{\cdot\,\varepsilon} K_1A \oplus K_1F \xrightarrow{(\mathrm{Nrd},-2)} K_1F$$

are exact.

(2) *One has*

$$\omega_3(\varepsilon) = (a, b, c),$$
$$\omega_4(\{e\} \cdot \varepsilon) = (e, a, b, c), \qquad e \in F^*.$$

Proof. (1) follows from the injectivity of the reduced norm $K_iD \to K_iF$ for $i = 0$, 1.

Corollary 5.3 shows that $\omega_3(\varepsilon)$ generates $\operatorname{Ker}\eta_2^3$. Since $(a, b, c) \in \operatorname{Ker}\eta_2^3$, we must have the first formula in (2). (If $(a, b, c) = 0$, then q is isotropic and $\operatorname{Ker}\eta_2^3 = 0$.) The second formula follows from the compatibility with multiplication. □

We now assume that $\widehat{q}$ is anisotropic. Via corollary 4.3, the map ω_4 of corollary 5.3 reads as a map

$$\tilde{\omega}_4\colon \mathrm{S}\Gamma(\widehat{q}) \twoheadrightarrow \operatorname{Ker}\eta_2^4.$$

We compute this map for plane elements $x \in \mathrm{S}\Gamma(\widehat{q})$.

Suppose $x \in L = C_0(\tilde{q})$ for a 2-dimensional subform $\tilde{q}$ of $\widehat{q}$. The map

$$L^* = C_0(\tilde{q})^* = \mathrm{S}\Gamma(\tilde{q}) \subset \mathrm{S}\Gamma(\widehat{q}) \to A^* \times F^*$$

coincides with the map

$$K_1L \to K_1A \oplus K_1F$$

given by the inclusion $L \subset A$ and the norm for L/F.

The form $\widehat{q}_L$ is isotropic. Denote by

$$\varepsilon_L \in \operatorname{Ker}(K_0(A \otimes L) \oplus K_0 L \xrightarrow{(\mathrm{Nrd},-2)} K_0 L)$$

the generator considered above.

By compatibility with transfer for L/F one finds

$$\tilde{\omega}_4(x) = \operatorname{Cor}_{L/F}((x) \cdot \omega_3(\varepsilon_L))$$

Note that if $\tilde{q}$ is a subform of q, then q_L is isotropic and therefore $\omega_3(\varepsilon_L) = 0$.

We compute $\omega_4(u)$ for arbitrary $u \in \mathrm{S}\Gamma(\widehat{q})$. If $u \in \mathrm{S}\Gamma(q)$, then u is a product of plane elements in $\mathrm{S}\Gamma(q)$ and $\tilde{\omega}_4(u) = 0$ by the last remark. In general we have

6.2. Lemma. *For any $u \in S\Gamma(q)$, there exist a one-dimensional subspace H of W and a plane element $x \in S\Gamma(\hat{q}|(F \oplus H))$ such that $u \in x \cdot S\Gamma(q)$.*

Proof. There exist by lemma 4.5 and the exact sequence (4.2) a vector $v_1 \in F \oplus W$ and vectors $v_2, \ldots, v_6 \in W$ such that $u = v_1 \cdots v_6$.

Write $v_1 = s(1,0) + th$ with s, $t \in F^*$, $0 \neq h \in W$, and let H be the subspace generated by h. Then $x = v_1 h$ has the desired properties. □

Since $\omega_4 | \mathrm{S}\Gamma(q) = 0$, we are reduced to computing $\tilde{\omega}_4$ on plane elements $x \in S\Gamma(\hat{q}|F \oplus H)$ as in lemma 6.2. Let h be a basis vector of H and let $a = -q(h)$. Then $L = C_0(\tilde{q}) = F(\sqrt{a})$ and q is of the form

$$q \simeq \langle -a, a_1, a_2, a_3, aa_1a_2a_3 \rangle.$$

Over L this gives

$$q_L \simeq \langle -1, a_1, a_2, a_3, a_1a_2a_3 \rangle \simeq \langle -1 \rangle \perp a_1 \langle\!\langle -a_1a_2, -a_1a_3 \rangle\!\rangle.$$

Hence

$$\omega_3(\varepsilon_L) = \operatorname{Res}_{L/F}((a_1, -a_1a_2, -a_1a_3))$$

comes from the ground field (although ε_L does not). So

$$\tilde{\omega}_4(x) = \operatorname{Cor}_{L/F}((x) \cdot \omega_3(\varepsilon_L)) = (N_{L/F}(x), a_1, -a_1a_2, -a_1a_3)$$

is a symbol.

We have proved:

6.3. Theorem. *If $\dim X = 3$, then $\operatorname{Ker} \eta_2^4$ consists of symbols.* □

Remark. The homomorphism $\tilde{\omega}_4$ is obviously trivial on the commutator subgroup $[\mathrm{S}\Gamma(\widehat{q}), \mathrm{S}\Gamma(\widehat{q})]$. Moreover it is easy to see that the map

$$\mathrm{S}\Gamma(\widehat{q}) / \mathrm{S}\Gamma(q) \to \mathrm{O}(\widehat{q}) / \mathrm{O}(q)$$

is bijective. Hence the map $\tilde{\omega}_4$ actually induces a surjective homomorphism

$$\Omega\colon\ \mathrm{O}(\widehat{q})/[\mathrm{O}(\widehat{q}), \mathrm{O}(\widehat{q})]\mathrm{O}(q) \twoheadrightarrow \ker(H^4F \to H^4F(X))$$

Merkurjev and Tignol have described this map in terms of symplectic involutions [28, 52].

The kernel of the spinor norm

$$\mathrm{O}(\widehat{q})/[\mathrm{O}(\widehat{q}), \mathrm{O}(\widehat{q})] \to F^*/(F^*)^2,$$

can be identified with SK_1A. The restriction of Ω to SK_1A depends only on $\widehat{q}$ (and not on the choice of 5-dimensional subform q). It fits into an exact sequence [27]

$$0 \to SK_1A \xrightarrow{\Omega} H^4F \to H^4F(\widehat{X})$$

where $\widehat{X}$ is the quadric for $\widehat{q}$.

Further details will appear in [25].

Part III. Coker η_2^i for higher-dimensional quadrics

7. A relationship between Ker η_2^i and Coker η_2^i

In this section, we prove a general result which we then apply to quadrics. To start with, let F have arbitrary characteristic exponent p and X be a smooth variety over F. We denote by $\delta = \delta(X/F)$ the prime-to-p part of the *gcd* of degrees of closed points of X. Let n be some integer divisible by δ and prime to p, $i \geq 1$ and let

$$\eta_n^i \colon H^i(F, \mu_n^{\otimes(i-1)}) \to H^0(X, \mathcal{H}^i(\mu_n^{\otimes(i-1)}))$$
$$\eta^i \colon H^i(F, i-1) \to H^0(X, \mathcal{H}^i(i-1))$$

be the natural maps.

7.1. Proposition. *Let m be another integer divisible by δ and prime to p. For $i \leq 3$, there is an exact sequence*

$$0 \to \operatorname{Ker} \eta_m^i \cap \operatorname{Im}\left(H^i(F, \mu_{mn}^{\otimes(i-1)}) \xrightarrow{n} H^i(F, \mu_m^{\otimes(i-1)})\right) \to \operatorname{Coker} \eta_n^i \to \operatorname{Coker} \eta^i$$

If m, n are powers of 2, this result also holds for $i = 4$.

Proof. We shall freely use the following fact, which is a consequence of Kummer theory for $i = 2$ (*resp.* the Merkurjev-Suslin theorem for $i = 3$, the Merkurjev-Suslin/Rost theorem for $i = 4$): the map $H^i(F, \mu_n^{\otimes(i-1)}) \xrightarrow{\iota_*} H^i(F, i-1)$ is injective, where $\iota' \colon \mu_n^{\otimes(i-1)} \to \mathbf{Q}/\mathbf{Z}(i-1)$ is the natural inclusion. We need:

7.2. Lemma. *Under the assumptions of proposition* 7.1*,*

a) $\operatorname{Ker} \eta_n^i$ *and* $\operatorname{Ker} \eta_{mn}^i$ *are killed by* δ*.*
b) $\operatorname{Ker} \eta_n^i \to \operatorname{Ker} \eta_{mn}^i$ *is an isomorphism.*
c) $\operatorname{Ker}(\operatorname{Coker} \eta_n^i \to \operatorname{Coker} \eta_{mn}^i) = \operatorname{Ker}(\operatorname{Coker} \eta_n^i \to \operatorname{Coker} \eta^i)$.

