Floer Homology and the Seiberg–Witten Equations

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Part I

Floer Homology and The Seiberg–Witten equations (SoSe 2023)

Chapter 1

Morse Homology in Finite Dimensions

1.1 Recollections from Morse Theory

Throughout this section, let M be a smooth n–manifold without boundary. Lecture 1, 4.4.23

Morse functions. Recall that a *Morse function* $f\colon M \to \mathbb{R}$ is a smooth function for which each critical point is non-degenerate, that is, for each $p \in \mathrm{Crit}(f)$ the Hessian

$$H_p(f)\colon T_pM \times T_pM \to \mathbb{R}, \quad H_p(v,w) = v(\tilde{w}(f)) \tag{1.1.1}$$

is non-degenerate as a symmetric bilinear form. The *Morse index* of $\mu(p)$ is maximal dimension of subspaces on which $H_p(f)$ is negative definite. According to the Morse Lemma (e.g. [Wal16, Prop. 4.8.1]), near a non-degenerate $p \in \mathrm{Crit}(f)$ one can find a *Morse chart* (U,φ) in which f is represented by

$$f \circ \varphi^{-1}(x_1,\ldots,x_n) = f(p) - x_1^2 - \cdots - x_{\mu(p)}^2 + x_{\mu(p)+1}^2 + \cdots + x_n^2. \tag{1.1.2}$$

An inspection of the local model shows, in particular, that non-degenerate critical points are isolated. It is known that the set of Morse functions is open and dense in $C^\infty(M)$ with the C^∞ topology (e.g. [Wal16, Theorem 4.7.1]).

Morse gradients. A vector field ξ is called a *Morse gradient* for a Morse function f is if $\xi(f) = df(\xi) > 0$ on $M \setminus \mathrm{Crit}(f)$ and near each $p \in \mathrm{Crit}(p)$ there is a Morse chart (U,φ) in which ξ takes the form

$$\varphi_*\xi(x_1,\ldots,x_n) = (-x_1,\ldots,-x_k,x_{k+1},\ldots,x_n). \tag{1.1.3}$$

The pair (f,ξ) is called a *Morse pair.* The heart of Morse theory is the study the interplay of the level sets of f and the integral curves of ξ, or equivalently $-\xi$. The latter is more common in the literature, since it is more in line with physics where processes tend minimize the internal energy (which would be measured by f) of a system as time evolves. Recall that an *integral curve* of $-\xi$ is a curve $\gamma\colon J \to M$ defined on some interval $J \subset \mathbb{R}$ satisfying the *negative flow equation*

$$\dot{\gamma}(t) = -\xi(\gamma(t)). \tag{1.1.4}$$

By the existence and uniqueness theorems for ODEs (c.f. [Wal16, Ch. 1.4]), every integral curve can be extended (as an integral curve) to a maximal interval. The image of a maximal integral curve of $-\xi$ will be called a *$-\xi$–trajectory.*

We note for later reference that f is decreasing along integral curves of $-\xi$, since

$$(f \circ \gamma)'(t) = -df(\dot{\gamma}(t)) = -df(\xi(\gamma(t))) \leq 0. \tag{1.1.5}$$

As for the existence problem, Morse gradients can be constructed for any Morse function using a simple partition of unity argument (c.f. [Mil65, Lemma 3.2]). However, the set of Morse gradients for a fixed Morse function is *neither open nor dense* in the space of all vector fields. In fact, Morse gradients are rather special.

Stable and unstable manifolds. For the moment, let us keep things simple and assume that M is closed. This has three convenient consequences for Morse pairs:

(1) All maximal integral curves of all vector fields on M are define on $\mathbb{R}$.

(2) Every Morse function on M has only finitely many critical points.

