Norm Varieties*

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0 Introduction

The purpose of these lectures is to present a part of Markus Rost's work on Norm Varieties. The primary goal is to prove the following result [MC/l, theorem 6.3] that is necessary to complete the inductive step in the proof of the Bloch – Kato conjecture.

0.1 Theorem (M. Rost)**.** *For any non-trivial n-symbol $\{a_1, \ldots, a_n\} \in K_n^M(k)/l$ there exists a splitting variety X such that*
1) X is a $\nu_{\leqslant n-1}$ variety
2) the sequence

$$H_{-1,-1}(X \times X) \xrightarrow{(p_1)_* - (p_2)_*} H_{-1,-1}(X) \longrightarrow H_{-1,-1}(k) = k^*$$

is exact.

An interested reader may find out more about both the history and the strategy of the proof of the conjecture in the introduction to the paper [MC/2]. A diagram illustrating various implications of results in the motivic cohomology that are used in the inductive step appears in the introduction to the notes [EMS].

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Observe that to get the proof of Bloch - Kato conjecture in general it suffices to prove the main theorem above for an arbitrary prime number $l > 2$ and an arbitrary base field k of the characteristic zero containing a primitive l-th root of unity. Moreover, the base field may be assumed to be l - special, see the definition 1.11 below. While the restriction on the characteristic is not essential for many steps of the construction, we impose it for the sake of simplicity. Thus we freely use the resolution of singularities technique by Hironaka [H]. We don't assume that l is odd because it gives no simplification.

The present proof of the theorem in weight n uses the Theorem 2.4 that in turn follows from the Bloch - Kato conjecture in weight $n-1$. Thus, rather then being a completely independent statement it is a part of inductive step

$$\boxed{\begin{array}{c}\text{Bloch - Kato}\\ \text{conjecture}\\ \text{in weight } n-1\end{array}} \Longrightarrow \boxed{\begin{array}{c}\text{Bloch - Kato}\\ \text{conjecture}\\ \text{in weight } n.\end{array}}$$

Here is the outline of the argument. In the first section we give all the necessary definitions and introduce the so-called group of reduced 0 - dimensional $\mathcal{K}_1$ - cycles on a smooth scheme to replace the $(-1,-1)$ - homology of the main theorem. We discuss a number of properties of these cycles and formulate the theorem 1.20 which is the central result of these notes. In short, it claims that splitting varieties of the special type exist, and that any such variety satisfies the claim of the main theorem. In conclusion of the section one we show that theorem 1.20 implies theorem 0.1 if the base field is l-special.

In the section two we describe an inductive construction of l-generic splitting varieties for a symbol. These varieties are constructed from symmetric powers and are, in fact, exactly the ones we want to produce. Toward the end we show that 1.20 implies 0.1 for a base field that is not necessary l-special.

The following two sections deal with the pseudo-Galios (i.e. 'Galois almost everywere') coverings. In the section three we define the η invariant of such coverings and show that it satisfies an appropriate degree formula. In the section four it is shown, by means of introduction of b - classes, that knowing η invariant of l-th (Cartesian) power of a variety over its l-th cyclic power is essentially the same as to know whether it is a $\nu_{\leqslant n-1}$ variety.

Finally in the section five we use Markus Rost's Chain Lemma to show that variety in question is indeed $\nu_{\leqslant n-1}$, and also to prove the Multiplication Principle for reduced 0 - dimensional $\mathcal{K}_1$ - cycles. In turn Multiplication Principle together with the Norm Principle (see [R]) obviously imply the remaining claim of the theorem 1.20 hence the main theorem as well.

1 Reduced 0 - dimensional $\mathcal{K}_1$ - cycles

We would like to transform the statement of the main theorem 0.1 into one about algebraic cycles.

Let X be a smooth, irreducible, projective variety of dimension d. Recall that the $\mathcal{K}$-cohomology groups of X are those of the Gersten complex

$$K^M_{d+1}(k(X)) \longrightarrow \coprod_{\operatorname{codim} x=1} K^M_d(k(x)) \longrightarrow \cdots \longrightarrow \coprod_{\operatorname{codim} x=d} K^M_1(k(x)).$$

The last cohomology group is

$$H^d(X, \mathcal{K}_{d+1}) = \operatorname{coker}\left(\coprod_{\dim \bar{y}=1} K_2^M(k(y)) \xrightarrow{\partial} \coprod_{x \text{ is a closed point of } X} K_1^M(k(x)) \right).$$

Finally recall that the latter group may also be denoted $A_0(X, \mathcal{K}_1)$ and, when written in this form, is called the group of 0 - dimensional $\mathcal{K}_1$ - cycles.

The connection between the motivic homology $H_{-1,-1}(X)$ and $A_0(X, \mathcal{K}_1)$ is a direct consequence of standard relations.

1.1 Lemma. *Let X be a smooth, irreducible, projective variety of dimension d. Then $H_{-1,-1}(X) = A_0(X, \mathcal{K}_1)$.*

Proof. Using duality (see [TC]) and usual isomorphism between motivic and $\mathcal{K}$-cohomology, as well as above remarks we compute

$$\begin{aligned}
H_{-1,-1}(X) &= Hom_{DM_-}(\boldsymbol{Z}(-1)[-1], M(X)) && \text{(definition)}\\
&= Hom_{DM_-}(\boldsymbol{Z}(-1)[-1], \mathcal{H}\mathrm{om}(M(X), \boldsymbol{Z}(d)[2d])) && \text{(duality)}\\
&= Hom_{DM_-}(\boldsymbol{Z}(-1)[-1] \otimes M(X), \boldsymbol{Z}(d)[2d]) &&\\
&= Hom_{DM_-}(M(X), \boldsymbol{Z}(d+1)[2d+1]) &&\\
&= H^{2d+1}(X, \boldsymbol{Z}(d+1)) && \text{(definition)}\\
&= H^d(X, \mathcal{K}_{d+1}) && \text{[MC/2, lemma 4.11]}\\
&= A_0(X, \mathcal{K}_1) && \text{(definition)}
\end{aligned}$$

with $\mathcal{H}\mathrm{om}$ being the internal Hom-object in the category DM_-.

Note that a proper morphism $f : X \to Y$ induces a map of Gersten complexes, hence the map $f_* : A_0(X, \mathcal{K}_1) \to A_0(Y, \mathcal{K}_1)$, consequently the groups $A_0(-, \mathcal{K}_1)$ are covariant, in particular, with respect to morphisms of projective varieties. Moreover the map f_* is compatible with the corresponding map of $(-1, -1)$ homology groups.

1.2 Notation. For a smooth, irreducible, projective variety X let $\bar{A}_0(X, \mathcal{K}_1)$ denote the group of *reduced* 0 - dimensional $\mathcal{K}_1$ - cycles

$$\operatorname{coker}\Big(A_0(X \times X, \mathcal{K}_1) \xrightarrow{(p_1)_* - (p_2)_*} A_0(X, \mathcal{K}_1) \Big).$$

Finally let us point out that the map $N : A_0(X, \mathcal{K}_1) \to A_0(\operatorname{Spec} k, \mathcal{K}_1) = k^\times$ induced by the structure map, is the sum of norm maps of Milnor K-groups, and that it obviously factors through $\bar{A}_0$. Now we can make a trivial but very important observation.

1.3 Remark. A projective variety X verifies the second requirement of the main theorem 0.1 if and only if the norm map $N : \bar{A}_0(X, \mathcal{K}_1) \to k^\times$ is injective.

Observe that the group $\bar{A}_0(X, \mathcal{K}_1)$ is generated by elements of the form $[x, \mu]$, where $x \in X$ is a closed points, $\mu \in k(x)^\times$. Such element may be thought of either as the image of μ under the canonical map $k(x)^\times = \bar{A}_0(\operatorname{Spec} k(x), \mathcal{K}_1) \to \bar{A}_0(X, \mathcal{K}_1)$ corresponding to embedding of x into X or simply as μ placed at x.

Let L/k be a field extension. A morphism $\phi : \operatorname{Spec} L \to X$ is determined by a point x of X and a field embedding $k(x) \hookrightarrow L$ over k. We will refer to such ϕ as *L - valued point of X*. If L/k is a finite extension then x must be a closed point of X. For such point the map ϕ_* defined above admits very explicit description. It is induced by the norm map

$$\begin{tikzcd} L^\times \arrow[r, "N_{L/k(x)}"] \arrow[d, equal] & k(x)^\times \arrow[d, "\text{canonical map}"] \\ A_0(\operatorname{Spec} L, \mathcal{K}_1) \arrow[r, "\phi_*"] & A_0(X, \mathcal{K}_1). \end{tikzcd}$$

This allows to give the following description of $\bar{A}_0(X, \mathcal{K}_1)$.

1.4 Lemma. *$\bar{A}_0(X, \mathcal{K}_1)$ is obtained from $A_0(X, \mathcal{K}_1)$ by factoring out all the relations of the form $\phi_*(\lambda) - \psi_*(\lambda)$ where L is any finite extension of k, $\lambda \in L^\times$, and $\phi, \psi : \operatorname{Spec} L \to X$ are any two L - valued points.*

Proof. Any two morphisms $\phi, \psi : \operatorname{Spec} L \to X$ determine the product morphism $(\phi, \psi) : \operatorname{Spec} L \to X \times X$. For any $\lambda \in L$ therefore $\phi_*(\lambda) - \psi_*(\lambda) = ((p_1)_* - (p_2)_*)(\phi, \psi)_*(\lambda)$ vanishes in $\bar{A}_0(X, \mathcal{K}_1)$. Conversely every element in the image of $(p_1)_* - (p_2)_*$ must be a sum of terms of that form.

Making the right choice of L we obtain

1.5 Corollary. *1) Assume that $x, x' \in X$ are closed points such that there exists an isomorphism $\sigma : k(x) \xrightarrow{\sim} k(x')$. Then for every $\lambda \in k(x)^\times$, $[x, \lambda] = [x', \sigma(\lambda)]$ in $\bar{A}_0(X, \mathcal{K}_1)$.*
2) Assume that $x, x' \in X$ are closed points such that there exists a field embedding $k(x') \hookrightarrow k(x)$. Then for every $\lambda \in k(x)^\times$, $[x, \lambda] = [x', N_{k(x)/k(x')}(\lambda)]$ in $\bar{A}_0(X, \mathcal{K}_1)$.

1.6 Corollary. *If X has a k - rational point x_0 then $N : \bar{A}_0(X, \mathcal{K}_1) \xrightarrow{\sim} k^\times$ is an isomorphism.*

Proof. The morphism $X \to \operatorname{Spec} k$ induces the map $N : \bar{A}_0(X, \mathcal{K}_1) \to k^\times$ that sends $[x, \mu]$ to $N_{k(x)/k}(\mu)$. It has right inverse that maps $\mu \in k^\times$ to $[x_0, \mu]$. It is enough to show that the latter is surjective. Let $x \in X$ be any closed point. Then according to the corollary 1.5 for each $\mu \in k(x)^\times$ we get $[x, \mu] = [x_0, N_{k(x)/k}(\mu)]$ in $\bar{A}_0(X, \mathcal{K}_1)$.