Proof. a) If $X(F) \neq \emptyset$, this follows from a classical specialization argument; in general, it follows from this case and a transfer argument.

b) follows from a).

c) We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \to & H^i(F, \mu_n^{\otimes(i-1)}) & \to & H^i(F, i-1) & \xrightarrow{n} & H^i(F, i-1) \\
& \Big\downarrow{\scriptstyle \eta_n^i} & & \Big\downarrow{\scriptstyle \eta^i} & & \Big\downarrow{\scriptstyle \eta^i} \\
0 \to & H^0(X, \mathcal{H}^i(\mu_n^{\otimes(i-1)})) & \to & H^0(X, \mathcal{H}^i(i-1)) & \xrightarrow{n} & H^0(X, \mathcal{H}^i(i-1))
\end{array}$$

where the exactness of the bottom row follows from a diagram chase and the injectivity of $H^{i-1}(F(x), \mu_n^{\otimes(i-2)}) \to H^{i-1}(F(x), i-2)$ for all codimension 1 points of X. Let $x \in \operatorname{Ker}(\operatorname{Coker} \eta_n^i \to \operatorname{Coker} \eta^i)$, $\tilde{x}$ a lift of x in $H^0(X, \mathcal{H}^i(\mu_n^{\otimes(i-1)}))$ and pick $\alpha \in H^i(F, i-1)$ such that $\eta^i \alpha =$ image of $\tilde{x}$. Since $n\alpha \in \operatorname{Ker} \eta^i$, we have $\delta n \alpha = 0$ by a), hence $mn\alpha = 0$. If β is the corresponding element of $H^i(F, \mu_{mn}^{\otimes(i-1)})$, it is clear that $\eta_{mn}^i \beta$ is the image of $\tilde{x}$ in $H^0(X, \mathcal{H}^i(\mu_{mn}^{\otimes(i-1)}))$. $\square$

Let $\iota\colon \mu_n^{\otimes(i-1)} \to \mu_{mn}^{\otimes(i-1)}$ (*resp.* $\pi\colon \mu_{mn}^{\otimes(i-1)} \to \mu_m^{\otimes(i-1)}$) denote the natural inclusion (*resp.* projection). Consider the commutative diagram with exact rows:

$$\begin{CD}
0 @>>> H^i(F,\mu_n^{\otimes(i-1)}) @>{\iota_*}>> H^i(F,\mu_{mn}^{\otimes(i-1)}) @>{\pi_*}>> H^i(F,\mu_m^{\otimes(i-1)})_0 @>>> 0\\
@. @V{\eta_n^i}VV @V{\eta_{mn}^i}VV @V{\eta_m^i}VV\\
0 @>>> H^0(X,\mathcal{H}^i(\mu_n^{\otimes(i-1)})) @>{\iota_*}>> H^0(X,\mathcal{H}^i(\mu_{mn}^{\otimes(i-1)})) @>{\pi_*}>> H^0(X,\mathcal{H}^i(\mu_m^{\otimes(i-1)}))
\end{CD}$$

where $H^i(F,\mu_m^{\otimes(i-1)})_0 = \operatorname{Im}\pi_*$. From lemma 7.2 b) and the snake lemma applied to this diagram, we derive an exact sequence

$$0 \to \operatorname{Ker}\eta_m^i \cap \operatorname{Im}\Big(H^i(F,\mu_{mn}^{\otimes(i-1)}) \xrightarrow{n} H^i(F,\mu_m^{\otimes(i-1)})\Big) \to \operatorname{Coker}\eta_n^i \to \operatorname{Coker}\eta_{mn}^i.$$

Proposition 7.1 now follows from lemma 7.2 c). □

7.3. Proposition. *Let* $\operatorname{char} F \neq 2$, $i \geq 0$ *and* $\alpha \in H^iF$ *be a decomposable element (i.e., a sum of symbols). Then* $\alpha \in \operatorname{Im}(H^i(F,\mu_4^{\otimes(i-1)}) \xrightarrow{p} H^iF)$ *if and only if* $(-1)\cdot\alpha = 0 \in H^{i+1}F$. *Here* p *is the map induced by the surjective homomorphism of Galois modules* $\mu_4^{\otimes(i-1)} \to \mu_2^{\otimes(i-1)} \simeq \mathbf{Z}/2$.

Proof. Let ∂ (*resp.* ∂') be the boundary map $H^iF \to H^{i+1}F$ corresponding to the short exact sequence $0 \to \mathbf{Z}/2 \to \mu_4^{\otimes i} \to \mathbf{Z}/2 \to 0$ (*resp.* $0 \to \mathbf{Z}/2 \to \mu_4^{\otimes(i-1)} \to \mathbf{Z}/2 \to 0$). It follows from [18, lemma 1] that $\partial\alpha - \partial'\alpha = (-1)\cdot\alpha$. If α is decomposable, then it obviously comes from $H^i(F,\mu_4^{\otimes i})$ (reduce to a symbol $(a_1)\cdot \ldots \cdot (a_i)$, where $a_k \in F^*$, and observe that each (a_k) comes from the Kummer class of a_k in $H^1(F,\mu_4)$), hence $\partial\alpha = 0$ and the lemma follows. □

7.4. Proposition. *Suppose* $\operatorname{char} F \neq 2$ *and* $\delta = 2$ *in proposition* 7.1 (*e.g.,* X *is a quadric*). *For* $i \leq 4$, *the group*

$$\operatorname{Ker}\eta_2^i \cap \operatorname{Im}\Big(H^i(F,\mu_4^{\otimes(i-1)}) \xrightarrow{2} H^iF\Big)$$

coincides with the group $(\operatorname{Ker}\eta_2^i)_0$ *of the introduction. We have an exact sequence*

$$0 \to (\operatorname{Ker}\eta_2^i)_0 \to \operatorname{Coker}\eta_2^i \to \operatorname{Coker}\eta^i.$$

Proof. Let q be a quadratic form defining X and $q' = \langle a,b\rangle$ be any binary subform of q. Then $\operatorname{Ker}\eta_2^i \subseteq \operatorname{Ker}(H^iF \to H^iF(\sqrt{-ab})) = (-ab)\cdot H^{i-1}F$. Since $H^{i-1}F$ is generated by symbols [31, 32], we see that every element of $\operatorname{Ker}\eta_2^i$ is decomposable. Finally, the last claim follows from this and proposition 7.1. □

Remark. The arguments in this section show in fact that proposition 7.4 follows for a given i from the Milnor conjecture in degree i (see the remark at the end of introduction).

8. Constructing some maps

In this section, we show how the Chow ring acts on étale and $\mathcal{H}$-cohomology with divisible coefficients. This action will be repeatedly used in the sequel.

Let X be a smooth variety over F, $r \geq 1$, i, $j \geq 0$ and k, $k' \in \mathbf{Z}$. We first recall that Galois cohomology of F acts on the Hochschild-Serre and coniveau spectral sequences

$$HS_2^{p,q}(\mathbf{Z}/2^r(k)) := H^p(F, H^q(\overline{X}, \mathbf{Z}/2^r(k))) \Rightarrow H^{p+q}(X_{\text{ét}}, \mathbf{Z}/2^r(k))$$
$$BO_1^{p,q}(\mathbf{Z}/2^r(k)) := \coprod_{x\in X^{(p)}} H^{q-p}(F(x), \mathbf{Z}/2^r(k-p)) \Rightarrow H^{p+q}(X_{\text{ét}}, \mathbf{Z}/2^r(k))$$

in the following sense. There are pairings

$$H^n(F, \mathbf{Z}/2^r(j)) \otimes HS_2^{p,q}(\mathbf{Z}/2^r(k)) \to HS_2^{p+n,q}(\mathbf{Z}/2^r(j+k))$$
$$H^n(F, \mathbf{Z}/2^r(j)) \otimes BO_1^{p,q}(\mathbf{Z}/2^r(k)) \to BO_1^{p,q+n}(\mathbf{Z}/2^r(j+k))$$

coming from the cup-product in Galois cohomology (and the restriction maps $H^*(F) \to H^*(F(x))$ for the coniveau spectral sequence). These pairings commute with differentials and induce compatible pairings

$$\begin{array}{l} H^n(F, \mathbf{Z}/2^r(j)) \otimes HS_s^{p,q}(\mathbf{Z}/2^r(k)) \to HS_s^{p+n,q}(\mathbf{Z}/2^r(j+k)) \\ H^n(F, \mathbf{Z}/2^r(j)) \otimes BO_s^{p,q}(\mathbf{Z}/2^r(k)) \to BO_s^{p,q+n}(\mathbf{Z}/2^r(j+k)) \end{array} \quad \text{for } s \le \infty$$

which are also compatible at the E_∞-level with the pairings on the abutments

$$H^n(F, \mathbf{Z}/2^r(j)) \otimes H^{p+q}(X_{\text{ét}}, \mathbf{Z}/2^r(k)) \to H^{p+q+n}(X_{\text{ét}}, \mathbf{Z}/2^r(j+k))$$

(obtained from the maps $H^i(F, \mathbf{Z}/2^r(k)) \to H^i(X, \mathbf{Z}/2^r(k))$ and cup-product in the étale cohomology of X). This is classical for the Hochschild-Serre spectral sequence; for the coniveau spectral sequence, it is immediate from its exact couple construction out of étale cohomology with supports (this supersedes the sophisticated argument of [17, end of introduction]).

If E/F is a finite extension, there are also compatible transfer maps.