(3) For a Morse pair (f, ξ), all maximal integral curves $\gamma\colon \mathbb{R} \to M$ of $-\xi$ have limits

$$\gamma(\pm\infty) = \lim_{t \to \pm\infty} \gamma(t) \in \mathrm{Crit}(f). \tag{1.1.6}$$

The last point suggests the following definition:

Definition 1.1 (Stable and unstable manifolds)**.** Let (f, ξ) be a Morse pair on a closed manifold M. The *stable* and *unstable manifolds* of $p \in \mathrm{Crit}(f)$ are defined as

$$W^u(p) = \{x \in M \mid \gamma_x(-\infty) = p\} \quad \text{and} \quad W^s(p) = \{x \in M \mid \gamma_x(+\infty) = p\} \tag{1.1.7}$$

where $\gamma_x\colon \mathbb{R} \to M$ is the unique maximal integral curve of $-\xi$ with $\gamma_x(0) = x$.

The name (un-)stable *manifold* is justified by the following lemma which is easy to prove for Morse pairs.

Theorem 1.2 (Stable manifold theorem for Morse pairs)**.** *Let (f, ξ) be a Morse pair on a closed n–manifold M and $p \in \mathrm{Crit}(f)$. Then $W^u(p)$ and $W^s(p)$ are smooth submanifolds of M and there are diffeomorphisms*

$$W^u(p) \cong \mathbb{R}^{\mu(p)} \quad \textit{and} \quad W^s(p) \cong \mathbb{R}^{n-\mu(p)}. \tag{1.1.8}$$

Proof. Exercise. *(Hint: Use Morse charts to compare the flow of ξ with that of the local models in* (1.1.3)*.)* ◻

Moduli spaces of trajectories. Since every point in M lies on a unique $-\xi$-trajectory, the collections $\{W^d(p)\}_{p \in \mathrm{Crit}(f)}$ and $\{W^a(p)\}_{p \in \mathrm{Crit}(f)}$ form partitions of M. We can refine them by fixing both limits.

Definition 1.3 (Moduli spaces of trajectories)**.** For a Morse pair (f, ξ) let

$$M(p, q) = W^u(p) \cap W^u(q) = \{x \in M \mid \gamma_x(-\infty) = p,\ \gamma_x(+\infty) = q\}. \tag{1.1.9}$$

Note that $\mathbb{R}$ acts on $M(p, q)$ by $(x, t) \mapsto \gamma_x(t)$. The orbit space $\hat{M}(p, q) = M(p, q)/\mathbb{R}$ is called the *moduli space of ξ–trajectories from p to q*. Points in $\hat{M}(p, q)$ are $-\xi$–trajectories running from p to q.

Unlike $W^u(p)$ and $W^s(q)$, the spaces $M(p, q)$ and $\hat{M}(p, q)$ are not guaranteed to be manifolds without further assumptions.

Definition 1.4. A Morse pair (f, ξ) satisfies the *Smale condition* if $W^u(p) \pitchfork W^s(q)$ for all $p, q \in \mathrm{Crit}(f)$. In this case we call (f, ξ) a *Morse–Smale pair*.

The central idea of Floer homology is to exploit some features of the moduli spaces $\hat{M}(p,q)$ for Morse–Smale pairs in more general situations. We begin with some trivial observations in our toy example.

(1) Obviously, $M(p,p) = \{p\}$ with trivial $\mathbb{R}$–action so that each $\hat{M}(p,p)$ is a singleton.

(2) For $p \neq q$ with $f(p) \leq f(q)$ we have $M(p,q) = \emptyset$, because f is strictly increasing along non-constant ξ–trajectories.

(3) If $p \neq q$ and $\mu(p) < \mu(q)$, then $M(p,q) = \emptyset$ follows from transversality, since

$$\dim W^u(p) + \dim W^s(q) = \mu(p) + (n - \mu(q)) < n.$$

(4) Similarly, if $p \neq q$ and $\mu(p) = \mu(q)$, then $M(p,q)$ is a 0–dimensional submanifold of M by transversality. Thus $M(p,q)$ is the union of *constant* $-\xi$–trajectories, which can never connect two different critical points. Again we find $M(p,q) = \emptyset$.