1.7 Corollary. *If X has a closed point of degree n then both the kernel and the cokernel of $N : \bar{A}_0(X, \mathcal{K}_1) \longrightarrow k^\times$ are annihilated by n.*

Proof. Let $x \in X$ be the point with $[k(x) : k] = n$. After extension of scalars to $k(x)$ the map in question becomes an isomorphism. The usual transfer argument completes the proof.

Now we recall the notion of a generic splitting variety.

1.8 Definition. Let $\{\underline{a}\} = \{a_1, \ldots, a_n\} \in K_n^M(k)/l$ be a non - zero symbol. A smooth variety X is a *splitting variety* for $\{\underline{a}\}$ if

1) $\{\underline{a}\} = 0$ in $K_n^M(k(X))/l$.

It is a *generic splitting variety* if in addition

2) for any L/k such that $\{\underline{a}\} = 0$ in $K_n^M(L))/l$ there is an L - valued point $\operatorname{Spec} L \to X$ over k.

1.9 Remark. Observe that for any $x \in X$ the map $K_n^M(k)/l \to K_n^M(k(x))/l$ factors through a (non-canonical) specialization map $K_n^M(k(X))/l \to K_n^M(k(x))/l$. Therefore if X is a splitting variety for $\{\underline{a}\}$ then $\{\underline{a}\} = 0$ in $K_n^M(k(x))/l$ for every point x of X.

Unfortunately the generic splitting varieties are only known to exist for $n \leqslant 3$ and also for arbitrary n provided $l = 2$. Observe however that if L'/L is a finite extension of degree prime to l and L' splits $\{\underline{a}\}$ then by the usual transfer argument L splits $\{\underline{a}\}$ as well. Therefore we can, without much loss, relax the definition as follows.

1.10 Definition. A smooth variety X is an l - *generic splitting variety* for a non - zero symbol $\{\underline{a}\} \in K_n^M(k)/l$ if

1) X is a splitting variety for $\{\underline{a}\}$ and

2) for any L/k such that $\{\underline{a}\} = 0$ in $K_n^M(L))/l$ there is a finite extension L'/L of degree prime to l and an L' - valued point $\operatorname{Spec} L' \to X$ over k.

Fields L as above are called splitting fields of $\{\underline{a}\}$.

1.11 Definition. A field F is called l - *special* provided F has no finite extensions of degree prime to l or equivalently if $Gal(F_{alg}/F)$ is a pro - l - group.

1.12 Remark. Let L be any field. Choose an algebraic closure L_{alg} of L. Let $G := Gal(L_{alg}/L)$ and G_l be a Sylow l - subgroup of the profinite group G. Set $\tilde{L} := L_{alg}^{G_l}$ to be subfield fixed by G_l. Then $\tilde{L}$ is l - special while the degree of every its finite subextension L'/L is prime to l. Such field $\tilde{L}$ is called *a maximal extension of L of degree prime to l.*

With a notion of l - special field at our disposal we can give an alternative description of l - generic splitting varieties

1.10′ Definition. A variety X is an l - *generic splitting variety* for a non - zero symbol $\{\underline{a}\} \in K_n^M(k)/l$ if

1) X is a splitting variety for $\{\underline{a}\}$ and

2) for any l - special splitting field F of $\{\underline{a}\}$ there is an F - valued point $\operatorname{Spec} F \to X$ over k.

Note that the definitions 1.10 and 1.10' are equivalent. Indeed if X is an l - generic splitting variety according to 1.10 and L is l - special then L'/L must be a trivial extension and any L' - valued point is an L - valued point. Conversely let X be an l - generic splitting variety according to 1.10' and L be a splitting field for $\{\underline{a}\}$. Let $\tilde{L}$ be a maximal extension of L of degree prime to l. Since $\{\underline{a}\}$ vanishes over L it does so over $\tilde{L}$ hence X has an $\tilde{L}$ - valued point. It is supported in a point x of X. Since $k(x)$ is finitely generated over k there exists some finite subextension L'/L such that field embedding $k(x) \hookrightarrow \tilde{L}$ factors through L'. Thus, by construction, X has an L' - valued point supported in x and L'/L is finite of degree prime to l.

Everywhere in these notes all the splitting varieties are always assumed to be smooth and projective.

1.13 Lemma. *Let $f : X \to X'$ be a birational morphism of projective varieties. Then for each point x' in the smooth locus of X' there exists $x \in X$ such that $f(x) = x'$ and $k(x) = k(x')$.*

Proof. Assume first that X' is smooth and $f : X' \to X$ is a blow - up with a smooth center. In this special case the claim holds for obvious reasons.

In the general case consider the inverse rational map $f^{-1} : X' - \to X$. Using the resolution of singularities one can find a tower of blow - ups $X_n \to \cdots \to X_1 \to X'$ such that X_n is smooth and $\pi : X_n \to X'$ is an isomorphism over the smooth locus of X'. In particular the fiber of g over x' consists of a single point x'' with the same residue field.

Then again using the resolution of singularities for the morphism $\pi \circ f^{-1}$ one constructs a tower of blow - ups with smooth centers $Y_m \to \cdots \to Y_1 \to X_n$ such that $\pi \circ f^{-1} : X_n = X' - \to X$ lifts to a morphism $g : Y_m \longrightarrow X$.

$$\begin{tikzcd}
 & & X \arrow[d, shift right, "f"'] \\
Y_m \arrow[urr, "g"] \arrow[r] & X_n \arrow[r, "\pi"] & X' . \arrow[u, dashed, shift right, "f^{-1}"']
\end{tikzcd}$$

According to the special case one can further lift $x'' \in X_n$ to $x''' \in Y_m$ with the same residue field. Setting $x := g(x''')$ we observe that $f(x) = x'$ and moreover that $k(x') = k(x''') \supset k(x) \supset k(x')$ hence the residue fields $k(x)$ and $k(x')$ are the same.

1.14 Notation. For a variety X set $FE_X := \{F/k : X \text{ has an } F \text{ - valued point}\}$. ($FE$ stands for 'field extension'.)

Using a tower of blow - ups as in the previous lemma one can readily prove the following.

1.15 Lemma. *Assume that $X - \to X'$ is a rational map of smooth projective varieties. Then $FE_X \subset FE_{X'}$.*

1.16 Remark. According to the lemma the property of being generic splitting variety for a given symbol is a birational invariant.

1.17 Notation. For a variety X set $FE^l_X := \{F/k : F \text{ is } l \text{ - special and } X \text{ has an } F \text{ - valued point}\}$.

We will repeatedly use the following technical statement.

1.18 Lemma. *Let $f : X \to X'$ be a dominant morphism of a degree prime to l of projective varieties of an equal dimension. Let L' be any field and $\psi : \operatorname{Spec} L' \to X'$ be a morphism supported in the smooth locus of X'. Then ψ may be lifted to a morphism $\phi : \operatorname{Spec} L \to X$ so that L/L' is a finite extension of degree prime to l.*

Proof. According to Raynaud - Gruson platification theorem [RG] there exists a blow - up $p : B_{Z'}X' \to X'$, not necessarily with a smooth center, such that the proper pull - back $p^! f$ of f is flat. Since $p^! f$ is flat proper, and generically finite (because so is f) then it is flat and finite.

Let $\tilde{X}'$ be a variety resolving singularities of $B_{Z'}X'$ and set $\tilde{X}$ to be the pull - back of the corresponding square.

$$\begin{tikzcd}
\operatorname{Spec} L \arrow[r, dotted, "\tilde{\phi}"] \arrow[d, dotted] & \tilde{X} \arrow[r] \arrow[d, "\tilde{f}"'] \arrow[dr, phantom, "\square"] & p^! X \arrow[r] \arrow[d, "p^! f"'] & X \arrow[d, "f"] \\
\operatorname{Spec} L' \arrow[r, dotted, "\tilde{\psi}"'] & \tilde{X}' \arrow[r] & B_{Z'}X' \arrow[r, "p"'] & X'
\end{tikzcd}$$

Since $\tilde{X}' \to X'$ is a birational morphism the lemma 1.13 allows to lift the morphism $\psi : \operatorname{Spec} L' \to X'$ to $\tilde{\psi} : \operatorname{Spec} L' \to \tilde{X}'$. Consider the fiber of $\tilde{f}$ over the L' - valued point $\tilde{\psi}$. It is a finite scheme of degree prime to l over $\operatorname{Spec} L'$ and hence has a closed point also of degree prime to l over $\operatorname{Spec} L'$. This point provides a morphism $\tilde{\phi} : \operatorname{Spec} L \to \tilde{X}$ that lifts $\tilde{\psi}$ with L/L' being a finite extension of degree prime to l. Composing $\tilde{\phi}$ with the other two morphisms in the top row of the diagram we get the required lifting of ψ.

1.19 Corollary. *Assume that $X - \to X'$ is a dominant rational map of smooth projective varieties of same dimension and of degree prime to l. Then $FE^l_X = FE^l_{X'}$.*

Proof. According to lemma 1.15 we may replace X by any birationally equivalent smooth projective variety. Thus by resolution of singularities we may assume that $f : X \to X'$ is a morphism. Then inclusion $FE^l_X \subset FE^l_{X'}$ is obvious.

To prove the opposite inclusion consider an l-special field L' such that X' has an L' - valued point. By lemma 1.18 X has an L - valued point for an appropriate extension L/L'. Since L' is l - special this extension is in fact trivial that is X also has an L' - valued point.

In conclusion of this section we state the theorem, the proof of which will occupy the remainder of the paper and show that it implies the main theorem in those cases we are mostly interested in.

1.20 Theorem (M. Rost)**.** *Let $n \geqslant 2$ and $0 \neq \{\underline{a}\} = \{a_1, \ldots, a_n\} \in K^M_n(k)/l$. Then*

1) there exists an absolutely irreducible projective l - generic splitting variety for $\{\underline{a}\}$ of the dimension $l^{n-1} - 1$.

Assume further that the field k is l-special. If X is a projective l - generic splitting variety for $\{\underline{a}\}$ of the dimension $l^{n-1} - 1$ then

2) X is a ν_{n-1} variety;

3) each element of $\bar{A}_0(X, \mathcal{K}_1)$ is of the form $[x, \lambda]$, where $x \in X$ is a closed point of degree $[k(x) : k] = l$, $\lambda \in k(x)^\times$.

1.21 Corollary. *Let k be l-special and X be as in the theorem. Then X is a $\nu_{\leqslant n-1}$ variety.*

Proof. We have to show that for every $1 \leqslant i < n - 1$ there is a ν_i variety equipped with a morphism to X. Consider a non - zero symbol $\{a_1, \ldots, a_{i+1}\} \in K^M_{i+1}(k)/l$. By the theorem it has an l - generic splitting variety X_i of the dimension $l^i - 1$. Its function field $k(X_i)$ splits $\{a_1, \ldots, a_{i+1}\}$ and hence splits $\{a_1, \ldots, a_n\}$. Therefore there exists a finite extension $F/k(X_i)$ of the degree prime to l and a F - valued point $\operatorname{Spec} F \to X$. Choosing a model for F and resolving singularities of the corresponding rational map we get a smooth projective variety X'_i of the same dimension as X_i and a pair of morphisms

$$\begin{tikzcd} & X'_i \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X_i & & X \end{tikzcd}$$

so that f is dominant of degree prime to l. By the corollary 1.19 $FE^l_{X_i} = FE^l_{X'_i}$ consequently X'_i is another l - generic splitting variety for an $(i+1)$ - symbol and because $\dim X'_i = \dim X_i = l^i - 1$ it is ν_i by the theorem. Thus we have constructed a morphism g from a ν_i variety to X.