We now play with these pairings a little bit. Using the cycle map

$$CH^j X/2^r \to H^{2j}_{\text{ét}}(X, \mathbf{Z}/2^r(j))$$

we get composite pairings

$$H^i(F, \mathbf{Z}/2^r(k)) \otimes CH^j X \to H^i(F, \mathbf{Z}/2^r(k)) \otimes H^{2j}_{\text{ét}}(X, \mathbf{Z}/2^r(j)) \xrightarrow{\cup} H^{i+2j}_{\text{ét}}(X, \mathbf{Z}/2^r(k+j))$$

hence, at the inductive limit:

$$H^i(F, k) \otimes CH^j X \to H^{i+2j}(X, k+j). \tag{8.1}$$

Recall now that $BO_2^{j,j}(\mathbf{Z}/2^r(j)) \simeq CH^j X/2^r$. We get therefore a pairing (similar to those in proposition 2.2):

$$H^i(F, \mathbf{Z}/2^r(k)) \otimes CH^j X \to H^j\big(X, \mathcal{H}^{i+j}(\mathbf{Z}/2^r(k+j))\big).$$

Passing to the inductive limit, we get pairings:

$$H^i(F, k) \otimes CH^j X \to H^j(X, \mathcal{H}^{i+j}(k+j)). \tag{8.2}$$

It is not difficult to check that the maps of proposition 2.4 are special cases of (8.2).

We have:

8.1. Lemma. *Let $(H^*(X, k+j)^{(m)})_{m\ge 0}$ be the decreasing filtration on $H^*(X, k+j)$ defined by the coniveau spectral sequence. Then*

a) *The image of the map (8.1) is contained in $H^{i+2j}(X, k+j)^{(j)}$.*

b) *The image of the map (8.2) consists of universal cycles.*

c) *The diagram*

$$\begin{array}{ccc} H^i(F,k)\otimes CH^jX & \longrightarrow & H^j(X,\mathcal{H}^{i+j}(k+j))_{d=0}\\ \Big\downarrow & & \Big\downarrow\\ H^{i+2j}(X,k+j)^{(j)} & \longrightarrow & BO_\infty^{j,i+j}\end{array}$$

commutes, where the subscript $d=0$ denotes universal cycles and $BO_\infty^{j,i+j}$ denotes the E_∞-term of the coniveau spectral sequence.

d) *For $i=0,1$, the triangle*

$$\begin{array}{ccc} H^i(F,k)\otimes CH^jX & \longrightarrow & H^j(X,\mathcal{H}^{i+j}(k+j))\\ \Big\downarrow & \swarrow & \\ H^{i+2j}(X,k+j) & & \end{array}$$

commutes, where the vertical map is (8.1)*, the horizontal map is* (8.2) *and the diagonal map is the edge homomorphism from the coniveau spectral sequence.*

Recall that, in a spectral sequence (E_r), an element $x\in E_r^{p,q}$ is called a *universal cycle* if $d_s^{p,q}x=0$ for all $s\geq r$ (this definition makes sense inductively).

Proof. It is enough to prove this for the maps defined at the finite coefficients level. By the multiplicativity of the coniveau spectral sequence, we are reduced to the case $i=k=0$. Then a), b), and c) follow from the fact that $BO_1^{p,q}=0$ for $p>q$. Finally, d) follows from c) and the definition of the edge homomorphism. □

Let A be a discrete topological G_F-module and M an abelian group. There is a pairing

$$H^i(F,A)\otimes M\to H^i(F,A\otimes M)$$

for all $i\geq 0$, coming from the natural map $M\to \mathrm{Hom}_{G_F}(A,A\otimes M)$.

8.2. Lemma.

a) *If M is torsion-free, the above pairing is an isomorphism.*
b) *If A is divisible, it is surjective.*
c) *If A and $H^i(F,A)$ are divisible, it is bijective.*

Proof. a) Reduce to the case where M is finitely generated, hence free.

b) Reduce to the case where M is torsion-free by using the commutative diagram:

$$\begin{array}{ccc} H^i(F,A)\otimes M & \twoheadrightarrow & H^i(F,A)\otimes(M/\,\mathrm{torsion})\\ \Big\downarrow & & \wr\Big\downarrow\\ H^i(F,A\otimes M) & \xrightarrow{\sim} & H^i(F,A\otimes(M/\,\mathrm{torsion})).\end{array}$$

c) Use the same diagram, noting that now the top horizontal map is bijective as well. □

Define a map

$$\alpha=\alpha^{i,j,k}\colon H^i(F,k)\otimes CH^jX\to H^i(F,H^{2j}(\overline{X},j+k)) \tag{8.3}$$

as the composite

$$\begin{aligned}H^i(F,k)\otimes CH^jX &\to H^i(F,CH^jX\otimes\mathbf{Q}/\mathbf{Z}(k))\\ &\to H^i(F,CH^j\overline{X}\otimes\mathbf{Q}/\mathbf{Z}(k))\xrightarrow{H^i(F,cl'^j_{\overline{X}}(k))} H^i(F,H^{2j}_{\text{ét}}(\overline{X},j+k)).\end{aligned}$$

We have:

8.3. Lemma. *Let $(H^*(X,k+j)^{[m]})_{m\geq 0}$ be the decreasing filtration on $H^*(X,k+j)$ defined by the Hochschild-Serre spectral sequence. Then*

a) *The image of the map* (8.1) *is contained in $H^{i+2j}(X,k+j)^{[i]}$.*
b) *The image of $\alpha^{i,j,k}$ consists of universal cycles.*
c) *The diagram*

$$\begin{tikzcd} H^i(F,k)\otimes CH^jX \arrow[r, "\alpha^{i,j,k}"] \arrow[d] & H^i(F,H^{2j}(\overline{X},k+j))_{d=0} \arrow[d] \\ H^{i+2j}(X,k+j)^{[i]} \arrow[r] & HS^{i,2j}_\infty \end{tikzcd}$$

commutes, where the subscript $d=0$ denotes universal cycles and $HS^{i,2j}_\infty$ denotes the E_∞-term of the Hochschild-Serre spectral sequence.

d) *Suppose that $H^{2j+1}(\overline{X},k+j)=0$. Then, for $i=1$, the triangle*

$$\begin{tikzcd} H^1(F,k)\otimes CH^jX \arrow[r, "\alpha^{1,j,k}"] \arrow[d] & H^1(F,H^{2j}(\overline{X},k+j)) \\ H^{1+2j}(X,k+j) \arrow[ur] & \end{tikzcd}$$

commutes, where the vertical map is (8.1) *and the diagonal map is the edge homomorphism from the Hochschild-Serre spectral sequence.*

Proof. We argue as above: it is enough to prove this for the maps defined at the finite coefficients level (we could have defined α as a limit, as before). By the multiplicativity of the Hochschild-Serre spectral sequence, we are reduced to the case $i=k=0$. Then a) is trivial, and b) and c) hold because $\alpha^{0,j,0}$ factors through the (F-rational) cycle map. Finally, d) follows as in the proof of lemma 8.1 from c) and the definition of the edge homomorphism. □

The map α is the most important of the three maps introduced above in (8.1), (8.2), and (8.3). We are interested in its kernel and cokernel.

8.4. Lemma. *If the geometric divisible cycle map is bijective, then*

$$\operatorname{Coker}\alpha^{i,j,k}\simeq\operatorname{Coker}\big(H^i(F,CH^jX\otimes\mathbf{Q}/\mathbf{Z}(k))\to H^i(F,CH^j\overline{X}\otimes\mathbf{Q}/\mathbf{Z}(k))\big).$$

If moreover G_F acts trivially on $CH^j\overline{X}$, this cokernel is a quotient of

$$\operatorname{Coker}(H^i(F,k)\otimes CH^jX\to H^i(F,k)\otimes CH^j\overline{X}).$$

Proof. The first claim follows from the definition of $\alpha^{i,j,k}$ and lemma 8.2. The second one follows from the commutative diagram

$$\begin{tikzcd} H^i(F,k)\otimes CH^jX \arrow[r, two heads] \arrow[d] & H^i(F,CH^jX\otimes\mathbf{Q}/\mathbf{Z}(k)) \arrow[d] \\ H^i(F,k)\otimes CH^j\overline{X} \arrow[r, two heads] & H^i(F,CH^j\overline{X}\otimes\mathbf{Q}/\mathbf{Z}(k)). \end{tikzcd}$$ □

8.5. Lemma. *Let X be a quadric and $k = i \le 3$. Then*

a) *If* $\dim X \neq 2j$ *or* $\dim X = 2j$ *and* $d(X) = 1$, $\alpha^{i,j,i}$ *is surjective.*

b) *Suppose* $\dim X = 2j$ *and* $d(X) = d \neq 1$. *Let* $E = F(\sqrt{d})$. *Then the composition*

$$H^i(E,i) \otimes CH^j X_E \xrightarrow{\alpha_E^{i,j,i}} H^i(E, H^{2j}(\overline{X}, i+j)) \xrightarrow{\mathrm{Cor}_{E/F}} H^i(F, H^{2j}(\overline{X}, i+j))$$

is surjective. For $i = 0$, $\alpha^{0,j,0}$ *is itself surjective.*

c) *If* $\dim X > 2j$, $\alpha^{i,j,i}$ *is bijective.*

Proof. In case a), the geometric cycle map is bijective by proposition 1.5 and G_F acts trivially on $CH^j\overline{X}$ by proposition 1.1, hence, by lemma 8.4, it is enough to see that the map

$$H^i(F,i) \otimes CH^j X \to H^i(F,i) \otimes CH^j\overline{X}$$

is surjective. But, since $i \le 3$, the group $H^i(F,i)$ is divisible by bijectivity of the 2-primary Galois symbol in this case. Surjectivity now follows from corollary 1.2, applied with $D = H^i(F,i)$.