More generally, we have the following.

Lemma 1.5. *Let (f,ξ) be a Morse–Smale pair on a closed manifold M. For $p \neq q \in \mathrm{Crit}(f)$ the spaces $M(p,q)$ and $\hat{M}(p,q)$ are smooth manifolds of dimensions*

$$\dim M(p,q) = \mu(p) - \mu(q) \quad \textit{and} \quad \dim \hat{M}(p,q) = \mu(p) - \mu(q) - 1. \tag{1.1.10}$$

Proof. We may assume that $f(p) > f(q)$ and $\mu(p) > \mu(q)$.

- ▶ The statements about $M(p,q)$ follow immediately from transversality.
- ▶ In order to study $\hat{M}(p,q)$ choose a regular value $a \in (f(q), f(p))$ and note that every ξ–trajectory intersects $f^{-1}(a)$ transversely in a single point.
- ▶ In particular, $M(p,q) \cap f^{-1}(a)$ is canonically a smooth submanifold of M of dimension $\mu(p) - \mu(q) - 1$.
- ▶ The map $\hat{M}(p,q) \to M(p,q) \cap f^{-1}(a)$ sending a $-\xi$–trajectory to its unique intersection with $f^{-1}(a)$ is a homeomorphism. Indeed, it is continuous (by ODE theory) with continuous inverse given by restricting the orbit map $M(p,q) \to \hat{M}(p,q)$ to $M(p,q) \cap f^{-1}(a)$.
- ▶ If $b \in (f(q), f(p))$ is another regular value, than translation along ξ trajectories gives a diffeomorphism $M(p,q) \cap f^{-1}(a) \cong M(p,q) \cap f^{-1}(b)$ so that we get a well–defined smooth structure on $\hat{M}(p,q)$. □

Compactness of moduli spaces. The key feature of the moduli space $\hat{M}(p,q)$ is that one can reasonably understand the nature of limit points. Here is the simplest instance: Lecture 2, 11.4.23

Proposition 1.6. *If $\mu(p) - \mu(q) = 1$, then $\hat{M}(p,q)$ is a compact 0–manifold and thus finite.*

Proof. We argue as in [Flo89, Lemma 2.1]:

- ▶ As before, let a be a regular value of f with $f(q) < a < f(p)$. It suffices to show that $M(p,q) \cap f^{-1}(a)$ is compact.
- ▶ Let $x_i \in M(p,q) \cap f^{-1}(a)$ be a sequence and $\gamma_i = \gamma_{x_i}$ the corresponding sequence of integral curves in $M(p,q)$.
- ▶ Since M is compact, so is $f^{-1}(a)$ and we can replace x_i by a convergent subsequence with limit $x_\infty \in f^{-1}(a)$.
- ▶ Suppose that $x_\infty \neq M(p,q)$, that is $\gamma_\infty = \gamma_{x_\infty}$ lies in $M(p',q')$ with $p' \neq p$ or $q' \neq q$.

- ▶ Repeating this argument with several different regular levels a eventually gives a sequence of critical points $p = p_0, p_1, \ldots, p_r = q$, $r \geq 2$, and trajectories in $M(p_{i-1}, p_i)$ in the closure of $M(p,q)$ in M.
- ▶ Since $\mu(p_i) > \mu(p_{i-1})$ for each i, this contradicts $\mu(p) - \mu(q) = 1$.
- ▶ We conclude that $x_\infty \in M(p,q) \cap f^{-1}(a)$. □

For $\mu(p) - \mu(q) \geq 2$ the moduli spaces need no longer be compact. This is already clearly visible on the "tilted 2–torus" in $\mathbb{R}^3$ and it's instructive to keep this standard example in mind for what follows. Indeed, in this example the moduli space of trajectories connecting the maximum to the minimum consists of four disjoint open intervals.

