ON KUMMER CHAINS IN ALGEBRAS OF DEGREE 3

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`preliminary version`

Contents

Introduction

This text is a sequel to [1] and we adopt the conventions of that article. (It is available on `www.math.uni-bielefeld.de/~rost/chain-deg-3.html`.)

`These are preliminary notes!`

Kummer elements

A is a central simple algebra of degree 3 over k, $\zeta \in k$ is a primitive cube root of 1.

For the characteristic polynomial of $x \in A$ we use the notation

$$N(t-x) = t^3 - T(x)t^2 + Q(x)t - N(x)$$

where $N\colon A \to F$ is the reduced norm of A.

Lemma 1. *For Kummer elements X, $Y \in A$ one has*

$$T(XYXY^{-1}) = T(XY)T(XY^{-1})$$

Proof. By Lemma 6 (see the appendix) one has

$$\begin{aligned} N(x+y) = N(x) &+ Q(x)T(y) - T(x)T(xy) + T(x^2y) \\ &+ T(x)Q(y) - T(xy)T(y) + T(xy^2) + N(y) \end{aligned}$$

Taking $x = XY$, $y = Y$ this yields

$$\begin{aligned} N(XY+Y) = N(XY) &+ Q(XY)\cdot 0 - T(XY)T(XY^2) + T(XYXY^2) \\ &+ T(XY)\cdot 0 - T(XY^2)\cdot 0 + 0 + N(Y) \end{aligned}$$

On the other hand

$$N(XY+Y) = N(X+1)N(Y) = (N(X)+1)N(Y) = N(XY) + N(Y)$$

Date: February 1, 2025.

Hence

$$T(XYXY^2) = T(XY)T(XY^2)$$

□

1. Chains of length 2

For Kummer elements X, $Y \in A^\times$ the symbol

$$X \xrightarrow{\zeta} Y$$

stands for

$$YX = \zeta XY$$

Lemma 2. *Suppose there exists a chain*

$$X \xrightarrow{\zeta} U \xrightarrow{\zeta} Y$$

Then XY^{-1} is a Kummer element.

Proof. Let $V = XY^{-1}$. Then $UVU^{-1} = \zeta^2 V$ and therefore V is a Kummer element. □

Lemma 3. *Let X, Y be Kummer elements and suppose XY^{-1} is a Kummer element.*

Then XY and YX commute. Moreover

$$N(X)X^{-1}YX^{-1} = N(Y)Y^{-1}XY^{-1} = T(XY) - XY - YX$$

Let

$$U = T(XY) + (\zeta - \zeta^2)(\zeta XY - \zeta^2 YX)$$
$$V = T(XY) + (\zeta^2 - \zeta)(\zeta^2 XY - \zeta YX)$$

Then

$$X \xrightarrow{\zeta} U \xrightarrow{\zeta} Y, \qquad X \xrightarrow{\zeta^2} V \xrightarrow{\zeta^2} Y$$

For generic X, Y, these conditions determine the elements U, V uniquely up to multiplication by scalars.

Proof. ... □

2. Chains of length 3

Let

$$\mathcal{K} = \{\, [X] \in \mathbf{P}(A) \mid T(X) = Q(X) = 0,\ N(X) \neq 0 \,\}$$

be the variety of (projective) Kummer elements. Let further

$$\mathcal{K}_r = \{\, ([X_i])_{i=0,\ldots,r} \in \mathbf{P}(A)^{r+1} \mid X_{i-1} \xrightarrow{\zeta} X_i,\ i = 1, \ldots, r \,\}$$

be the variety of chains of length r and let

$$h_r \colon \mathcal{K}_r \to \mathcal{K} \times \mathcal{K}$$
$$h_r\big(([X_i])_{i=0,\ldots,r}\big) = \big([X_0], [X_r]\big)$$

be the projections.

Theorem 4. (1) *The morphism h_2 is generically an immersion.*
(2) $\deg(h_3) = 2$
(3) *For $r \geq 4$, the morphism h_r has a rational section.*

(1) follows from Lemma 3, and (3) is shown in [1].