1.22 Remark. For $i = 0$ the same argument applied to $X_0 = \operatorname{Spec} k(\sqrt[l]{a_1})$ shows that X has an F - valued point where $F/k(\sqrt[l]{a_1})$ is a finite extension of degree prime to l. However since k is l - special this extension can only be the trivial one, that is X must have a $k(\sqrt[l]{a_1})$ - valued point.

1.23 Corollary. *Let k be l-special and X be as in the theorem. Then the norm map $N : \bar{A}_0(X, \mathcal{K}_1) \to k^\times$ is injective.*

Proof. Consider $[x, \lambda] \in \ker N$ with $[k(x) : k] = l$. Let σ be a generator of $Gal(k(x)/k) \cong \boldsymbol{Z}/l$. By the Hilbert' 90 Theorem $N_{k(x)/k}(\lambda) = 1$ implies $\lambda = \mu^{1-\sigma}$ for some $\mu \in k(x)^\times$. Thus $[x, \lambda] = [x, \mu] - [x, \sigma(\mu)] = 0$ by corollary 1.5

1.24 Remark. Evidently the corollaries 1.21 and 1.23 along with the remark 1.3 allow us to conclude that the theorem 1.20 implies the main theorem 0.1 in the case of an l - special base field.

2 Symmetric powers

In order to prove the existence clause of the theorem 1.20 we use the following construction suggested by V. Voevodsky. It is based on the notion of symmetric powers that we briefly recall below.

Let Y be a quasi - projective variety. The symmetric group Σ_m acts on the m - fold product Y^m and we let S^mY (or $Symm^mY$) denote the quotient variety Y^m/Σ_m.

For every normal and irreducible scheme T one can identify the set of morphisms $Hom(T, S^mY)$ with the set of all effective cycles $Z \subset Y \times T$ such that each component of Z is finite surjective over T and that the degree of Z over T is m.

Assume that Y is smooth and absolutely irreducible hence S^mY is absolutely irreducible and normal. The identity morphism $id : S^mY \to S^mY$ then corresponds to the *incidence cycle* $Z \subset Y \times S^mY$. In fact Z is a closed subscheme equal to the image of the closed embedding $Y \times S^{m-1}Y \to Y \times S^mY$ mapping (y, z) to $(y, z + y)$.

Let $p : Y \times S^{m-1}Y \to Y \times S^mY \to S^mY$ be the composition of the above morphism with the projection onto the second factor. It is finite surjective of the degree m.

Consider the largest open subscheme $Y^m \setminus \Delta$ of Y^m on which Σ_m acts freely. (Δ denotes the union of all the 'diagonals' in Y^m.) Set $U := (Y^m \setminus \Delta)/\Sigma_M \subset S^mY$. From the diagram

$$\begin{tikzcd} & p^{-1}(U) \subset Y \times S^{m-1}Y \arrow[dd, "p|_{p^{-1}U}"] \\ Y^m \setminus \Delta \arrow[ur] \arrow[dr] & \\ & U \subset S^mY \end{tikzcd}$$

were both slant arrows are Galois étale coverings we see that $p|_{p^{-1}U}$ is a finite étale map of degree m and that U is smooth.

Note that $p_*(\mathcal{O}_{Y \times S^{m-1}Y})$ is a coherent $\mathcal{O}_{S^mY}$ - algebra and that the sheaf $\mathcal{A} := p_*(\mathcal{O}_{Y \times S^{m-1}Y})|_U$ is a locally free $\mathcal{O}_U$ - algebra of rank m. Set $V := \operatorname{Spec}(S^*\mathcal{A}^\#)$ to be the m - dimensional vector bundle over U corresponding to $\mathcal{A}$. (Here $\mathcal{A}^{\#}$ denotes the dual sheaf and S^* denotes its symmetric algebra.)

Since $\mathcal{A}$ is a locally free algebra, there is a well - defined norm function $N : \mathcal{A} \to \mathcal{O}_U$. Moreover locally N is a homogenous polynomial function of degree m that is $N \in S^m(\mathcal{A}^{\#})$.

We will inductively construct l - generic splitting varieties of the theorem 1.20. The case of $n = 2$ is well - known, one can choose a splitting variety to be the Severi - Brauer variety of a cyclic algebra associated to the symbol $\{a_1, a_2\}$.

So we assume that $n > 2$, let Y in the preceding construction be a smooth projective absolutely irreducible l - generic splitting variety for $\{a_1, \ldots, a_{n-1}\}$ of the dimension $l^{n-2} - 1$, and set $m := l$.

Let $W \subset V$ be the hypersurface defined by the equation $N - a_n = 0$.

2.1 Lemma. *W is smooth over U (and hence smooth) and absolutely irreducible.*

Proof. Since N locally is a form of degree l in l variables, and $a_n \neq 0$, the first claim follows from the Jacobi criterion. To prove the second we may replace k by its algebraic closure. Assume that W is not irreducible. Hence there exists a point $u \in U$ such that the homogenous polynomial $N_u - a_n$ with coefficients in $\mathcal{O}_u$ is reducible. The the lemma 2.2 below would imply that $N_u = M^m$ is a power of a non-trivial linear form $M : \mathcal{A}_u \to \mathcal{O}_u$. Observe however that kernel of M is a codimension 1 subspace in A_u while the norm map N_u can not have a degeneracy locus of this form.

2.2 Lemma. *Let N be a form of prime degree m in n variables with coefficients in a UFD B/k and let $a \neq 0$ in k. The following conditions are equivalent.*

1) The polynomial $N - a \in B[X_1, \ldots, X_n]$ is irreducible;

2) The polynomial $N - aT^m \in B[T, X_1, \ldots, X_n]$ is irreducible;

3) N does not equal aM^m for any linear form M.

Proof. The equivalence of the first two conditions is obvious. The last two are equivalent thanks to the Gauss lemma applied to $B[X_1, \ldots, X_n]$ and its fraction field.

By the resolution of singularities we can embed W as an open subvariety into a smooth, projective, and absolutely irreducible variety X. Note that $\dim X = \dim W = \dim V - 1 = \dim U + l - 1 = l \dim Y + l - 1 = l(l^{n-2} - 1) + l - 1 = l^{n-1} - 1$ just as required. It remains to show that X is indeed an l - generic splitting variety for $\{\underline{a}\}$. This will be done in several steps.

2.3 Lemma. *Let F/k be any field extension such that $W(F) \neq \emptyset$. Then $\{a_1, \ldots, a_n\} = 0$ in $K_n^M(F)$.*

Proof. To specify an F - valued point x of W one may specify the underlying F - valued point $\tilde{x}$ of U and a rational point $\hat{x}$ in the fiber $V_{\tilde{x}}$ such that $N(\hat{x}) - a_n = 0$. (Note that $V_{\tilde{x}} \cong \boldsymbol{A}^l_F$.) The point $\tilde{x}$ corresponds to a cycle $x_1 + \ldots + x_k$ on Y_F such that $\sum [F(x_i) : F] = l$. Then the point $\hat{x}$ has 'coordinates' $(\lambda_1, \ldots, \lambda_k) \in F(x_1) \times \cdots \times F(x_k)$ and by assumption $a_n = N(\hat{x}) = \prod N_{F(x_i)/F}(\lambda_i)$.

By construction Y has an $F(x_i)$ - valued point for each $1 \leqslant i \leqslant k$ hence $\{a_1, \ldots, a_{n-1}\} = 0$ in $K_{n-1}^M(F(x_i))/l$. Thus in $K_n^M(F))/l$

$$\{a_1, \ldots, a_n\} = \sum N_{F(x_i)/F}(\{a_1, \ldots, a_{n-1}, \lambda_i\} = 0.$$

Applying the previous lemma to $F = k(W) = k(X)$ we conclude that X is a splitting variety for $\{\underline{a}\}$.

2.4 Theorem (V. Voevodsky)**.** *Assume that Bloch - Kato conjecture holds in weight $(n-1)$. Let $\{a_1, \ldots, a_{n-1}\} \in K^M_{n-1}(k)/l$ be any non - zero symbol. Assume that k is l - special and Y is a $\nu_{\leqslant n-2}$ splitting variety for the symbol $\{a_1, \ldots, a_{n-1}\}$. Then*

$$\big\{a \text{ such that } \{a_1, \ldots, a_{n-1}, a\} = 0 \text{ in } K^M_n(k)/l\big\} = (k^\times)^l N(\bar{A}_0(Y, \mathcal{K}_1))$$

The inclusion $\supset$ is straightforward in view of 1.20 applied to Y. The other one is non - trivial and will (hopefully) appear in a later version of [MC/l].

Finally we are able to show that for every l - special field F that splits $\{\underline{a}\}$ X has an F - valued rational point. Two cases are possible.

First case. F does not split $\{a_1, \ldots, a_{n-1}\}$. Then by 2.4 applied to Y_F we get $a_n \in (F^\times)^l N(\bar{A}_0(Y_F, \mathcal{K}_1))$. Hence by the theorem 1.20 part 3, that holds for Y by inductive assumption, there exists $y \in Y_F$ such that $[F(y) : F] = l$ and $\lambda \in F(y)^\times$ so that $a_n = a^l N_{F(y)/F}(\lambda) = N_{F(y)/F}(a\lambda)$. This data determines an F - valued point of W thus the one of X.

Second case. (This argument is due to V. Voevodsky.) F splits $\{a_1, \ldots, a_{n-1}\}$ hence Y_F has a rational point. By the lemma 2.5 below Y_F has some l distinct rational points $y_1, \ldots, y_l$ that determine an F-rational point $y = y_1 + \ldots + y_l$ of U_F. This point along with $(1, \ldots, 1, a_n) \in V_y$ determines an F - rational point in the fiber W_y. Hence W_F has a rational point. As above this data determines an F - valued point of W thus the one of X.

2.5 Lemma. *Let F be l - special, and Y be a smooth projective variety over F of dimension at least 1. If $Y(F) \neq \emptyset$ then $\#Y(F) = \infty$*

Proof. We may assume that Y/F is a curve. Let $y_1, \ldots, y_k$ be distinct rational points on Y. We need to exhibit one more point. Consider a divisor $\sum_1^k n_i y_i$ such that $n_i > 0$, $\sum n_i > 2g-2$, and $(\sum n_i, l) = 1$. By the Riemann - Roch theorem we can find a rational function f such that $(f)_\infty = \sum_1^k n_i y_i$. Let $(f)_0 = \sum m_j z_j$. Note that all z_j are different from all y_i and that $\sum m_j [F(z_j) : F] = \sum n_i$ is prime to l. Hence for at least one j the degree $[F(z_j) : F]$ is prime to l. F has no finite extensions of degree prime to l thus z_j is another rational point.