Back to the general setting. Elaborating on the compactness argument in the above proof gives a general statement about the closure of $M(p,q)$ in M: the (topological) boundary of $M(p,q)$ is made up of *broken trajectories*, that is, sequences of trajectories in $M(p_{i-1}, p_i)$ where $p = p_0, \ldots, p_r = q$ are critical points and $r \leq \mu(p) - \mu(q)$ (c.f. [Jos17, Theorem 8.4.1]). One can also prove that all such broken trajectories are contained in the closure of $M(p,q)$ and can be approximated by unbroken trajectories in $M(p,q)$ in a controlled way. This culminates in the following "compactness theorem" for the moduli spaces which is proved, for example, in [Weh12]:

Theorem 1.7 (Compactness theorem)**.** *Let (f, ξ) be a Morse–Smale pair and $p, q \in \mathrm{Crit}(f)$. The moduli spaces $\hat{M}(p,q)$ have compactifications given by*

$$\bar{M}(p,q) = \hat{M}(p,q) \cup \bigcup_{r=2}^{\mu(p)-\mu(q)} \bigcup_{p=p_0,p_1,\ldots,p_r=q} \hat{M}(p_0,p_1) \times \cdots \times \hat{M}(p_{r-1},p_r) \tag{1.1.11}$$

with a suitable topology. The space $\bar{M}(p,q)$ has the structure of a smooth $(\mu(p) - \mu(q) - 1)$–manifold with corners.

A reasonably down-to-earth reference for the topology on $\bar{M}(p,q)$ is [AD14, Ch. 3.2]. We will only need a special case which is also proved in [Jos17, Theorem 8.5.1] and [AD14, Theorem 3.2.7].

Corollary 1.8. *Let $\mu(p) - \mu(q) = 2$. Then $\bar{M}(p,q)$ is a 1–dimensional manifold with boundary*

$$\partial \bar{M}(p,q) = \bigcup_{r \in \mathrm{Crit}(f)} \hat{M}(p,r) \times \hat{M}(r,q). \tag{1.1.12}$$

The Morse–Floer complex. Continuing with a Morse–Smale pair (f, ξ) on a closed n–manifold M, the *(mod 2) Morse–Floer complex* is generated by the critical points

$$CF_k(M; f, \xi) = CF_k(M; f, \xi; \mathbb{Z}_2) = \bigoplus_{\mu(p)=k} \mathbb{Z}_2 = \bigoplus_{p \in \mathrm{Crit}_k(f)} \mathbb{Z}_2 \langle p \rangle \tag{1.1.13}$$

with the *Floer differential* given by counting points in 0–dimensional moduli spaces $\hat{M}(p,q)$:

$$d \colon CF_{k+1}(f,\xi) \to CF_k(f), \quad d\langle p \rangle = \sum_{\mu(q)=k} \#_2 \hat{M}(p,q) \langle q \rangle. \tag{1.1.14}$$

Here $\#_2$ is number of points modulo 2.

Proposition 1.9. *Let (f, ξ) be a Morse–Smale pair on a closed n–manifold M. Then the Floer differential on $CF_\bullet(M; f, \xi)$ satisfies $d^2 = 0$.*

Proof. Let $p \in \mathrm{Crit}(f)$ with $\mu(p) = k+2$. A direct calculation gives

$$d^2 \langle p \rangle = \cdots = \sum_{\mu(q)=k} \sum_{\mu(r)=k+1} \#_2 \hat{M}(p,r) \#_2 \hat{M}(r,q) \langle q \rangle \tag{1.1.15}$$

and we have to show that

$$\sum_{\mu(r)=k+1} \#_2 \hat{M}(p,r) \#_2 \hat{M}(r,q) = 0 \tag{1.1.16}$$

for all $q \in \mathrm{Crit}(f)$ with $\mu(q) = k$. But this follows from Corollary 1.8: the left hand side of (1.1.16) is just $\#_2 \partial \bar{M}(p,q)$ which is zero, because every compact 1–manifold with boundary has an even number of boundary points. □

Remark 1.10. We restrict to mod 2 coefficients to avoid discussions of orientations. Setting up the theory with integer coefficients is not overly complicated once one has grasped the essence of the constructions (see [Jos17, Ch. 8.6]). But it adds a layer of bookkeeping which can obscure the central ideas.