In case b), α_E is surjective by a). It remains to prove that

$$H^i(E, H^{2j}(\overline{X}, i+j)) \xrightarrow{\mathrm{Cor}_{E/F}} H^i(F, H^{2j}(\overline{X}, i+j))$$

is surjective. By the geometric cycle map, the coefficients can be identified with $CH^j\overline{X} \otimes \mathbf{Q}/\mathbf{Z}(i) \simeq \mathrm{Ind}_{G_E}^{G_F} \mathbf{Q}/\mathbf{Z}(i)$ by corollary 1.3. Using Shapiro's lemma, the corestriction map can therefore be identified with

$$H^i(E,i) \oplus H^i(E,i) \xrightarrow{\Sigma} H^i(E,i)$$

which is obviously surjective. Finally, the last claim is corollary 1.3.

In case c), the three maps out of which α is defined are bijective: the first one is by lemma 8.2 c), the second one by proposition 1.1 b) and the last one by proposition 1.5. □

In section 11 we shall need the following lemma and its corollary for $(j,k) = (2,1)$.

8.6. Lemma. *Let X be a quadric over F. Then:*

a) *If F contains all 2-primary roots of unity, $\alpha^{0,j,k}$ is surjective for any j, k. The same holds if k is even and $F(\sqrt{-1})$ contains all 2-primary roots of unity.*

b) *In general,* $\operatorname{Coker} \alpha^{0,j,k}$ *is*

  (i) 0 *if* $\dim X > 2j$;

  (ii) *cyclic of order* ≤ 2 *if* $\dim X < 2j$;

  (iii) *cyclic of order* $\le 2^t$ *if* $\dim X = 2j$ *and* $d(X) = 1$, *where t is the integer of proposition* 1.1 d).

Proof. By lemma 8.4,

$$\operatorname{Coker} \alpha^{0,j,k} \simeq \operatorname{Coker}\big(H^0(F, CH^j X \otimes \mathbf{Q}/\mathbf{Z}(k)) \to H^0(F, CH^j\overline{X} \otimes \mathbf{Q}/\mathbf{Z}(k))\big).$$

Assume first that we are not in the case $\dim X = 2j$ and $d = d(X) \neq 1$. Applying proposition 1.1 and corollary 1.2 with $D = \mathbf{Q}/\mathbf{Z}(k)$, we see that the map on coefficients

$$CH^j X \otimes \mathbf{Q}/\mathbf{Z}(k) \to CH^j\overline{X} \otimes \mathbf{Q}/\mathbf{Z}(k)$$

can be identified with multiplication by $\left|\operatorname{Coker}(CH^jX \to CH^j\overline{X})\right|$ on $\mathbf{Q}/\mathbf{Z}(k)$, except if $\dim X = 2j$ and $d(X) = 1$, where it is $\operatorname{diag}(1, 2^t)$ on $\mathbf{Q}/\mathbf{Z}(k) \oplus \mathbf{Q}/\mathbf{Z}(k)$ with t as in prop. 1.1 d). If F contains all 2-primary roots of unity, this shows that $\alpha^{0,j,k}$ is surjective, and similarly if k is even and $F(\sqrt{-1})$ contains all 2-primary roots of unity. For arbitrary F, if $\dim X > 2j$ then $CH^jX \to CH^j\overline{X}$ is surjective (prop. 1.1) hence $\alpha^{0,j,k}$ is an isomorphism. If $\dim X < 2j$ (*resp.* $\dim X = 2j$ and $d(X) = 1$), the map $CH^jX \to CH^j\overline{X}$ has cokernel cyclic of order ≤ 2 (*resp.* $\leq 2^t$), so that $\operatorname{Coker}\alpha^{0,j,k}$ is cyclic of order ≤ 2 (*resp.* $\leq 2^t$).

Assume now that $\dim X = 2j$ and $d = d(X) \neq 1$; let $E = F(\sqrt{d})$. By corollary 1.3, $\alpha^{0,j,k}$ can be identified with the map $H^0(F,k) \to H^0(E,k)$, so is surjective when F and E have the same number of 2-primary roots of unity, in particular if F contains all 2-primary roots of unity; similarly if k is even and $F(\sqrt{-1})$ contains all 2-primary roots of unity. ∎

8.7. Corollary. *Let X be a quadric over F. Consider the composition*

$$\gamma\colon H^j(X, \mathcal{H}^j(j+k)) \to H^{2j}(X, j+k) \to H^0(F, H^{2j}(\overline{X}, j+k))$$

where the first (resp. second) map is an edge homomorphism of the coniveau (resp. Hochschild-Serre) spectral sequence. Then:

a) *If F contains all 2-primary roots of unity or if k is even and $F(\sqrt{-1})$ contains all 2-primary roots of unity, γ is surjective.*
b) *In general,* $\operatorname{Coker}\gamma$ *is:*
 (i) 0 *if* $\dim X > 2j$*;*
 (ii) *cyclic of order* ≤ 2 *if* $\dim X < 2j$*;*
 (iii) *cyclic of order* $\leq 2^t$ *if* $\dim X = 2j$ *and* $d(X) = 1$*, where t is the integer of proposition* 1.1 d)*.*
 (iv) *cyclic of order* $\leq 2^{t_E}$ *if* $\dim X = 2j$ *and* $d(X) = d \neq 1$*, where t_E is the integer of proposition* 1.1 d) *for X_E, with $E = F(\sqrt{d})$.*

Proof. By lemmas 8.1 and 8.3, the composition

$$H^0(F,k) \otimes CH^jX \to H^j(X, \mathcal{H}^j(j+k)) \xrightarrow{\gamma} H^0(F, H^{2j}(\overline{X}, j+k))$$

coincides with $\alpha^{0,j,k}$. Therefore corollary 8.7 follows directly from lemma 8.6, except when $\dim X = 2j$, $d(X) = d \neq 1$ and F does not contain enough roots of unity. But in this case, we have a commutative diagram:

$$\begin{array}{ccccc}
H^0(E,k) \otimes CH^jX_E & \longrightarrow & H^j(X_E, \mathcal{H}^j(j+k)) & \xrightarrow{\gamma_E} & H^0(E, H^{2j}(\overline{X}, j+k)) \\
 & & \Big\downarrow{\scriptstyle \operatorname{Cor}_{E/F}} & & \Big\downarrow{\scriptstyle \operatorname{Cor}_{E/F}} \\
H^0(F,k) \otimes CH^jX & \longrightarrow & H^j(X, \mathcal{H}^j(j+k)) & \xrightarrow{\gamma} & H^0(F, H^{2j}(\overline{X}, j+k))
\end{array}$$

where the left Cor is best described from the E_1-term of the coniveau spectral sequence. By lemma 8.6, $\operatorname{Coker}\gamma_E$ is cyclic of order $\leq 2^{t_E}$. But

$$\operatorname{Cor}_{E/F}\colon H^0(E, H^{2j}(\overline{X}, j+k)) \to H^0(F, H^{2j}(\overline{X}, j+k))$$

is surjective: this follows from the bijectivity of the geometric cycle map and the fact that the Galois module $CH^j\overline{X}$ is isomorphic to $\operatorname{Ind}_{G_E}^{G_F}\mathbf{Z}$. This concludes the proof of corollary 8.7. ∎

9. The case $i = 2$: Proof of theorem 4

Let X be a smooth variety over F such that $H^1(\overline{X}, \mathbf{Q}/\mathbf{Z}) = 0$. We have a cross of exact sequences

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & H^2(F,1) \arrow[d] \arrow[dr, "\eta^2"] & & \\
0 \arrow[r] & H^1(X,\mathcal{H}^1(1)) \arrow[r] \arrow[dr, "\alpha^{0,1,0}"'] & H^2(X,1) \arrow[r] \arrow[d] & H^0(X,\mathcal{H}^2(1)) \arrow[r] & 0 \\
& & H^0(F,H^2(\overline{X},1)) \arrow[d] & & \\
& & H^3(F,1) \arrow[d] & & \\
& & H^3(X,1) & &
\end{tikzcd}$$

where the horizontal (*resp.* vertical) line is given by the coniveau (*resp.* Hochschild-Serre) spectral sequence with coefficients $\mathbf{Q}/\mathbf{Z}(1)$. The group $H^1(X, \mathcal{H}^1(1))$ can be identified with $CH^1X \otimes \mathbf{Q}/\mathbf{Z}$ and $H^1(X, \mathcal{H}^1(1)) \to H^2(X, 1)$ with cl'^1_X. The composite

$$H^1(X, \mathcal{H}^1(1)) \to H^2(X, 1) \to H^0(F, H^2(\overline{X}, 1))$$

defined by the above cross can therefore be identified with the map $\alpha^{0,1,0}$ (8.3) of section 8.