As for the morphism h_3, the fibre over the generic point $([X], [Y]) \in \mathcal{K} \times \mathcal{K}$ has the following description:

$$\begin{aligned} 0 = t^2 - t\big(3 + \zeta T(XYX^{-1}Y^{-1}) + \zeta^2 T(YXY^{-1}X^{-1})\big) \\ + T(XY)T(X^{-1}Y^{-1})T(XY^{-1})T(X^{-1}Y) \end{aligned}$$

One finds:

Lemma 5. *For Kummer elements X, $Y \in A$ one has*

$$\begin{gathered} 3 + T(XYX^{-1}Y^{-1}) + T(YXY^{-1}X^{-1}) = \\ T(XY)T(X^{-1}Y^{-1}) + T(XY^{-1})T(X^{-1}Y) \end{gathered}$$

Proof. One uses again the formula for $N(x+y)$ in Lemma 6, this time with $x = X$ and $y = 1 + Y + Y^2$. Note here that $N(1 + Y + Y^2) = 1 - 2N(Y) + N(Y)^2$ for Kummer elements (which can be also deduced formally from Lemma 6). □

The function $T(XYX^{-1}Y^{-1})$ is not in the function field K generated by $T(XY)$, $T(X^{-1}Y^{-1})$, $T(XY^{-1})$, $T(X^{-1}Y)$, because these functions are invariant under reversing the product in the algebra A, while $T(XYX^{-1}Y^{-1})$ is not. However, if I am not mistaken, $T(XYX^{-1}Y^{-1})$ satisfies a quadratic equation over K.

`Todo: Find this relation, and give a nice description.`

Perhaps one can deduce it again from Lemma 6 which seems to be really useful. For instance using the last expression (of degree 4) one finds $Q(XY) = T(X^2Y^2)$ for Kummer elements in a degree 3 algebra.

Maybe it is a good idea to consider the cubic subalgebras

$$L = k \oplus Xk \oplus X^2k, \quad H = k \oplus Yk \oplus Y^2k$$

of A and to analyze for $\lambda_i \in L$ and $\mu_i \in H$ the product

$$N(\lambda_1)N(\mu_1)N(\lambda_2)\cdots N(\mu_r) = N(\lambda_1\mu_1\lambda_2\cdots\mu_r)$$

by expanding the right hand side using Lemma 6 with respect to sums of non-commutative monomials in X, Y.

Appendix

This is copied from [2].

Let F be a field and let A be a central simple algebra of degree 4 over F. For the characteristic polynomial of $x \in A$ we use the notation

$$N(t-x) = t^4 - T(x)t^3 + Q(x)t^2 - S(x)t + N(x)$$

where $N\colon A \to F$ is the reduced norm of A.

Lemma 6. *For x, $y \in A$ one has*

$$\begin{aligned}
T(x+y) &= T(x) + T(y)\\
Q(x+y) &= Q(x) + T(x)T(y) - T(xy) + Q(y)\\
S(x+y) &= S(x) + Q(x)T(y) - T(x)T(xy) + T(x^2y)\\
&\quad + T(x)Q(y) - T(xy)T(y) + T(xy^2) + S(y)\\
N(x+y) &= N(x) + S(x)T(y) - Q(x)T(xy) + T(x)T(x^2y) - T(x^3y)\\
&\quad + Q(x)Q(y) - T(x)T(xy)T(y) + T(x)T(xy^2) + T(x^2y)T(y)\\
&\qquad + Q(xy) - T(x^2y^2)\\
&\quad + T(x)S(y) - T(xy)Q(y) + T(xy^2)T(y) - T(xy^3) + N(y)
\end{aligned}$$

Proof. In the power series ring $A[[t]]$ one has

$$1 + t(x+y) = (1+tx)\big[1 - t^2\frac{x}{1+tx}\frac{y}{1+ty}\big](1+ty)$$

The middle term expands as follows:

$$1 - t^2\frac{x}{1+tx}\frac{y}{1+ty} = 1 - t^2xy + t^3x(x+y)y - t^4x(x^2+xy+y^2)y + \cdots$$

Taking norms gives in $F[[t]]/(t^5)$

$$\begin{aligned}
N\big(1+t(x+y)\big) = N(1+tx)N(1+ty)\big[1 &- t^2T(xy) + t^3T(x^2y+xy^2)\\
&+ t^4\big(Q(xy) - T(x^3y + x^2y^2 + xy^3)\big)\big]
\end{aligned}$$

Multiplying out yields the claims. □

References

[1] M. Rost, *The chain lemma for Kummer elements of degree 3*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 3, 185–190, MR 2000c:12003.

[2] ———, *Quadratic elements in a central simple algebra of degree four*, Notes, http://www.math.uni-bielefeld.de/~rost/lines.html, 2003.

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