We define the *Morse–Floer homology* groups

$$HF_*(M; f, \xi) = H_*(CF_\bullet(M; f, \xi)) = \ker d / \operatorname{im} d. \tag{1.1.17}$$

The following theorem is proved in [Mil65, §7]:

Theorem 1.11. *Let (f, ξ) be a Morse–Smale pair on a closed manifold M. Then $CF_\bullet(M; f, \xi)$ is isomorphic to the cellular chain complex of a CW replacement*[1] *of M. In particular,*

$$HF_*(M; f, \xi) \cong H_*(M; \mathbb{Z}_2). \tag{1.1.18}$$

Proof (sketch). Those familiar with the machinery of [Mil65] already know how this works.

(1) Replace (f, ξ) by a Morse pair (g, ξ) with $\mathrm{Crit}(g) = \mathrm{Crit}(f)$ and $g(p) = \mu(p)$ for all $p \in \mathrm{Crit}(g)$ (c.f. [Mil65, Theorem 4.1]).

(2) According to [Mil65, Theorem 3.15] the space $M_k = g^{-1}(-\infty, k + \frac{1}{2}]$ is homotopy equivalent to M_{k-1} with one k–cells attached for each critical point of index k.

(3) It follows that the chain groups of $CF_\bullet(M; g, \xi)$ are isomorphic to those of ther cellular complex of a CW replacement for M. The Morse–Floer differential is identified with the cellular differential in [Mil65, Corollary 7.3]. In particular, we have $d^2 = 0$ for the Morse–Floer differential.

(4) Lastly, $CF_\bullet(M; g, \xi) = CF_\bullet(M; f, \xi)$, since the complex really only depends on ξ. □

As a result of Floer [Flo89], one can make sense of *Floer complexes* $CF_\bullet(S; f, \xi)$ and *Floer homology groups* $HF_*(S; f, \xi)$ for *compact isolated -ξ–invariant subsets* $S \subset M$. The complex is generated by $\mathrm{Crit}(f) \cap S$ and the differential counts points in 0–dimensional moduli spaces of $-\xi$–trajectories. We will discuss this further in the next section.

Before we move on, we record a few trivial but important observations:

(1) For a Morse pair (f, ξ) the critical points of f are the same as the zeros of ξ.

(2) The Morse index $\mu(p)$ can also be recovered from ξ alone. Indeed, there is a well-defined *linearization*

$$D_p\xi \colon T_pM \to T_pM, \quad D_q\xi(v)(f) = v(\xi(f)) \tag{1.1.19}$$

and $\mu(p)$ agrees with the number of negative eigenvalues of ξ

[1] A *CW replacement* of a space X is a CW complex that is homotopy equivalent to X.

(3) In order to define an ungraded Morse–Floer complex, it would be enough to know

$$\mu(p,q) = \mu(p) - \mu(q) = \dim M(p,q). \tag{1.1.20}$$

The only contribution of the Morse index $\mu(p)$ itself is an *absolute* $\mathbb{Z}$–grading on $CF_\bullet(M,f,\xi)$.

The upshot is that that the Morse–Floer complex really only depends on ξ. The function f only plays a secondary role.