9.1. Proposition. *For X as above, there is an exact sequence*

$$0 \to \operatorname{Coker} \eta^2 \to \operatorname{Coker} \alpha^{0,1,0} \to H^3(F, 1) \to H^3(X, 1).$$

Proof. This follows from the "lemma of the 700th" [33]. □

9.2. Corollary. *For any (projective) quadric X over F, η^2 is surjective.*

Proof. $\alpha^{0,1,0}$ is surjective for all quadrics by lemma 8.5. □

10. The case $i = 3$: Proof of theorem 5

Let X be a smooth variety over F such that $H^1(\overline{X}, \mathbf{Q}/\mathbf{Z}) = H^3(\overline{X}, \mathbf{Q}/\mathbf{Z}) = 0$. In the following cross of exact sequences the (broken) horizontal sequence comes from the coniveau spectral sequence and the vertical sequence from the Hochschild-Serre spectral sequence with coefficients $\mathbf{Q}/\mathbf{Z}(2)$.

$$\begin{tikzcd}
 & & H^3(F,2) \arrow[d] \arrow[dr, "\eta^3"] & \\
0 \arrow[r] & H^1(X,\mathcal{H}^2(2)) \arrow[r] \arrow[dr, "\beta"'] & H^3(X,2) \arrow[r] \arrow[d] & H^0(X,\mathcal{H}^3(2)) \arrow[d] \\
 & & H^1(F,H^2(\overline{X},2)) \arrow[d] & H^2(X,\mathcal{H}^2(2)) \arrow[d] \\
 & & H^4(F,2) \arrow[d] & H^4(X,2) \\
 & & H^4(X,2). &
\end{tikzcd}$$

In this diagram, one can identify the group $H^2(X,\mathcal{H}^2(2))$ with $CH^2X \otimes \mathbf{Q}/\mathbf{Z}$ and accordingly the map $H^2(X,\mathcal{H}^2(2)) \to H^4(X,2)$ with cl'^2_X. The above cross defines a composite homomorphism:

$$\beta\colon H^1(X,\mathcal{H}^2(2)) \to H^3(X,2) \to H^1(F,H^2(\overline{X},2)).$$

10.1. Proposition. *For X as above, there is a natural surjection*

$$\operatorname{Coker}\eta^3 \to \operatorname{Ker} cl'^2_X \to 0.$$

Denote by $(\operatorname{Coker}\eta^3)^1$ *the kernel of this surjection. Then there is a natural exact sequence*

$$0 \to (\operatorname{Coker}\eta^3)^1 \to \operatorname{Coker}\beta \to H^4(F,2) \to H^4(X,2).$$

Proof. The first claim is obvious and the second again follows from the lemma of the 700th [33]. ☐

10.2. Corollary. *For any (projective) quadric X over F, there is a natural isomorphism* $\operatorname{Coker}\eta^3 \xrightarrow{\sim} \operatorname{Ker} cl'^2_X$.

Proof. We prove that β is surjective for all quadrics. For this, we remark that the composition

$$H^1(F,1) \otimes CH^1X \to H^1(X,\mathcal{H}^2(2)) \xrightarrow{\beta} H^1(F,H^2(\overline{X},2)),$$

where the first map is (8.2), coincides with $\alpha^{1,1,1}$ by lemmas 8.1 and 8.3. If $\dim X \neq 2$ or $\dim X = 2$ and $d(X) = 1$, lemma 8.5 says that $\alpha^{1,1,1}$ is surjective, hence β is surjective. If $\dim X = 2$ and $d = d(X) \neq 1$, we use the same trick as in the proof of corollary 8.7: the composite $\operatorname{Cor}_{E/F} \circ \alpha_E^{1,1,1}$ is surjective, still by lemma 8.5, where $E = F(\sqrt{d})$. But there is a transfer map $H^1(X_E,\mathcal{H}^2(2)) \xrightarrow{\operatorname{Cor}_{E/F}} H^1(X,\mathcal{H}^2(2))$ making the following diagram commute:

$$\begin{tikzcd}
H^1(E,1) \otimes CH^1X_E \arrow[r] & H^1(X_E,\mathcal{H}^2(2)) \arrow[r, "\beta_E"] \arrow[d, "\operatorname{Cor}_{E/F}"'] & H^1(E,H^2(\overline{X},2)) \arrow[d, "\operatorname{Cor}_{E/F}"'] \\
H^1(F,1) \otimes CH^1X \arrow[r] & H^1(X,\mathcal{H}^2(2)) \arrow[r, "\beta"] & H^1(F,H^2(\overline{X},2)),
\end{tikzcd}$$

which concludes the proof in this case. ☐

10.3. Proposition (*cf.* [17, th. 3])**.** *For a quadric X over a field of characteristic $\neq 2$, the cycle map*

$$cl'^2_X\colon CH^2X\otimes\mathbf{Q}/\mathbf{Z}\to H^4(X,2)$$

is injective, except if X is defined by an anisotropic Albert form. In this case, $\operatorname{Ker} cl'^2_X\simeq \mathbf{Z}/2$.

Proof. In view of corollary 10.2, we may assume X anisotropic (prop. 2.5). If $\dim X>4$ or $\dim X=4$ and $d(X)\neq 1$, the map

$$CH^2X\otimes\mathbf{Q}/\mathbf{Z}\to H^0(F,CH^2\overline{X}\otimes\mathbf{Q}/\mathbf{Z})$$

is an isomorphism by proposition 1.1 and the claim follows from the injectivity of the geometric cycle map (proposition 1.5). If $\dim X=1$, $CH^2X=0$ and the claim is trivial. If $\dim X=3$ and Y is a hyperplane section of X, then the restriction map $CH^2X\otimes\mathbf{Q}/\mathbf{Z}\to CH^2Y\otimes\mathbf{Q}/\mathbf{Z}$ is an isomorphism by proposition 1.1 again, so that injectivity for $\dim X=2$ implies injectivity for $\dim X=3$. Let us assume $\dim X=2$. As in [17], we may assume $d(X)=1$ since, if $d=d(X)\neq 1$, the map $CH^2X\to CH^2X_{F(\sqrt{d})}$ is bijective (prop. 1.1 and 1.6). In this case, CH^2X is generated by h^2, where $h\in CH^1X$ is the class of a hyperplane section (*ibid.*). Since $\operatorname{Ker} cl'^2_X$ is killed by 2 (cor. 10.2 and prop. 2.5), it is enough to prove:

10.4. Lemma. *If X is an anisotropic quaternion quadric (= defined by a 2-fold Pfister form), the image of $cl^2_X(h^2)\in H^4X$ in $H^4(X,2)$ is $\neq 0$.*

Proof. Indeed, $cl^2_X(h^2)=c\cdot cl^1_X(h)$ by [17, lemma 2], where c is the Brauer class of the quadric, and $cl^1_X(h)\neq 0$ by [51, lemma 5.3.2 c)]. Therefore $cl^2_X(h^2)$ maps to a nonzero element of $H^2(F,H^2\overline{X})\simeq H^2F$. Its image in $H^2(F,H^2(\overline{X},2))\simeq H^2(F,1)$ is not 0 since the map $H^2F\to H^2(F,1)$ is injective (compare beginning of proof of prop. 7.1). Therefore the image of $cl^2_X(h^2)$ in $H^4(X,2)$ cannot be 0. □

It remains to deal with the case of an anisotropic Albert quadric. In this case we have $\operatorname{Ker}\eta^3_2=0$ [1, Satz 5.6] and $\operatorname{Coker}\eta^3_2\simeq\mathbf{Z}/2$ [17, theorem 2 d)], hence from proposition 7.4 and corollary 10.2 we get an injection

$$\mathbf{Z}/2\simeq\operatorname{Coker}\eta^3_2\hookrightarrow\operatorname{Ker} cl'^2_X. \tag{10.1}$$

But propositions 1.5 and 1.1 d) yield a commutative diagram with exact row

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}/4 \arrow[r] & CH^2X\otimes\mathbf{Q}/\mathbf{Z} \arrow[r] \arrow[d, "cl'^2_X"'] & CH^2\overline{X}\otimes\mathbf{Q}/\mathbf{Z} \arrow[r] \arrow[d, "cl'^2_{\overline{X}}"', "\wr"] & 0\\
& & H^4(X,2) \arrow[r] & H^4(\overline{X},2) &
\end{tikzcd}$$

It follows that $\operatorname{Ker} cl'^2_X$ is cyclic; as it is of exponent 2 by corollary 10.2 and proposition 2.5 b), the map (10.1) is an isomorphism. □

Theorem 5 follows from corollary 10.2 and proposition 10.3.

11. The case $i = 4$

This case is similar to the former two, but more complicated.

For any smooth variety X over F define

$$\overline{H}^4(X,3) = \frac{H^4(X,3)}{H^4(F,3) + H^2(X,\mathcal{H}^2(3)) + \bigoplus_{[E:F]<+\infty} H^2(E,2) \otimes CH^1 X_E},$$

where $H^2(X,\mathcal{H}^2(3)) \to H^4(X,3)$ is the edge homomorphism of the coniveau spectral sequence and the map $\bigoplus_{[E:F]<+\infty} H^2(E,2) \otimes CH^1 X_E \to H^4(X,3)$ comes from (8.1) via transfer.