Finally we will show (again using parts 2 and 3 of the Rost's theorem 1.20) that the construction described above produces a variety satisfying the claim of the main theorem 0.1 for a field k that is not necessary l-special.

Let k be any field of characteristic zero and $\{\underline{a}\} = \{a_1, \ldots, a_n\} \in K^M_n(k)/l$ be any non - zero n - symbol. The case of $n = 2$ is well - known and we assume that $n > 2$. Let k' denote a maximal extension of k of degree prime to l.

Let X_1 be the Severi - Brauer variety of $\{a_1, a_2\}$. Let X_i for $2 \leqslant i \leqslant n-1$ be consecutively constructed from one another by means of the procedure described above. We already know, among other properties, that X_i is a splitting variety for $\{a_1, \ldots, a_{i+1}\}$ for each i.

2.6 Proposition. *X_{n-1} is a $\nu_{\leqslant n-1}$ variety.*

Recall that a smooth projective variety X is said to be ν_m if $d = \dim X = l^m - 1$ and $\deg_k s_d(X)$ is a multiple of l but not of l^2, were s_d is d - th Milnor class and $\deg_k$ stands for the degree of a zero - cycle with respect to the base field k. Note that the property to be ν_m therefore depends on the base field.

2.7 Lemma. *Let X be a smooth projective variety over F.*

1) Let F'/F be any field extension such that $X_{F'}$ is irreducible. Then X is ν_m over F if and only if $X_{F'}$ is ν_m over F'.

2) Let F/F'' be a finite extension of degree prime to l. Then X is ν_m over F if and only if X is ν_m over F''.

Proof. Obviously all the varieties under consideration have the same dimension. Moreover $\deg_{F'} s_d(X_{F'}) = \deg_F s_d(X)$ proves the first claim, while $\deg_{F''} s_d(X) = [F : F''] \deg_F s_d(X)$ proves the second one.

Note that the construction of splitting varieties given above is stable with respect to an extension of scalars. In particular $(X_i)_{k'}$ are splitting varieties for the non - zero symbols $\{a_1, \dots, a_{i+1}\} \in K^M_{i+1}(k')/l$. Since k' itself is l - special we proceed as in 1.21 and find ν_i varieties X'_i over k' that fit into the diagrams

$$\begin{tikzcd} & X'_i \arrow[dl, "f_i"'] \arrow[dr, "g_i"] & \\ (X_i)_{k'} & & (X_{n-1})_{k'}. \end{tikzcd}$$

All these diagrams must be defined over some finite subextension k''/k, that is they could be obtained by an extension of scalars from k'' to k' from

$$\begin{tikzcd} & X''_i \arrow[dl, "f'_i"'] \arrow[dr, "g'_i"] & \\ (X_i)_{k''} & & (X_{n-1})_{k''}. \end{tikzcd}$$

In particular each $X'_i = (X''_i)_{k'}$ hence by the lemma X''_i is ν_i over k'' and thus it is ν_i over k as well. Composing g'_i with the projection $(X_{n-1})_{k''} \to X_{n-1}$ we get the required map $X''_i \to X_{n-1}$ from a ν_i - variety to X_{n-1} for each $1 \leqslant i < n-1$. Similar argument shows that X_{n-1} itself is a ν_{n-1} - variety. Thus X is $\nu_{\leqslant n-1}$.

2.8 Remark. It was noted by A. Vishik that using the Landweber-Novikov operations in algebraic cobordisms one can prove that every ν_{n-1} - variety is in fact a $\nu_{\leqslant n-1}$ - variety. That fact makes the above argument unnecessary.

2.9 Proposition. *The norm map $N : \bar{A}_0(X_{n-1}, \mathcal{K}_1) \to k^\times$ is injective.*

Proof. Set $E := k(\sqrt[l]{a_1})$. Since E splits $\{a_1, a_2\}$ and *char* $k = 0$ the Severi - Brauer variety X_1 has infinitely many E - rational points. The argument preceding 2.5 shows that each X_i has infinitely many E - rational points. Thus by corollary 1.7 kernel of N is annihilated by $[E : k] = l$. On the other hand $\ker N$ vanishes after extension to k' hence the orders of all its elements are prime to l. Thus $\ker N = 1$.

3 Rost degree formula

To prove the second claim of the theorem 1.20 we will develop a version of degree formula invented by Markus Rost. With that goal in mind we begin by defining the degree for zero - cycles on an open subscheme relative to the ambient projective variety.

Let S/k be any projective (not necessarily smooth) variety of the dimension d. The degree homomorphism $\deg_S : CH_0(S) \to \boldsymbol{Z}$ is nothing but the proper push - forward π_* induced by the structure morphism $\pi : S \to \operatorname{Spec} k$. Let $I(S) := \deg_S CH_0(S)$ denote the subgroup of $\boldsymbol{Z}$ generated by the degrees of the closed points.

For a proper morphism $i : S_0 \to S$ of projective varieties there is the usual commutative diagram of push - forwards

$$\begin{tikzcd} CH_0(S_0) \arrow[rr, "i_*"] \arrow[dr, "\deg_{S_0}"'] & & CH_0(S) \arrow[dl, "\deg_S"] \\ & Z. & \end{tikzcd}$$

Let S/k be a projective variety, $S_0 \subset S$ be a closed subscheme, $U = S \setminus S_0$ be the complementary open subscheme, with the inclusion morphisms i and j respectively. From the diagram

$$\begin{tikzcd} CH_0(S_0) \arrow[r, "i_*"] \arrow[d, "\deg_{S_0}"'] & CH_0(S) \arrow[r, "j^*"] \arrow[d, "\deg_S"] & CH_0(U) \arrow[r] \arrow[d, dotted, "\deg U"] & 0 \\ I(S_0) \arrow[r] & \boldsymbol{Z} \arrow[r] & \boldsymbol{Z}/I(S_0) & \end{tikzcd}$$

we get a homomorphism $\deg_U : CH_0(U) \to \boldsymbol{Z}/I(S_0)$.

The following lemma summarizes the basic properties of the homomorphism $\deg_U$.

3.1 Lemma.
1) Let $S \supset S_0 \supset S_0'$ be a projective variety and two of its closed subschemes, let $U := S \setminus S_0$ and $U' := S \setminus S_0'$. Then

$I(S_0') \subset I(S_0)$ *and*

$\forall Z \in CH_0(U') \quad \deg_{U'}(Z) \equiv \deg_U(Z|_U) \mod I(S_0).$

2) Let $f : S \to S'$ be a morphism of projective varieties, $S_0' \subset S'$ be a closed subscheme. Set $S_0 := f^{-1}(S_0')$, $U' := S' \setminus S_0'$, and $U := S \setminus S_0$. Then $I(S_0) \subset I(S_0')$ and the diagram

$$\begin{tikzcd} CH_0(U) \arrow[r, "(f|_U)_*"] \arrow[d, "\deg_U"'] & CH_0(U') \arrow[d, "\deg_{U'}"] \\ \boldsymbol{Z}/I(S_0) \arrow[r] & \boldsymbol{Z}/I(S_0') \end{tikzcd}$$

commutes.

3) Let $f : S \to S'$ be a morphism of projective varieties of equal dimension. Let $S_0 \subset S$, $S_0' \subset S'$ be closed subschemes, and set $U := S \setminus S_0$, $U' := S' \setminus S_0'$. Assume that $f^{-1}(S_0') \subset S_0$ and hence $f(U) \subset U'$. Finally assume that U' is smooth. Then for every cycle $Z \in CH_0U'$

$$\deg_U\left((f|_U)^*(Z)\right) \equiv \deg f \ \deg_{U'}(Z) \mod I(S_0) + I(S_0').$$

Proof. (Sketch) For 1) note that $S_0' \subset S_0$ implies that $U \subset U'$, $I(S_0') \subset I(S_0)$, so all the claims make sense and follow from the definition. Similarly 2) follows from the commutative diagram

$$\begin{tikzcd} CH_0(S_0) \arrow[r] \arrow[d, "(f|_{S_0})_*"'] & CH_0(S) \arrow[r] \arrow[d, "f_*"] & CH_0(U) \arrow[r] \arrow[d, "(f|_U)_*"] & 0 \\ CH_0(S_0') \arrow[r] & CH_0(S') \arrow[r] & CH_0(U') \arrow[r] & 0. \end{tikzcd}$$

For 3) observe, first of all, that for $Z \in CH_0(U')$

$$(f|_U)^*(Z) = (f|_{f^{-1}(U')})^*(Z)|_U$$

hence by 1)

$$\deg_U (f|_U)^*(Z) \equiv \deg_{f^{-1}(U')}(f|_{f^{-1}(U')})^*(Z) \mod I(S_0).$$

Thus replacing U by $f^{-1}(U')$ we may assume that $U = f^{-1}(U')$. Now $f|_U : U \to U'$ is proper and the projection formula yields

$$(f|_U)_*((f|_U)^*(Z)) = (\deg f)Z \in CH_0(U').$$

Finally according to 2) we get

$$\begin{aligned}(\deg f)\deg_{U'}(Z) &\equiv \deg_{U'}(f|_U)_*((f|_U)^*(Z)) \\ &\equiv \deg_U((f|_U)^*(Z)) \mod I(S_0) + I(S_0').\end{aligned}$$

Next we construct an invariant for pseudo - Galois coverings.

3.2 Definition. Let $p : X \to S$ be a finite surjective morphism of integral schemes. Let G be a finite group acting on X over S. The covering p is called pseudo - Galois provided $k(X)/k(S)$ is a Galois field extension and the natural map $G \to Gal(k(X)/k(S))$ is an isomorphism.

3.3 Remark. Under conditions of the definition there is an induced birational morphism $\bar{p} : X_G \to S$. If in addition S is normal then according to Zariski's Main Theorem $\bar{p}$ is an isomorphism.

3.4 Remark. It is well known and easy to check that every diagram of the form

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

where vertical morphisms are Galois coverings with the same group G and the top horizontal morphism is G - equivariant is in fact Cartesian.

3.5 Notation. Let $S_{unr} \subset S$ be the open subscheme over which the morphism p is étale, let $S_{ram} := S \setminus S_{unr}$ be the closed ramification subscheme.

To simplify matters we only consider pseudo - Galois coverings with $G = \boldsymbol{Z}/l$. We assume that *char* $k \neq l$ and that k contains a primitive root of unity. Furthermore we choose an identification $\mu_l = \boldsymbol{Z}/l$.

The Kummer sequence $1 \longrightarrow \mu_l \longrightarrow \mathbb{G}_m \xrightarrow{l} \mathbb{G}_m \longrightarrow 1$ induces an epimorphism $H^1_{et}(S_{unr}, \mu_l) \twoheadrightarrow {}_l Pic(S_{unr})$.

Finally starting with p we get an étale Galois covering $p^{-1}(S_{unr}) \longrightarrow S_{unr}$, the corresponding element in $H^1_{et}(S_{unr}, \boldsymbol{Z}/l) = H^1_{et}(S_{unr}, \mu_l)$, its image in ${}_l Pic(S_{unr})$, and thus an invertible sheaf $L(X/S)$ on S.