Suppose that $H^1(\overline{X},\mathbf{Q}/\mathbf{Z}) = H^3(\overline{X},\mathbf{Q}/\mathbf{Z}) = 0$. We then have two exact sequences

$$(11.1) \quad 0 \to H^1(X,\mathcal{H}^3(3)) \to \frac{H^4(X,3)}{H^2(X,\mathcal{H}^2(3))} \to H^0(X,\mathcal{H}^4(3)) \longrightarrow \xrightarrow{d_2^{0,4}} H^2(X,\mathcal{H}^3(3)) \to H^5(X,3)$$

(coniveau)

$$(11.2) \quad 0 \to \operatorname{Ker}\Big(H^2(F,H^2(\overline{X},3)) \xrightarrow{d_3^{2,2}} H^5(F,3)\Big) \to \to \frac{H^4(X,3)}{H^4(F,3)} \to H^0(F,H^4(\overline{X},3))$$

(Hochschild-Serre).

The exact sequence (11.2) identifies $\operatorname{Ker} d_3^{2,2}$ to a subgroup of $\dfrac{H^4(X,3)}{H^4(F,3)}$. For the purpose of stating proposition 11.1 below, let $H^2(X,\mathcal{H}^2(3))_0$ denote its intersection with $\operatorname{Im}\left(H^2(X,\mathcal{H}^2(3)) \to H^4(X,3) \to \dfrac{H^4(X,3)}{H^4(F,3)}\right)$ (the first map in this composition being an edge homomorphism of the coniveau spectral sequence).

11.1. Proposition. *For X as above, there are exact sequences*

$$\overline{H}^4(X,3) \to \operatorname{Coker}\eta^4 \to H^2(X,\mathcal{H}^3(3)) \to H^5(X,3)$$

and

$$\frac{\operatorname{Ker}\Big(H^2(F,H^2(\overline{X},3)) \xrightarrow{d_3^{2,2}} H^5(F,3)\Big)}{H^2(X,\mathcal{H}^2(3))_0 + \bigoplus_{[E:F]<+\infty} H^2(E,2) \otimes CH^1 X_E} \to \overline{H}^4(X,3) \to \frac{H^0(F,H^4(\overline{X},3))}{H^2(X,\mathcal{H}^2(3))},$$

where the factor

$$\bigoplus_{[E:F]<+\infty} H^2(E,2) \otimes CH^1 X_E$$

comes from

$$\sum_{[E:F]<+\infty} \operatorname{Cor}_{E/F} \circ \alpha_E^{2,1,2}.$$

Proof. This follows from the exact sequences (11.1) and (11.2), using lemma 8.3. $\square$

Consider the cross of exact sequences

$$\begin{tikzcd}
 & & H^1(F,1)\otimes CH^2X \arrow[d] \arrow[dr, "\nu"'] & \\
\overline{H}^4(X,3) \arrow[r] & \operatorname{Coker}\eta^4 \arrow[r] & H^2(X,\mathcal{H}^3(3)) \arrow[r] \arrow[d] & H^5(X,3) \\
 & & (CH^3X)_{\mathrm{torsion}} \arrow[d] & \\
 & & 0 &
\end{tikzcd} \tag{11.3}$$

coming from propositions 2.4 (for $i=3$) and 11.1. From diagram (11.3) and the lemma of the 700th, we deduce a complex

$$\overline{H}^4(X,3) \to \operatorname{Coker}\eta^4 \to (CH^3X)_{\mathrm{torsion}} \to \operatorname{Coker}\nu \tag{11.4}$$

which is exact at $(CH^3X)_{\mathrm{torsion}}$ and whose homology at $\operatorname{Coker}\eta^4$ is a quotient of $\operatorname{Ker}\nu$.

11.2. Lemma. *Let X be a quadric. Then $\overline{H}^4(X,3)$ is cyclic of order ≤ 4; it is 0 if $\dim X=1$, $\dim X>4$ or if F contains all 2-primary roots of unity.*

Proof. By lemma 8.5, $\alpha^{2,1,2}$ is surjective, except if $\dim X=2$ and $d=d(X)\neq 1$; in this case, $\operatorname{Cor}_{E/F}\circ\alpha_E^{2,1,2}$ is surjective. It follows that for any quadric, the group on the left of $\overline{H}^4(X,3)$ in the second exact sequence of proposition 11.1 is 0. Therefore $\overline{H}^4(X,3)$ injects into $\dfrac{H^0(F,H^4(\overline{X},3))}{H^2(X,\mathcal{H}^2(3))}$. But the latter group is cyclic of order as in the statement of lemma 11.2, by corollary 8.7 and proposition 1.1. □

11.3. Lemma. *If $\dim X=1$ or $\dim X>4$, the map ν of diagram (11.3) is injective. If $\dim X>4$, there is a canonical isomorphism*

$$\operatorname{Coker}\nu \simeq H^5(F,3)\oplus\big(H^3(F,2)\otimes CH^1X\big).$$

Proof. The case $\dim X=1$ is obvious, since then $CH^2X=0$. Suppose $\dim X>4$. By lemma 8.1 d), the map ν coincides with (8.1). By lemma 8.5 c), $\alpha^{1,2,1}$ is bijective. Therefore, by lemma 8.3 d), ν is (split-) injective. This argument shows that, in the Hochschild-Serre spectral sequence for coefficients $\mathbf{Q}/\mathbf{Z}(3)$, the term $E_2^{1,4}$ is isomorphic to $H^1(F,1)\otimes CH^2X$ and consists of universal cycles. The same holds for all the terms $E_2^{p,2}\simeq H^p(F,2)\otimes CH^1X$ by the same argument, since $CH^1X \xrightarrow{\sim} CH^1\overline{X}$ for $\dim X\geq 3$. Finally the term $E_2^{0,4}$ also consists of universal cycles by lemma 11.2. Hence we get a canonical isomorphism:

$$H^5(X,3)\simeq H^5(F,3)\oplus\big(H^3(F,2)\otimes CH^1X\big)\oplus\big(H^1(F,1)\otimes CH^2X\big)$$

and the second part of the lemma follows. □

12. Proof of theorems 6 and 7

Proof of theorem 6. Claim (1) follows from (11.4), lemma 11.2 and lemma 11.3. We now check (2) and (3). Let q be a quadratic form defining the quadric X.

a) If $\dim q = 3$, $\operatorname{Coker} \eta^4 = 0$ by (11.4), lemma 11.2 and lemma 11.3.
b) If $\dim q > 6$, part (1) of theorem 6 implies that $\operatorname{Coker} \eta^4$ injects into the group $(CH^3X)_{\text{torsion}}$, hence has order ≤ 2 by Karpenko's results (proposition 1.6). If $\dim q > 12$, $(CH^3X)_{\text{torsion}} = 0$ by the same results, and so $\operatorname{Coker} \eta^4 = 0$.
c) If $\dim q = 8$, $d_\pm q = 1$, and q is not similar to a Pfister form, then CH^3X is torsion-free by [20, th. 8.3] and $\operatorname{Coker} \eta^4 = 0$ by (1).

The case of a 2-fold Pfister neighbour and that of an i-fold Pfister neighbour for $i \ge 4$ is covered by a) and b) respectively.

It remains to prove claim (3) in the case of a 3-fold neighbour. We may assume the quadric to be of dimension at least 5 by proposition 2.5 c). Then the claim follows from:

12.1. Lemma. *Suppose* $\dim X > 4$ *and* X *is defined by a neighbour of a* 3*-fold Pfister form. Then the map of theorem* 6 (1)

$$(CH^3X)_{\text{torsion}} \to H^5(F,3) \oplus \big(H^3(F,2) \otimes CH^1X\big)$$

is injective.

Proof. By the same argument as in the proof of lemma 11.3, we get a canonical isomorphism

$$H^3(X,2) \simeq H^3(F,2) \oplus \big(H^1(F,1) \otimes CH^1X\big)$$

compatible with the maps of section 8. The diagram analogous to (11.3)

$$\begin{tikzcd} & H^1(F,1) \otimes CH^1X \arrow[d] & \\ 0 \arrow[r] & H^1(X,\mathcal{H}^2(2)) \arrow[r] \arrow[d] & H^3(X,2) \\ & (CH^2X)_{\text{torsion}} \arrow[d] & \\ & 0 & \end{tikzcd}$$

yields an injection

$$(CH^2X)_{\text{torsion}} \hookrightarrow H^3(F,2).$$

Tensoring this injection by CH^1X yields another injection (since CH^1X is torsion-free)

$$(CH^2X)_{\text{torsion}} \otimes CH^1X \hookrightarrow H^3(F,2) \otimes CH^1X.$$

The way these maps are defined shows that the diagram

$$\begin{tikzcd} (CH^2X)_{\text{torsion}} \otimes CH^1X \arrow[r, hook] \arrow[d] & H^3(F,2) \otimes CH^1X \arrow[d] \\ (CH^3X)_{\text{torsion}} \arrow[r] & H^5(F,3) \oplus (H^3(F,2) \otimes CH^1X) \end{tikzcd}$$

commutes, where the left vertical map is multiplication, the right vertical map is inclusion, the top horizontal map has just been defined and the bottom horizontal map comes from theorem 6 (1). Since $CH^1X \simeq \mathbf{Z}$ and $(CH^2X)_{\text{torsion}} \simeq \mathbf{Z}/2$ by propositions 1.1 and 1.6, this shows that the latter map is nonzero. Since, by proposition 1.6, $(CH^3X)_{\text{torsion}}$ has order at most 2, this map is injective, which is what we wanted (moreover, we get that $CH^1X \otimes (CH^2X)_{\text{torsion}} \to (CH^3X)_{\text{torsion}}$ is bijective, which fits with what Karpenko's method gives [21]). □

Proof of theorem 7. In the case where X is defined by an anisotropic Albert form q, we define a homomorphism

$$\rho\colon H^1F \longrightarrow \operatorname{Coker}\eta_2^4$$

by cup-product with a representative of $\operatorname{Coker}\eta_2^3 \simeq \mathbf{Z}/2$.