3.6 Definition. Assume that $p : X \to S$ is a pseudo - Galois covering with the group $G = \boldsymbol{Z}/l$, S is projective, and the assumptions made above hold. Assume further that there exists a closed subscheme $S_{bad} \subset S$ such that

(a) $I(S_{bad}) \subset l\boldsymbol{Z}$;

(b) $S_{good} := S \setminus S_{bad}$ is smooth;

(c) over S_{good} the morphism p is étale.

This data determines an invertible sheaf $L(X/S) \in {}_l Pic(S_{good})$ and a zero - cycle $Z(X/S)$ defined as $c_1(L(X/S))^{\dim S} \in CH_0(S_{good})$. Finally we define the η - invariant of the covering p as

$$\eta(X/S) := \deg_{S_{good}}(c_1(L(X/S))^{\dim S}) \in \boldsymbol{Z}/l.$$

3.7 Remark. $\eta(X/S)$ does not depend on the choice of a closed subscheme S_{bad}. If $\tilde{S}_{bad}$ is another such subscheme one could compute η using $\tilde{S}_{bad} \cup S_{bad}$ and evidently get the same result. $\eta(X/S)$ depends on the choice of a primitive root of unity $\zeta \in \mu_l$. Once ζ is replaced by ζ^s, $L(X/S)$ gets replaced by $L(X/S)^{\otimes s}$, and $\eta(X/S)$ by $(s^{\dim_S})\eta(X/S)$. This will not cause any difficulties as long as the same choice is maintained throughout.

3.8 Theorem (Markus Rost Degree Formula)**.** *Assume that k is a field of characteristic zero, that X/S and X'/S' are two pseudo - Galois coverings with Galois groups $G = G' = \boldsymbol{Z}/l$, that both S and S' are projective of the same dimension d, and that $\eta(X/S)$ and $\eta(X'/S')$ are defined. Then for any G - equivariant rational map $g : X - \to X'$*

$$\eta(X/S) = \deg g \, \eta(X'/S') \in \boldsymbol{Z}/l.$$

Proof. Note that g induces a morphism from a neighborhood of the generic point of S to S'. Hence there is a unique rational map $f : S - \to S'$ compatible with g and clearly $\deg f = \deg g$.

3.9 Lemma. *Let X/S and X'/S' be pseudo - Galois coverings with the same group G fitting into an equivariant diagram of morphisms and rational maps*

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ S \arrow[r, dashed, "f"] & S'. \end{tikzcd}$$

1) Assume that g is everywhere defined and that S is normal. Then f is everywhere defined.
2) Assume that f is everywhere defined and that X is normal. Then g is everywhere defined.

Proof. 1) The morphism g induces a morphism $\bar{g} : X_G \to X'_G$. Since S is normal p induces an isomorphism $\tilde{p} : X_G \to S$. Hence birational map f may be defined everywhere by $\tilde{p}' \circ \bar{g} \circ (\tilde{p})^{-1}$, where $\tilde{p}' : X'_G \to S'$ is induced by p'.

2) Let $\tilde{X}$ be the normalization of S' in $k(X')$. Since X'/S' is finite there is a morphism $\rho : \tilde{X} \to X'$ over S'. Because X is normal g induces a morphism $\tau : X \to \tilde{X}$. Evidently the morphism $\tau \circ \rho$ represents the rational map g.

First we prove the special case of the theorem. Assume that g is everywhere defined and that S is normal. By the lemma f is everywhere defined and we get the diagram of morphisms

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ S \arrow[r, "f"] & S'. \end{tikzcd}$$

Replacing S_{bad} by $S_{bad} \cup f^{-1}(S'_{bad})$ if necessary we may assume that $S_{bad} \supset f^{-1}(S'_{bad})$. In the equivariant diagram

$$\begin{tikzcd} p^{-1}(S_{good}) \arrow[r, "g|_{p^{-1}(S_{good})}"] \arrow[d, "p"'] & p^{-1}(S'_{good}) \arrow[d, "p'"] \\ S_{good} \arrow[r, "f|_{S_{good}}"] & S'_{good} \end{tikzcd}$$

both vertical arrows are étale coverings with the same Galois group G. Hence the left one is the pull - back of the right one and the diagram is Cartesian. In particular $L(X/S) = (f|_{S_{good}})^* L(X'/S')$. (Recall that both are in ${}_lPic(S_{good})$.) Therefore $Z(X/S) = (f|_{S_{good}})^*(Z(X'/S'))$ in $CH_0(S_{good})$. Finally the part 3) of lemma 3.1 completes the proof of this special case.

Now to the general case.

3.10 Lemma. *Assume that X/S is any pseudo - Galois covering with the Galois group $G = \boldsymbol{Z}/l$. Let $\phi : \tilde{S} \to S$ be a birational morphism. Set $\tilde{X}$ to be the normalization of $\tilde{S}$ in the finite field extension $k(X) \supset k(S) = k(\tilde{S})$.*
1) There exists a unique morphism $\psi : \tilde{X} \to X$ that completes the diagram

$$\begin{tikzcd} \operatorname{Spec} k(\tilde{X}) \arrow[r, equal] \arrow[d] & \operatorname{Spec} k(X) \arrow[d] \\ \tilde{X} \arrow[r, "\psi"] \arrow[d, "\tilde{p}"'] & X \arrow[d, "p"] \\ \tilde{S} \arrow[r, "\phi"] & S \end{tikzcd}$$

moreover ψ is G - equivariant and $\tilde{X}/\tilde{S}$ is a pseudo - Galois covering with the group G.
2)Assume in addition that S and $\tilde{S}$ are projective, $\tilde{S}$ is smooth and $\eta(X/S)$ is defined. Then $\eta(\tilde{X}/\tilde{S})$ is defined as well and $\eta(X/S) = \eta(\tilde{X}/\tilde{S}) \in \boldsymbol{Z}/l$.

Proof. The first claim is trivial by construction.

To prove the second one we need to check the conditions (a), (b), and (c) of the definition 3.6. Set $\tilde{S}_{bad} := \phi^{-1}(S_{bad})$ hence $\tilde{S}_{good} = \phi^{-1}(S_{good})$. Since $I(\tilde{S}_{bad}) \subset I(S_{bad}) \subset l\boldsymbol{Z}$

condition (a) holds. Since $\tilde{S}$ is smooth so is $\tilde{S}_{good}$ hence (b). For (c) observe that $\tilde{S}_{good}$ is smooth and hence normal, thus $\tilde{S}_{good} = (\tilde{p}^{-1}(\tilde{S}_{good}))_G$. In the following G - equivariant diagram the right vertical arrow is an étale Galois covering. Moreover since the action of G on $p^{-1}(S_{good})$ is free the action of G on $\tilde{p}^{-1}(\tilde{S}_{good})$ is free as well.

$$\begin{tikzcd}
\tilde{p}^{-1}(\tilde{S}_{good}) \arrow[r, "\psi"] \arrow[d, "\tilde{p}"'] & p^{-1}(S_{good}) \arrow[d, "p"] \\
\tilde{S}_{good} \arrow[r, "\phi"] & S_{good}
\end{tikzcd}$$

Hence $\tilde{p} : \tilde{p}^{-1}(\tilde{S}_{good}) \to (\tilde{p}^{-1}(\tilde{S}_{good}))_G = \tilde{S}_{good}$ is an étale Galois covering too. Finally the equality $\eta(X/S) = \eta(\tilde{X}/\tilde{S})$ in $\boldsymbol{Z}/l$ follows from the special case of the theorem.

Now let X, S, X', S', g be as in the statement of the theorem. Set X'' to be the closure of the graph of g in $X \times X'$. Two projections induce the birational morphism $\psi' : X'' \to X$ and the morphism $g' : X'' \to X'$, so that $g' = g \circ \psi'$ as rational maps. Moreover G acts on X'', and ψ', g' are equivariant. Set $S'' := (X'')_G$, choose a birational morphism $\tilde{S} \to S''$ with $\tilde{S}$ smooth. Let $\tilde{X}$ be the normalization of $\tilde{S}$ in $k(X) \supset k(S) = k(\tilde{S})$ as in the lemma. We have the diagram

$$\begin{tikzcd}
 & \tilde{X} \arrow[ddl, bend right, "\psi"'] \arrow[d] \arrow[ddr, bend left, "\tilde{g}"] \arrow[ddd] & \\
 & X'' \arrow[dl, "\psi'"'] \arrow[dr, "g'"] \arrow[ddd] & \\
X \arrow[rr, dashed, "g"'] \arrow[ddd] & & X' \arrow[ddd] \\
 & \tilde{S} \arrow[ddl, bend right, "\phi"'] \arrow[d] \arrow[ddr, bend left, "\tilde{f}"] & \\
 & S'' \arrow[dl, "\phi'"'] \arrow[dr, "f'"] & \\
S \arrow[rr, dashed, "f"'] & & S'
\end{tikzcd}$$

where ϕ is the obvious composition, ψ is the morphism that comes from the lemma, morphism $\tilde{X} \to X''$ also comes from the lemma, $\tilde{g}$ is another composition, and the remaining morphisms are the obvious ones.

Applying the lemma to $\tilde{X}/\tilde{S}$ and X/S we conclude that $\eta(\tilde{X}/\tilde{S})$ is defined and that

$$\eta(\tilde{X}/\tilde{S}) = \eta(X/S).$$

The coverings $\tilde{X}/\tilde{S}$ and X'/S' meet the conditions for the special case so

$$\eta(\tilde{X}/\tilde{S}) = \deg \tilde{g}\, \eta(X'/S').$$

Since $\deg \tilde{g}$ clearly equals $\deg g$ these two relations complete the proof.

Now we compute the η invariant for coverings of the special type.

3.11 Definition. Let S/k be an arbitrary scheme, L be an invertible sheaf of $\mathcal{O}_S$ - modules and $\alpha \in \Gamma(S, L^{\otimes l})$ be a global section. (Recall that we assume k to contain an l - th primitive root of unity.) Let $\boldsymbol{A}(L) := \mathrm{Spec}\,(S^*(L^{\#}))$ denote the line bundle corresponding to L. The sheaf $L_{\boldsymbol{A}(L)}$ has a canonical section T corresponding to the diagonal embedding $\boldsymbol{A}(L) \xrightarrow{\Delta} \boldsymbol{A}(L) \times_k \boldsymbol{A}(L) = \boldsymbol{A}(L_{\boldsymbol{A}(L)})$. Finally let $S(\sqrt[l]{\alpha})$ be the Cartier divisor in $\boldsymbol{A}(L)$ defined by the global section $T^{\otimes l} - \alpha \in \Gamma(\boldsymbol{A}(L), L^{\otimes l}_{\boldsymbol{A}(L)})$.

If L is trivial over some open affine $U \subset S$, then α determines a regular function a on U. Hence over U the scheme $S(\sqrt[l]{\alpha})$ is given by the equation $T^l - a = 0$ in $U \times \boldsymbol{A}^1$. In particular $S(\sqrt[l]{\alpha}) \to S$ is flat and finite of degree l.

3.12 Lemma. *Assume that S is smooth and irreducible and that $\alpha \notin \Gamma(S, L)^{\otimes l}$. Then $\phi : S(\sqrt[l]{\alpha}) \to S$ is a pseudo - Galois covering with the group $G = \boldsymbol{Z}/l$.*

Proof. First we verify that $S(\sqrt[l]{\alpha})$ is integral. This may be checked locally. Over an affine U as above $S(\sqrt[l]{\alpha})$ coincides with $\mathrm{Spec}\, A[T]/T^l - a$ where $A = k[U]$. $\mathrm{Spec}\, A[T]/T^l - a$ is integral if and only if $T^l - a \in A[T]$ is irreducible. However if the latter polynomial is reducible then $a = b^l$ for $b \in k(U) = k(S)$ hence $\alpha = \beta^{\otimes l}$ for a rational section β of L. Noting that $l(\beta) = (\alpha)$ and so β has no poles we conclude that β is regular, and $\alpha \in \Gamma(S, L)^{\otimes l}$, a contradiction.

$\mathbb{G}_m$ acts naturally on $\boldsymbol{A}(L)$ and so does $\mu_l \subset \mathbb{G}_m$. As is evident from the local description $S(\sqrt[l]{\alpha})$ is μ_l - invariant and moreover $\mu_l \xrightarrow{\sim} Gal(k(U(\sqrt[l]{a}))/k(U)) = Gal(k(S(\sqrt[l]{\alpha}))/k(S))$. The identification $\mu_l = \boldsymbol{Z}/l$ completes the proof.

Evidently the covering ϕ is unramified away from the vanishing subscheme of $V(\alpha)$ of α. Thus the following corollary is almost straightforward.

3.13 Corollary. *Assume that S is smooth, projective, and irreducible, and that $\alpha \notin \Gamma(S, L)^{\otimes l}$. Assume further that $I(V(\alpha)) \subset l\boldsymbol{Z}$. Then $\eta(S(\sqrt[l]{\alpha})/S)$ is defined and equals $\deg(-c_1 L)^{\dim S} \mod l\boldsymbol{Z}$.*

Proof. Since ϕ is étale over $S_{good} := S \setminus V(\alpha)$ then η is defined. Since the invertible sheaf corresponding to this covering is the dual sheaf $L^\vee$ then $\eta(S(\sqrt[l]{\alpha})/S) = \deg(c_1 L^\vee)^{\dim S} = \deg(-c_1 L)^{\dim S}$.

4 Computations with b - classes

4.1 Definition. Let X/k be a smooth absolutely irreducible projective variety. The group $G = \boldsymbol{Z}/l$ acts on the irreducible variety X^l by cyclic permutations of factors. We call the factor variety $C^l(X) := (X^l)_G$ *the l-th cyclic power of X.*

4.2 Remark. Note that $C^l(X)$ is a normal projective variety and that the projection $p : X^l \to C^l(X)$ is a pseudo - Galois covering with the group G. Let $\Delta : X \to X^l$ be the diagonal embedding of the fixed - point subscheme and Δ_X be its image. Then $X^l \setminus \Delta_X \to C^l(X) \setminus p(\Delta_X)$ is an étale Galois covering with the group G. In particular $C^l \setminus p(\Delta_X)$ is smooth. We thus set $C^l(X)_{bad} := p(\Delta_X)$.

4.3 Lemma. *The morphism $X \overset{\Delta}{\hookrightarrow} X^l \overset{p}{\to} C^l(X)$ is a closed embedding identifying X with $p\,(\Delta_X)$.*

Proof. The statement is local with respect to X. For $X = \mathrm{Spec}\, A$ we need to show that

$$(A^{\otimes l})^G \longrightarrow A^{\otimes l} \xrightarrow{mult} A$$

is surjective. This is so because for every $a \in A$ the composition maps $(A^{\otimes l})^G \ni (1/l)\sum_1^l (1 \otimes \cdots \otimes a \otimes \cdots \otimes 1)$ to a. (Recall that *char* $k = 0$.)

It follows that $C^l(X)_{bad}$ is isomorphic to X hence $\eta_l(X) := \eta(X^l/C^l(X))$ is defined if and only if $I(X) \subset l\boldsymbol{Z}$.

The invariant η_2 may be computed via the following result.

4.4 Theorem (Rost)**.** *Let X/K be smooth absolutely irreducible projective variety of dimension d. Then* $\deg(c_d(-T_X)) \in 2\boldsymbol{Z}$. *If in addition $I(X) \subset 2\boldsymbol{Z}$ then*

$$\eta_2(X) = \frac{\deg(c_d(-T_X))}{2} \mod 2\boldsymbol{Z}.$$

Proof. See Merkurjev's notes on degree formula [M].

We will be mostly interested in the case $l > 2$. Let $c = 1 + c_1 + \cdots + c_d : K_0(X) \to CH^*(X)$, where $d = \dim X$, be the total Chern class. We formally write $c = (1 - x_1)\dots(1 - x_d)$ with $\deg x_i = 1$.

4.5 Definition. *Total b - class of X* is $b = b^{(l)} := (1 - x_1^{l-1})\dots(1 - x_d^{l-1}) =: 1 + b_1 + ...$

4.6 Remark. Note that the operation $b : K_0(X) \to CH^*(X)$ is multiplicative that is $b(V \oplus V') = b(V)b(V')$ and that $b(L) = 1 - (-c_1(L))^{l-1}$ for a line bundle L. By the splitting principle these two properties completely determine b. Also note that $b_i = 0$ unless $l - 1|i$ and that $b_i = c_i$ for $l = 2$.

4.7 Theorem (Rost)**.** *Let X/K be a smooth absolutely irreducible projective variety of the dimension d. Let l be a fixed prime. Then* $\deg(b_d(-T_X)) \in l\boldsymbol{Z}$. *If in addition $I(X) \subset l\boldsymbol{Z}$ then*

$$\eta_l(X) = \frac{\deg(b_d(-T_X))}{l} \mod l\boldsymbol{Z}.$$

In particular, $\eta_l(X) = 0 \mod l\boldsymbol{Z}$ *if d is not a multiple of $l - 1$.*

Proof. See Rost's 'Notes on Degree Formula' web page [CL].

4.8 Proposition. *Assume that in conditions of the theorem $d = l^n - 1$. Then*

$$\deg(b_d(-T_X)) = (-1)^{l(n-1)} \deg(s_d(-T_X)) \mod l^2\boldsymbol{Z}.$$

In particular, if η_l is defined then

$$\eta_l(X) = \pm\frac{\deg(s_d(-T_X))}{l} \mod l\boldsymbol{Z}.$$

Proof. It is enough to prove the statement over the algebraic closure of k. Essentially one may either assume that $k = \mathbb{C}$ and use the topological complex cobordism theory or use the algebraic cobordism theory of Morel and Levin [LM].

Let $\Omega = MU_*$ be the Lazard ring of bordism classes. We need to show that the following map is the zero map.

$$\begin{array}{ccc} \Omega_d & \longrightarrow & \boldsymbol{Z}/l \\ [X] & \longmapsto & \dfrac{\deg(b_d(-T_X))}{l} - (-1)^{l(n-1)}\dfrac{\deg(s_d(-T_X))}{l} \end{array}$$

After localization outside l the component Ω_d is additively generated by decomposable elements and the class of any hypersurface of degree l in $\boldsymbol{P}^{d+1}$.

Case of decomposable $[X]$. Suppose $X = X_1 \times X_2$ with $d_1 := \dim X_1, d_2 := \dim X_2 < d$. Because $s_d(Y) = 0$ whenever $\dim Y < d$ we get

$$\begin{aligned} s_d(-T_X) = s_d(-T_{X_1\times X_2}) = s_d(p_1^*(-T_{X_1}) \oplus p_2^*(-T_{X_2})) \\ = p_1^*(s_d(-T_{X_1})) + p_2^*(s_d(-T_{X_2})) = 0 \end{aligned}$$

Since the total b - class is multiplicative and commutes with pull - backs

$$\begin{aligned} b_d(-T_X) = b_d(-T_{X_1\times X_2}) = \sum_{i+j=d} p_1^*(b_i(-T_{X_1}))p_2^*(b_j(-T_{X_2})) \\ = p_1^*(b_{d_1}(-T_{X_1}))p_2^*(b_{d_2}(-T_{X_2})) \end{aligned}$$

because all other terms vanish by dimensional reasons. Recall however that by the theorem 4.7 each factor is a multiple of l hence the product vanishes in $\boldsymbol{Z}/l$. We see that $[X_1 \times X_2]$ indeed maps to zero.

Case of a hypersurface. Let $X \subset \boldsymbol{P}^{d+1}$ be a hypersurface of degree l. The ideal sheaf I of X is isomorphic to $\mathcal{O}_{\boldsymbol{P}^{d+1}}(-l)$ hence the normal sheaf $N = (I|_X)^\vee$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^{d+1}}(l)|_X$ From the two usual exact sequences

$$0 \longrightarrow \mathcal{O}_{\boldsymbol{P}^{d+1}} \longrightarrow \mathcal{O}_{\boldsymbol{P}^{d+1}}(1)^{d+2} \longrightarrow T_{\boldsymbol{P}^{d+1}} \longrightarrow 0$$

$$0 \longrightarrow T_X \longrightarrow T_{\boldsymbol{P}^{d+1}}|_X \longrightarrow N \longrightarrow 0.$$

we conclude that $[-T_X] = V|_X \in K_0(X)$ where

$$V := [\mathcal{O}_{\boldsymbol{P}^{d+1}}(l)] - (d+2)[\mathcal{O}_{\boldsymbol{P}^{d+1}}(1)] + [\mathcal{O}_{\boldsymbol{P}^{d+1}}] \in K_0(\boldsymbol{P}^{d+1}).$$

Therefore $s_d(-T_X) = s_d(V)\cdot X$ and $b_d(-T_X) = b_d(V)\cdot X$.

Let $H = c_1(\mathcal{O}_{\boldsymbol{P}^{d+1}}(1))$ be the class of a hypersurface in $CH^1(\boldsymbol{P}^{d+1})$. Then (recall $d = l^n - 1$) we get $s_d(V) = (lH)^d - (l^n+1)H^d = (l^d - l^n - 1)H^d$ so $s_d(-T_X) = (l^d - l^n - 1)(H^d \,.\, X)$ hence $\deg(s_d(-T_X)) = (l^d - l^n - 1)\deg(H^d \,.\, X) = (l^d - l^n - 1)l$ and finally we conclude that

$$\frac{\deg(s_d(-T_X))}{l} = -1 \mod l\boldsymbol{Z}.$$

Next we observe that $b_d(V)\,.\,X$ is the 0 - degree component of $b(V)\,.\,X$. We may write $b(V) = (1+(lH)^{l-1})/(1+H^{l-1})^{l^n+1}$. It follows that $\deg(b_d(-T_X))/l$ is the coefficient at H^d in $b(V)$ and we only need to know it modulo l. Since

$$b(V) = \frac{(1+(lH)^{l-1})}{(1+H^{l-1})^{l^n+1}} = \frac{1}{(1+H^{(l-1)l^n})(1+H^{l-1})} \mod l$$

$b(V)$ has the same coefficient at H^d as $1/(1+H^{l-1})$ and we conclude that

$$\frac{\deg(b_d(-T_X))}{l} = (-1)^{1+l+\ldots l^{n-1}} = (-1)^{n(l-1)}$$

and we are done.