Recall [17] that such a representative e is constructed as follows: let $K = F(X)$ and $q_K \sim a\tau$, where $a \in K^*$ and τ is a 2-fold Pfister form. Then $e = (a)\cdot e^2(\tau)$.

The composition $H^1F \to \operatorname{Coker}\eta_2^4 \to \operatorname{Coker}\eta^4$ has infinite image if we pass from F to the rational function field $F(t)$. Indeed, F is infinite since q is anisotropic. Then, for any $\alpha \in F$, the element

$$(t-\alpha)\cdot e \in H^4F(X)(t) \hookrightarrow H^4(F(X)(t),3)$$

is unramified over $F(t)$, and its residue at α is $e \in H^3(F(X),2)$. This shows that the classes of the $(t-\alpha)\cdot e$ in $\operatorname{Coker}\eta^4_{F(t)}$ are linearly independent.

In the case of a virtual Albert form q of discriminant d, we define a homomorphism

$$\rho'\colon E^* \longrightarrow \operatorname{Coker}\eta_2^4$$

simply as the composite

$$E^* \xrightarrow{\rho} \operatorname{Coker}\eta^4_{2,q_E} \xrightarrow{\operatorname{Cor}_{E/F}} \operatorname{Coker}\eta_2^4$$

where $\operatorname{Coker}\eta^4_{2,q_E}$ is the cokernel attached to the form q_E ($E = F(\sqrt{d})$). We show just as above that $\operatorname{Coker}\eta^4_{F(t)}$ is infinite, now using rational points of $\mathbf{A}^1_E$ not defined over F.

These arguments can be summarized by noting that the groups $\operatorname{Coker}\eta^i$ are cycle modules in the sense of [42] and using the exact sequence

$$0 \to \operatorname{Coker}\eta^4_F \to \operatorname{Coker}\eta^4_{F(t)} \to \bigoplus_{y\in\mathbf{A}^1_{(0)}} \operatorname{Coker}\eta^3_{F_y} \to 0$$

cf. the exact sequence $(*)$ in the introduction. □

Exercise. Let X be an anisotropic quadric, $a \in F^*$ and $E = F(\sqrt{a})$ a quadratic extension such that X_E is isotropic.

a) Show that extension of scalars defines a homomorphism

$$\theta^n_E\colon H^0(X,\mathcal{H}^n) \to H^nE$$

for all $n \geq 0$.

b) Define a 0-sequence

$$H^3F \xrightarrow{\cdot(a)} \operatorname{Ker}\theta^4_E \to \operatorname{Ker}(CH^3X \to CH^3X_E)$$

(proceed by a diagram chase, as in [27] for the definition of the map $\operatorname{Ker}\eta_2^4 \to \operatorname{Ker}(H^2(X,\mathcal{K}_3) \to H^2(X_E,\mathcal{K}_3)))$.

c) Compare the map $\operatorname{Ker}\theta^4_E \to \operatorname{Ker}(CH^3X \to CH^3X_E)$ of b) with the map $\operatorname{Coker}\eta^4_2 \to (CH^3X)_{\text{torsion}}$ deduced from theorem 6.

Appendix A. Quadrics of dimension 0 and 1

A.1. Proposition. *Let* $\dim X = 1$ *and suppose the Milnor conjecture holds in degree* $i-2$. *Then* $\operatorname{Coker}\eta^i = 0$.

Proof. It follows from the cross of exact sequences

$$\begin{array}{ccccc} & & H^i(F,i-1) & & \\ & & \downarrow & & \\ 0 \to H^1(X,\mathcal{H}^{i-1}(i-1)) \to & & H^i(X,i-1) & & \to H^0(X,\mathcal{H}^i(i-1)) \to 0 \\ & & \downarrow & & \\ & & H^{i-2}(F,H^2(\overline{X},i-2)) & & \end{array}$$

(coniveau and Hochschild-Serre), by using the fact that the composition

$$H^{i-2}(F,i-2)\otimes CH^1X \to H^1(X,\mathcal{H}^{i-1}(i-1)) \to H^{i-2}(F,H^2(\overline{X},i-2))$$

is surjective. As above, this is deduced from the divisibility of $H^{i-2}(F,i-2)$ and corollary 1.2. ☐

In the following we present analogues of proposition A.1 and proposition 3 for 0-dimensional quadrics. Let $L = F(\sqrt{d})$ be a quadratic extension with Galois group $G = \operatorname{Gal}(L/F)$.

A.2. Proposition. *Suppose the Milnor conjecture holds in degree* $i-1$. *Then*

a) *The restriction map*

$$H^i(F,i-1) \to (H^i(L,i-1))^G$$

is surjective.

b) *The restriction map*

$$H^iF \to (H^iL)^G$$

has kernel $(d)\cdot H^{i-1}F$ *and cokernel isomorphic to*

$$\{\, x \in (d)\cdot H^{i-1}F \mid (-1)\cdot x = 0 \,\}.$$

Proof. a) Let σ be the generator of G and consider the exact sequence of Galois-modules

$$0 \to \mathbf{Z} \to \mathbf{Z}[G] \xrightarrow{1-\sigma} \mathbf{Z}[G] \to \mathbf{Z} \to 0.$$

Tensoring with $\mathbf{Q}/\mathbf{Z}(i-1)$ gives a spectral sequence converging to 0, with E_1-term:

$$\begin{array}{l} 0 \longrightarrow H^0(F,i-1) \xrightarrow{\text{Res}} H^0(L,i-1) \xrightarrow{1-\sigma} H^0(L,i-1) \xrightarrow{\text{Cor}} H^0(F,i-1) \longrightarrow 0 \\ \qquad\qquad \dots \\ 0 \longrightarrow H^{i-1}(F,i-1) \xrightarrow{\text{Res}} H^{i-1}(L,i-1) \xrightarrow{1-\sigma} H^{i-1}(L,i-1) \xrightarrow{\text{Cor}} H^{i-1}(F,i-1) \longrightarrow 0 \\ 0 \longrightarrow H^i(F,i-1) \xrightarrow{\text{Res}} H^i(L,i-1) \xrightarrow{1-\sigma} H^i(L,i-1) \xrightarrow{\text{Cor}} H^i(F,i-1) \longrightarrow 0 \\ \qquad\qquad \dots \end{array}$$

Here the d^2-differentials go 2 steps right and 1 step up. The norm map

$$H^{i-1}(L, i-1) \xrightarrow{\mathrm{Cor}} H^{i-1}(F, i-1)$$

is computed as the norm map

$$K^{\mathrm{M}}_{i-1}L \otimes \mathbf{Q}/\mathbf{Z} \xrightarrow{N} K^{\mathrm{M}}_{i-1}F \otimes \mathbf{Q}/\mathbf{Z},$$

which is obviously surjective. Hence

$$H^i(F, i-1) \xrightarrow{\mathrm{Res}} H^i(L, i-1) \xrightarrow{1-\sigma} H^i(L, i-1)$$

is exact, *i.e.*, η^i is surjective.

This discussion holds for any cyclic extension L/F with the degree $[L:F]$ prime to $\operatorname{char} F$.

b) The statement on the kernel is classical [1]; the statement on the cokernel follows from this one and a), by using proposition 7.4, which is shown to apply in this case just as in section 7. □

Appendix B. Application to real quadrics

In this section, we examine the case when the ground field is the field $\mathbf{R}$ of real numbers. The real anisotropic quadric of dimension d, denoted Q_d, is defined by the $(d+2)$-dimensional form $(d+2)\langle 1\rangle$. As all anisotropic quadratic forms over $\mathbf{R}$ are Pfister neighbours, the anisotropic quadrics can be grouped according to the level of their function fields (*cf.* [10, §3]). As observed in [10, §3], the unramified cohomology groups are invariant for the set of quadrics grouped in this manner. In [10], it was proved that the unramified cohomology group $H^0(Q_d, \mathcal{H}^3)$ is 0 for $d \leq 6$ and that the map $\mathbf{Z}/2 \simeq H^3(\mathbf{R}) \to H^0(Q_d, \mathcal{H}^3)$ is bijective for $d > 7$. This result was used to show that the cycle map in codimension two

$$CH^2 Q_d/2 \to H^4 Q_d$$

is injective for all quadrics Q_d (compare also [17, th. 3]).