5 The Chain Lemma

Let J be an invertible sheaf on X. A non - zero l - form $\gamma : J^{\otimes l} \to \mathcal{O}_X$ may be viewed as an element of $\Gamma(X, J^{\otimes(-l)})$. Let $U \subset X \setminus V(\gamma)$ be an open subscheme trivializing J and let $u \in \Gamma(U, J)$ be a non - vanishing section. Then $\gamma = au^{\otimes(-l)}$ for an appropriate $a \in \Gamma(U, \mathcal{O}_X^\times)$. Since a is well defined up to an $l - th$ power the form γ gives rise to a well - defined element $\gamma_U \in \Gamma(U, \mathcal{O}_X^\times)/\Gamma(U, \mathcal{O}_X^\times)^l$.

Choose $x \in X \setminus V(\gamma)$. The above construction applied to neighborhoods of x provides an element $\gamma_x \in \mathcal{O}_x^\times/(\mathcal{O}_x^\times)^l$. Denote by $\gamma(x) \in k(x)^\times/(k(x)^\times)^l$ the corresponding element. Choosing x to be the generic point we get $\gamma(X) \in k(X)^\times/(k(X)^\times)^l$ assigned to the form γ. By abuse of notation we will write just γ instead of $\gamma(X)$ since no confusion will occur.

Let $J_1, \dots, J_n$ be invertible sheaves equipped with non - zero l - forms $\gamma_1, \dots, \gamma_n$ respectively. To this collection of sheaves and forms we can assign the symbol $\{\gamma_1, \dots, \gamma_n\} \in K_n^M(k(X))/l$.

5.1 Theorem (Rost's Chain Lemma). *Let $\{a_1, \dots, a_n\} \in K_n^M(k)/l$ be a non-trivial n-symbol. Then there exists a smooth projective cellular variety S/k and a collection of invertible sheaves $J = J_1, J_1' \dots, J_{n-1}, J_{n-1}'$ equipped with non -zero l - forms $\gamma = \gamma_1, \gamma_1', \dots, \gamma_{n-1}, \gamma_{n-1}'$ respectively satisfying the following conditions.*

1. $\dim S = l(l^{n-1} - 1) = l^n - l$;
2. $\{a_1, \dots, a_n\} = \{a_1, \dots, a_{n-2}, \gamma_{n-1}, \gamma_{n-1}'\} \in K_n^M(k(S))/l$, *for each* $2 \leqslant i \leqslant n-1$ $\{a_1, \dots, a_{i-1}, \gamma_i\} = \{a_1, \dots, a_{i-2}, \gamma_{i-1}, \gamma_{i-1}'\} \in K_i^M(k(S))/l$, *and in particular* $\{a_1, \dots, a_n\} = \{\gamma, \gamma_1', \dots, \gamma_{n-1}'\} \in K_n^M(k(S))/l$;
3. $\gamma \notin \Gamma(S, J)^{\otimes(-l)}$, *as is evident from (2);*
4. *for any $s \in V(\gamma_i)$ or $V(\gamma_i')$ the field $k(s)$ splits $\{a_1, \dots, a_n\}$;*
5. $I(V(\gamma_i)), I(V(\gamma_i')) \subset l\boldsymbol{Z}$ *for all i, as follows from (4);*
6. $\deg(c_1(J)^{\dim S})$ *is relatively prime to l.*

Proof. See Markus Rost's 'Notes on Degree Formula' web page and also [R].

Our first application of the Chain Lemma is

5.2 Proposition. *Let X be an absolutely irreducible l - generic splitting variety for a non-zero symbol $\{\underline{a}\} = \{a_1, \dots, a_n\} \in K_n^M(k)/l$ of the dimension $d = l^{n-1} - 1$. Then X is a ν_{n-1} - variety.*

Proof. We adopt all the notation from the statement of the Chain Lemma. By construction $k(S)(\sqrt[l]{\gamma})$ splits $\{\underline{a}\}$. Let F_∞ be a maximal extension of $k(S)$ of degree prime to l. Then $F_\infty(\sqrt[l]{\gamma})$ is l-special and also splits $\{\underline{a}\}$. Hence there exists a morphism $\operatorname{Spec} F_\infty(\sqrt[l]{\gamma}) \to X$ over k. Since X is of finite type this morphism may be factored through $\operatorname{Spec} F(\sqrt[l]{\gamma}) \to X$ for a certain finite subextension $k(S) \subset F \subset F_\infty$. Starting with the embedding $k(S) \subset F$ we choose a model for F and then resolve singularities to obtain a smooth projective variety $\tilde{S}$ equipped with a dominant morphism $h : \tilde{S} \longrightarrow S$ of a degree prime to l, and a rational map $\phi : \tilde{S}(\sqrt[l]{\gamma}) - \to X$. (Since $k(\tilde{S}(\sqrt[l]{\gamma})) = k(\tilde{S})(\sqrt[l]{\gamma}) = F(\sqrt[l]{\gamma})$.)

Let σ be a generator of $G = \boldsymbol{Z}/l = \mu_l$. By construction we get an equivariant diagram of pseudo - Galois coverings

$$\begin{tikzcd}
\tilde{S}(\sqrt[l]{\gamma}) \arrow[r, dashed, "{(\phi,\phi\sigma,\ldots,\phi\sigma^{l-1})}"] \arrow[d] & X^l \arrow[d] \\
\tilde{S} \arrow[r, dashed] & C^l(X)
\end{tikzcd}$$

with the bottom map induced by the top one. Note that $\dim S = \dim C^l(X)$. We will apply Rost's degree formula to this diagram.

First observe that by the Chain Lemma the form γ is not an l-th power and $I(V(\gamma)) \subset l\boldsymbol{Z}$ hence $\eta(\tilde{S}(\sqrt[l]{\gamma})/\tilde{S})$ is defined and

$$\begin{aligned}
\eta(\tilde{S}(\sqrt[l]{\gamma})/\tilde{S}) &= \deg(c_1(h^*(J))^{\dim \tilde{S}}) \\
&= \deg(h^*(c_1(J))^{\dim S}) \\
&= \deg(h_*(h^*(c_1(J))^{\dim S})) \\
&= \deg h \deg(c_1(J)^{\dim S}).
\end{aligned}$$

Note that both factors are prime to l by the construction and by the Chain Lemma respectively.

Next recall that by the proposition 4.8

$$\eta(X^l/C^l(X)) = \pm\frac{s_d(-T_X)}{l} \mod l\boldsymbol{Z}.$$

Finally by the Degree Formula

$$\deg h \deg(c_1(J)^{\dim S}) = \deg g \frac{s_d(-T_X)}{l} \mod l\boldsymbol{Z}$$

where $g := (\phi, \phi\sigma, \ldots, \phi\sigma^{l-1})$. We readily conclude that neither of the factors on the right is a multiple of l. In particular X is ν_{n-1}.

5.3 Remark. For a variety X/k let $k_c(X)$ denote *the field of constants of* X that is the algebraic closure of k in $k(X)$. It is well known, and easy to verify, that X is absolutely irreducible if and only if $k_c(X) = k$. Also note that a rational map $X - \rightarrow Y$ induces an embedding $k_c(Y) \hookrightarrow k_c(X)$.

5.4 Proposition. *Assume that k is l - special field. Then every l - generic splitting variety X/k for a symbol $\{\underline{a}\}$ is absolutely irreducible.*

Proof. Let Y be an absolute irreducible l - generic splitting variety X/k for $\{\underline{a}\}$ that exists according the first part of 1.20. Then there exists an extension $F/k(Y)$ of degree prime to l and a point $\operatorname{Spec} F \to X$. Fields involved form a diagram of embeddings

$$\begin{tikzcd}
 & F \arrow[dl, dash, "\text{prime to } l"'] \arrow[ddr, dash] & \\
k(Y) \arrow[ddr, dash] & & \\
 & & k_c(X). \arrow[dl, dash, "\text{power of } l"] \\
 & k &
\end{tikzcd}$$

Since Y is absolutely irreducible $k(Y) \otimes_k k_c(X)$ is a subfield of F. Thus degree count shows that $k_c(X) = k$ and X is absolutely irreducible.

The second claim of the theorem 1.20 now follows from the above two propositions.

The second application of the Chain Lemma will be concerned with the so - called "multiplication principle".

Consider the variety S of the Chain Lemma. Let $s \in S \setminus \bigcup_1^{n-1}(V(\gamma_i) \cup V(\gamma_i'))$ be a rational point. Specialization of $\{\gamma(S), \gamma_1'(S), \dots, \gamma_{n-1}'(S)\}$ from $K_n^M(k(S))/l$ to $K_n^M(k(s))/l$ amounts to evaluation hence

$$\{a_1, \dots, a_n\} = \{\gamma(s), \gamma_1'(s), \dots, \gamma_{n-1}'(s)\} \text{ in } K_n^M(k)/l.$$

In particular $k(\sqrt[l]{\gamma(s)})$ splits the symbol $\{\underline{a}\}$. (It may be shown that specialization to a rational point of S provides a universal way to rewrite the symbol.)

5.5 Theorem. *Let k be l - special. Let E/k be a cyclic extension of degree l splitting $\{\underline{a}\}$. Then there is a rational point $s \in S$ such that $k(\sqrt[l]{\gamma(s)}) = E$.*

Proof. Recall that there is a dominant morphism of smooth projective varieties $\tilde{S} \to S$ of degree prime to l along with an equivariant diagram of pseudo - Galois coverings

$$\begin{tikzcd}
\tilde{S}(\sqrt[l]{\gamma}) \arrow[r, dashed, "g=(\phi,\phi\sigma,\dots,\phi\sigma^{l-1})"] \arrow[d] & X^l \arrow[d] \\
\tilde{S} \arrow[r, dashed] & C^l(X).
\end{tikzcd}$$

Moreover $\deg g$ is prime to l and, *a fortiori* g is dominant. Using the resolution of singularities we can find a birational morphism $\hat{S} \to \tilde{S}$ from a smooth projective variety such that the composition $\hat{S} \to \tilde{S} - \to C^l(X)$ is everywhere defined. Then the previous diagram induces the following one

$$\begin{tikzcd}
\hat{S}(\sqrt[l]{\gamma})_{\mathrm{norm}} \arrow[r, "\hat{g}"] \arrow[d] & X^l \arrow[d] \\
\hat{S} \arrow[r, "\hat{f}"] & C^l(X)
\end{tikzcd}$$

that is also equivariant and consists of pseudo - Galois coverings. Indeed, the normalization does not change these properties. The bottom map is everywhere defined by construction and so is the top one by lemma 3.9.