We extend these results in this section, and use them to get extra information on the cohomology of real quadrics. Concerning unramified cohomology, results can be expressed concisely in a striking way:

B.1. Theorem. *Let $F = \mathbf{R}$ and $i \leq 4$. Then, for any $d > 0$, the map $H^i(\mathbf{R}) \to H^0(Q_d, \mathcal{H}^i)$ is surjective. If $d > 2^i - 2$ it is bijective; if $d \leq 2^i - 2$, $H^0(Q_d, \mathcal{H}^i) = 0$.*

Proof. For $i = 2$ it is classical and for $i = 3$ this is the main result of [10]. Note however that these cases also follow directly from proposition 7.4, theorem 6 and the fact that there are no anisotropic Albert quadrics over $\mathbf{R}$ in view of

B.2. Lemma. *With notation as above, if $F = \mathbf{R}$ then $(\operatorname{Ker} \eta_2^i)_0 = 0$ for all quadrics X and all $i \geq 1$.*

Proof. Cup-product by (-1) is injective on $H^*\mathbf{R}$. □

Let us now prove theorem B.1 in the case $i = 4$. For $d > 14$ and for $6 < d \leq 14$, this follows from theorem 3.4 and corollary 8. For $3 < d \leq 6$, we may assume $d = 3$; then $cd(\mathbf{R}(Q_3)) = 3$ [47], hence $H^4(\mathbf{R}(Q_3)) = 0$ and *a fortiori* $H^0(Q_3, \mathcal{H}^4) = 0$. The argument is the same for $d \leq 2$. □

Theorem B.1 implies in particular that, in the coniveau spectral sequence

$$E_2^{p,q} = H^p(Q_d, \mathcal{H}^q) \Rightarrow H^{p+q}Q_d,$$

$E_2^{0,q}$ consists of universal cycles for $q \le 4$. This in turn implies:

B.3. Theorem.

a) *For any $d \ge 1$, we have:*

$$\dim H^1(Q_d, \mathcal{H}^2) + \dim H^0(Q_d, \mathcal{H}^3) = \dim H^3 Q_d;$$

$$\dim CH^2 Q_d \otimes \mathbf{Z}/2 + \dim H^1(Q_d, \mathcal{H}^3) + \dim H^0(Q_d, \mathcal{H}^4) = \dim H^4 Q_d.$$

b) *The "edge homomorphism" $H^2(Q_d, \mathcal{H}^3) \to H^5 Q_d$ is injective.*

□

B.4. Corollary. *We have:*

$$\dim H^1(Q_d, \mathcal{H}^2) = \begin{cases} 0 & \textit{for } d = 1 \\ 1 & \textit{for } d = 2 \\ 2 & \textit{for } 3 \le d \le 6 \\ 1 & \textit{for } d > 6 \end{cases}$$

and

$$\dim H^1(Q_d, \mathcal{H}^3) = \begin{cases} 0 & \textit{for } d \le 3 \\ 1 & \textit{for } 4 \le d \le 6 \\ 2 & \textit{for } 7 \le d \le 14 \\ 1 & \textit{for } d > 14. \end{cases}$$

Sketch of proof. An inductive argument allows one to compute $\dim H^* Q_d$. The method is to first compute the cohomology of the corresponding affine quadric by reduction to the complex case. The answer is [19]:

$$\dim H^n Q_d = \begin{cases} \left[\dfrac{n}{2}\right] + 1 & \text{if } d \ge n \\[2ex] \left[\dfrac{2d-n}{2}\right] + 1 & \text{if } d \le n. \end{cases}$$

The corollary follows from this fact, from theorem B.1, theorem B.3, and from (for the value of $\dim CH^2 Q_d \otimes \mathbf{Z}/2$) propositions 1.1 and 1.6. □

Further results on $H^2(Q_d, \mathcal{H}^3)$, $H^1(Q_d, \mathcal{H}^4)$ and the cycle map $cl^3_{Q_d}$ can be obtained at a higher cost [19].

Appendix C. The unramified Witt ring

In this appendix, we prove theorem 9, which we now recall:

C.1. Theorem. *Let $i \le 4$. For any quadric X, the composition*

$$\operatorname{Coker} \chi^i \to \operatorname{Coker} \eta_2^i \to \operatorname{Coker} \eta^i$$

is injective, where

$$\chi^i \colon I^i F / I^{i+1} F \longrightarrow I^i_{nr}(F(X)/F) / I^{i+1}_{nr}(F(X)/F)$$

is the map induced by extension of scalars and the first map in the composition is induced by the cohomological invariant $\bar{e}^i$.

C.2. Lemma. *Let C be a conic over F and $K = F(C)$. Let $c \in H^2F$ be the class of the quaternion algebra corresponding to C. Then, for $i \leq 4$,*

a) $\operatorname{Ker} \eta_2^i = c \cdot H^{i-2}F$.

b) *The natural map*
$$I^{i+1}F \to I^{i+1}_{nr}(K/F)$$
is surjective.

Proof. a) has been given earlier in the paper for $i \leq 3$ and is proven in [30, prop. 3.15] for $i = 4$. Let us prove b). By a result of Parimala [37, 53, 10], the map $W(F) \to W_{nr}(K/F)$ is surjective. Let $\varphi \in I^{i+1}_{nr}(K/F)$ and, by induction on i, let $\psi \in I^iF$ be such that $\psi_K = \varphi$. We have $e^i(\psi) \in \operatorname{Ker} \eta_2^i$, hence $e^i(\psi) = c \cdot x$ for some $x \in H^{i-2}F$ by a). By the surjectivity of e^{i-2} and the injectivity of $\bar{e}^i$, we can write $\psi = \tau \otimes \theta \perp \psi'$, where τ is the quaternion form corresponding to c, $e^{i-2}(\theta) = x$ and $\psi' \in I^{i+1}F$. Then $\psi'_K = \varphi$, as desired. □

Notation. We write $\bar{I}^iF = I^iF/I^{i+1}F$ and, for a finitely generated extension L/F, $\bar{I}^i_{nr}(L/F) = I^i_{nr}(L/F)/I^{i+1}_{nr}(L/F)$.

C.3. Lemma. *Let X be a quadric (of dimension > 0), and let C be a conic lying on X. Let τ be the quaternion form associated to C. Then, for $i \leq 4$, there is a surjection*
$$\tau \bar{I}^{i-2}F(X) \cap \bar{I}^i_{nr}(F(X)/F) \twoheadrightarrow \operatorname{Coker} \chi^i.$$

Proof. Let $K = F(C)$. Consider the commutative diagram
$$\begin{array}{ccc}
\bar{I}^i_{nr}(F(X)/F) & \longrightarrow & \bar{I}^i_{nr}(K(X)/F) \\
{\scriptstyle \chi^i}\uparrow & & \wr\uparrow \\
\bar{I}^iF & \twoheadrightarrow & \bar{I}^i_{nr}(K/F)
\end{array}$$

In this diagram, the lower horizontal map is surjective by lemma C.2 b) and the right vertical one is an isomorphism because $K(X)/K$ is purely transcendental. A diagram chase then yields a surjection
$$\operatorname{Ker}(\bar{I}^i_{nr}(F(X)/F) \to \bar{I}^i_{nr}(K(X)/F)) \twoheadrightarrow \operatorname{Coker} \chi^i.$$

To conclude the proof, it suffices to observe that $\operatorname{Ker}(\bar{I}^iF(X) \to \bar{I}^iK(X)) = \tau\bar{I}^{i-2}F(X)$ by lemma C.2 a) and the bijectivity of $\bar{e}^{i-2}$ and $\bar{e}^i$. □

Proof of theorem C.1. Consider the commutative diagram
$$\begin{array}{ccccc}
\operatorname{Coker} \chi^i & \xrightarrow{\bar{e}^i} & \operatorname{Coker} \eta_2^i & \xrightarrow{\lambda} & \operatorname{Coker} \eta^i \\
\uparrow & & \uparrow & & \uparrow \\
\bar{I}^i_{nr}(F(X)/F) & \xrightarrow{\bar{e}^i} & H^i_{nr}(F(X)/F) & \xrightarrow{\lambda} & H^i_{nr}(F(X)/F, 2) \\
{\scriptstyle \chi^i}\uparrow & & {\scriptstyle \eta_2^i}\uparrow & & {\scriptstyle \eta^i}\uparrow \\
\bar{I}^iF & \xrightarrow{\bar{e}^i} & H^iF & \xrightarrow{\lambda} & H^i(F, 2)
\end{array}$$
where λ is the natural map. Let $x \in \bar{I}^i_{nr}(F(X)/F)$. Choose C as in lemma C.3. By this lemma, we can modify x by an element from I^iF so that $x = \tau \otimes \theta$. Let

$y \in H^i(F,2)$ be such that $\lambda\bar{e}^i(x) = \eta^i(y)$. Let E/F be a quadratic extension such that C_E, hence X_E, is isotropic. Then

$$\tau_E \sim 0$$

hence

$$x_{E(X)} = 0$$

and

$$\eta^i(y)_{E(X)} = \eta^i(y_E) = 0.$$

Since X_E is isotropic, $E(X)/E$ is purely transcendental and $y_E = 0$ (prop. 2.5 a)). Hence $2y = 0 \in H^i(F,2)$ by the usual transfer argument, and $y = \lambda(z)$ for some $z \in H^iF$. By the surjectivity of $\bar{e}^i$, write $z = \bar{e}^i(t)$ for $t \in \bar{I}^iF$. Then

$$\lambda\bar{e}^i(x) = \eta^i\lambda\bar{e}^i(t) = \lambda\bar{e}^i(\chi^i(t))$$

hence $x = \chi^i(t)$ by the injectivity of $\bar{e}^i$ and λ. □

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