On the other hand, since E is also l - special and splits $\{\underline{a}\}$ there is an E - valued point $\psi : \operatorname{Spec} E \to X$ that gives rise to the diagram

$$\begin{tikzcd}
\operatorname{Spec} E \arrow[r, "(\psi,\psi\sigma,\ldots,\psi\sigma^{l-1})"] \arrow[d] & X^l \arrow[d] \\
\operatorname{Spec} k \arrow[r] & C^l(X).
\end{tikzcd}$$

Let the rational point $z \in C^l(X)$ be the image of the bottom map. Since the diagonal of X^l has no rational points z belongs to both the smooth locus of $C^l(X)$ and the unramified locus of $X^l \to C^l(X)$. Hence the diagram is Cartesian and the fiber over z consists of a single E - rational point of X^l.

Since $\hat{f}$ is a dominant morphism of degree prime to l and z is smooth, using lemma 1.18 we can lift z to a rational point $\hat{s} \in \hat{S}$. Note that $V(\gamma)$ has no rational points, and that $\tilde{S}(\sqrt[l]{\gamma}) \to \hat{S}$ is unramified away from $V(\gamma)$. Therefore the fiber over $\hat{s}$ is the same in both $\hat{S}(\sqrt[l]{\gamma})$ and $\hat{S}(\sqrt[l]{\gamma})_{\mathrm{norm}}$. Moreover since the diagram in question is Cartesian locally near z this fiber is a single point with the residue field $k(\sqrt[l]{\gamma(\hat{s})}) = E$. Set s to be the image of $\hat{s}$ under the projection $\hat{S} \to S$. Since both $\hat{s}$ and s are rational we conclude that $k(\sqrt[l]{\gamma(s)}) = k(\sqrt[l]{\gamma(\hat{s})}) = E$.

As an easy corollary we get following statement, also referred to as *Chain Lemma.*

5.6 Theorem. *Let k be l - special and $E_1, \ldots, E_n$ be cyclic splitting fields of degree l for a non-trivial symbol $\{a_1, \ldots, a_n\} \in K_n^M(k)/l$. Then there exist $a'_1, \ldots, a'_n \in k^\times$ such that $\{a_1, \ldots, a_n\} = \{a'_1, \ldots, a'_n\}$ and that E_i splits $\{a'_1, \ldots, a'_i\}$ for each $1 \leqslant i \leqslant n$.*

Proof. Using induction on j we will show that we can rewrite the given symbol so that the last condition holds true for $1 \leqslant i \leqslant j$.

The case of $j = 1$ is settled by the previous theorem.

Induction step from $j-1$ to j. Applying the assumption to $E_2, \ldots, E_j$ we may rewrite the symbol so that E_2 splits $\{a_1\}$, $\ldots$, E_j splits $\{a_1, \ldots, a_{j-1}\}$.

By the previous theorem we may find a rational point $s \in S$ such that $E_1 = k(\sqrt[l]{\gamma(s)})$. We set $a'_1 := \gamma(s)$, $a'_2 := \gamma'_1(s)$, $\ldots$, $a'_n := \gamma'_{n-1}(s)$. Then E_1 splits $\{a'_1\}$ and for each $1 < i \leqslant j$ the field E_j splits $\{a_1, \ldots, a_{j-1}\}$ hence also splits $\{a_1, \ldots, a_{j-1}, \gamma'_{j-1}\} = \{\gamma(s), \gamma'_1(s), \ldots, \gamma'_{j-1}\} = \{a'_1, \ldots, a'_j\}$.

Observe that the third part of the theorem 1.20 would be an immediate corollary of the following two statements. (Recall that we assume k to be l - special and X to be $l^{n-1} - 1$ dimensional l - generic splitting variety for an n - symbol $\{\underline{a}\}$.)

5.7 Proposition (Multiplication Principle)**.** *Let $[x, \lambda], [x', \lambda'] \in \bar{A}_0(X, \mathcal{K}_1)$ be such that $[k(x) : k] = [k(x') : k] = l$. Then there exist $x'' \in X$, $\lambda'' \in k(x'')^\times$ such that $[k(x'') : k] = l$ and $[x, \lambda] + [x', \lambda'] = [x'', \lambda'']$.*

5.8 Proposition (Norm Principle)**.** *Let $[x, \lambda] \in \bar{A}_0(X, \mathcal{K}_1)$ be such that $[k(x) : k] = l^m$, where $m > 1$. Then there exist $x_i \in X$, $\lambda_i \in k(x_i)^\times$ such that $[k(x_i) : k] < [k(x) : k]$ for all i and $[x, \lambda] = \sum_i [x_i, \lambda_i]$.*

Below we give a proof of multiplication principle leaving out the norm principle till the spring term [R].

We may rewrite $\{\underline{a}\} = \{a'_1, a'_2, \dots\}$ so that $k(x)$ splits $\{a'_1\}$ and $k(x')$ splits $\{a'_1, a'_2\}$. Let $D := \binom{a'_1, a'_2}{k}$ be the cyclic algebra and let $Y := SB(D)$ be its Severi-Brauer variety.

The following two facts in one form or another are well established in the folklore so we only sketch their proofs.

5.9 Lemma. *Multiplication principle holds for Y.*

Proof. Let $[y, \lambda]$ in $\bar{A}_0(Y, \mathcal{K}_1)$ be such that $[k(y) : k] = l$. Then $k(y)$ may be identified with a maximal subfield of D and moreover $N([y, \lambda]) = \mathrm{Nrd}\,(\lambda) \in k^\times$. Recall that according to [MS]

$$N : A_0(Y, \mathcal{K}_1) \xrightarrow{\sim} \mathrm{Nrd}\,(D^\times) \subset k^\times$$

is an isomorphism. (Thus in this special case $\bar{A}_0(Y, \mathcal{K}_1) = A_0(Y, \mathcal{K}_1)$.)

Let $[y', \lambda']$ be the other summand. Form $\lambda\lambda'$ in $D^\times$ and choose $y'' \in Y$ such that $\lambda\lambda' \in k(y'')$. Since $N([y, \lambda])N([y', \lambda']) = \lambda\lambda' = N([y'', \lambda\lambda'])$ and N is an isomorphism we conclude that

$$[y, \lambda] + [y', \lambda'] = [y'', \lambda\lambda'].$$

5.10 Lemma. *Let $f : \tilde{Z} \to Z$ be a dominant morphism of smooth projective varieties of the degree relatively prime to l. Then $f_* : \bar{A}_0(\tilde{Z}, \mathcal{K}_1) \to \bar{A}_0(Z, \mathcal{K}_1)$ is an isomorphism.*

Proof. Recall that the base field k is assumed to be l - special. Hence for each generator $[z, \lambda]$ of $\bar{A}_0(Z, \mathcal{K}_1)$ one can find, according to 1.18, a point $\tilde{z} \in \tilde{Z}$ that maps to z so that $k(\tilde{z}) = k(z)$. Thus $f_*([\tilde{z}, \lambda]) = [z, \lambda]$ and we conclude that f_* is surjective.

To prove the injectivity we first show that the composition $f^* f_*$ coincides with multiplication by $\deg f$.

Choose any generator $[\tilde{z}, \lambda]$ of $\bar{A}_0(\tilde{Z}, \mathcal{K}_1)$. Let $z = f(\tilde{z})$. As above one can find $\tilde{z}'$ in the fiber over z having the residue field $k(\tilde{z}') = k(z)$. According to the corollary 1.5 we get $[\tilde{z}, \lambda] = [\tilde{z}', N_{k(\tilde{z})/k(z)}(\lambda)]$. Thus replacing one by the other we may assume that $\tilde{z}$ and $z = f(\tilde{z})$ have isomorphic residue fields. Consider any open $U \subset Z$ over which f is finite. One can show that $\bar{A}_0(Z, \mathcal{K}_1)$ and $\bar{A}_0(\tilde{Z}, \mathcal{K}_1)$ are generated by points from U and $\tilde{U} := f^{-1}(U)$ respectively. Hence we may assume that $\tilde{z} \in \tilde{U}$. In this case the fiber of f over z is finite and consists of points $\tilde{z}_1 = \tilde{z}, \dots, \tilde{z}_k$. Finally an explicit computation shows that

$$\begin{aligned} f^* f_*([\tilde{z}, \lambda]) = f^*([z, \lambda]) = \sum_1^k [\tilde{z}_i, \lambda] &= \sum_1^k [\tilde{z}, N_{k(\tilde{z}_i)/k(z)}(\lambda)] \\ &= \sum_1^k [\tilde{z}, \lambda^{[k(\tilde{z}_i):k(z)]}] = \Big(\sum_1^k [k(\tilde{z}_i) : k(z)]\Big)[\tilde{z}, \lambda] = (\deg f)[\tilde{z}, \lambda]. \end{aligned}$$

In particular we conclude that $\ker f_*$ is annihilated by $\deg f$. On the other hand the diagram

$$\begin{array}{ccc} \bar{A}_0(\tilde{Z}, \mathcal{K}_1) & \xrightarrow{f_*} & \bar{A}_0(Z, \mathcal{K}_1) \\ \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle N} \\ k^\times & \xrightarrow{=} & k^\times \end{array}$$

along with the corollary 1.7 demonstrates that $\ker f_* \subset \ker N$ is annihilated by the degree of any closed point that is by some power of l. Since $(degf, l) = 1$ we conclude that $\ker f = 0$ that is f_* is injective as well.

To prove the multiplication principle for the generic splitting variety X above we first note that $k(Y)$ splits $\{\underline{a}\}$. Therefore we can construct a smooth projective variety $\tilde{Y}$ along with a dominant morphism $p : \tilde{Y} \to Y$ of degree relatively prime to l such that there exists a morphism $\pi : \tilde{Y} \to X$.

Let $y, y' \in Y$ be such that $k(y) = k(x)$, $k(y') = k(x')$. According to 5.9 one can find another point $y'' \in Y$ of degree l and $\lambda'' \in k(y'')^\times$ such that $[y, \lambda] + [y', \lambda'] = [y'', \lambda'']$ in $\bar{A}_0(Y, \mathcal{K}_1)$. Points y, y', y'' may be lifted as in the proof of 5.10 to $\tilde{y}, \tilde{y'}, \tilde{y''} \in \tilde{Y}$ with the same residue fields and moreover $[\tilde{y}, \lambda] + [\tilde{y'}, \lambda'] = [\tilde{y''}, \lambda'']$ in $\bar{A}_0(\tilde{Y}, \mathcal{K}_1)$. Pushing $\tilde{y}, \tilde{y'}, \tilde{y''}$ down to X we finally get z, z', z'' such that $k(z) = k(\tilde{y}) = k(y) = k(x)$, similarly $k(z') = k(y')$, and $k(z'') = k(y'')$, and with the sought after relation

$$[x, \lambda] + [x', \lambda'] = [z, \lambda] + [z', \lambda'] = [z'', \lambda''].$$

Epilogue

Vigilant reader may have noticed that the expression "Norm Varieties" does not appear anywhere in the text. Let us point out that the term *norm variety* was coined to describe a variety given by an equation $N(x) = a$ where N is any norm map, while x and a are whatever circumstances dictate. Hence both the splitting varieties constructed from the symmetric powers and the varieties $S(\sqrt[l]{\alpha})$ deserve that name. Which ones the paper is entitled after is anybody's guess.

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