1.2 Floer homology of isolated invariant sets

As indicated earlier, the central idea of Floer theory is that mimicking the construction of the Morse–Floer complexes can be fruitful beyond the setting of closed, finite dimensional manifolds. As a first example, we drop the compactness assumption on M. This adds two major complications for Morse pairs (f,ξ):

- $\mathrm{Crit}(f)$ no longer needs to be finite.
- $-\xi$ might have integral curves which escape to infinity (both in finite and infinite time)

To begin with, a minor change of perspective will be more convenient in the long run. Instead of focusing on single integral curves of $-\xi$, we henceforth consider the *(local) flow* generated by $-\xi$. Recall that this is the smooth map $\phi\colon U \to M$ uniquely determined by

$$\partial_t \phi_t(x) + \xi(\phi_t(x)) = 0, \quad \phi_0(x) = x \tag{1.2.1}$$

where $U \subset M \times \mathbb{R}$ is the open neighborhood of $M \times \{0\}$ whose intersection with $\{x \times \mathbb{R}\}$ is the domain of the maximal integral curve γ_x of $-\xi$. Note that $\phi_t(x) = \gamma_x(t)$.

Definition 1.12. Let (f,ξ) be a Morse pair on M and ϕ the flow generated by $-\xi$.

(a) $S \subset M$ is called *ϕ–invariant* if $x \in S$ implies $\gamma_x(t) \in S$ for all t.

(b) $N \subset M$ is called *ϕ–isolating* if if *ϕ–invariant part*

$$\mathrm{Inv}(N) = \{x \in N \,|\, \phi_t(x) \in N \text{ for all } t\} \tag{1.2.2}$$

is contained in the interior of N.

(c) $S \subset M$ is called *isolated ϕ–invariant* if $S = \mathrm{Inv}(N)$ for some ϕ–isolating set $N \subset M$.

For the remainder of this section let $S \subset M$ be a compact isolated ϕ–invariant set and $N \subset M$ a ϕ–isolating neighborhood. Our goal is to adapt the construction for closed manifolds to define *Floer complexes* and *Floer homology groups*

$$HF_*(S,\phi) = H_*(CF_\bullet(S,\phi)). \tag{1.2.3}$$

To that end, we make a series of observations:

(1) The stable and unstable manifolds $W^{s/u}(p)$ can be defined for all $p \in \mathrm{Crit}(f)$ essentially as before, but without control at the other ends. They are still immersed submanifolds of M of dimensions $\mu(p)$ and $n - \mu(p)$, respectively, which is enough to make sense of the Smale condition $W^u(p) \pitchfork W^s(p)$.

(2) Assuming the Smale condition, the statement and proof of Lemma 1.5 go through without changes, making $M(p,q) = W^u(p) \cap W^s(q)$ and $\hat{M}(p,q) = M(p,q)/\mathbb{R}$ smooth manifolds of dimensions $\mu(p,q)$ and $\mu(p,q)-1$, respectively. However, the compactness argument in Proposition 1.6 for $\mu(p,q) = 1$ no longer applies to $\hat{M}(p,q)$ for $\mu(x,y) = 1$.

(3) Since S ϕ-invariant and compact, for each $x \in S$ the integral curve $t \mapsto \phi_t(x)$ is contained in S, defined for all times and has limits

$$\lim_{t\to\pm} \phi_t(x) \in \mathrm{Crit}(f) \cap S. \tag{1.2.4}$$

So even without mentioning stable and unstable manifolds, we can define

$$M_S(p,q) = \left\{ x \in S \,\middle|\, \lim_{t\to-\infty} = p \text{ and } \lim_{t\to+\infty} \phi_t(x) = q \right\}, \quad p,q \in \mathrm{Crit}(f) \cap S \tag{1.2.5}$$

and $\hat{M}_S(p,q) = M_S(p,q)/\mathbb{R}$. We can also think of $\hat{M}_S(p,q)$ as a subspace of $\hat{M}(p,q)$.

(4) For $p,q \in \mathrm{Crit}(f) \cap S$ with $\mu(p,q) = 1$, one can show as in Proposition 1.6 that $M_S(p,q)$ is compact using the compactness of $f^{-1}(a) \cap S$ for a regular value $a \in (f(q), f(p))$. The argument is carried out in [Flo89, Lemma 2.1].

With this is mind, we define the *Floer complex* of (S,ϕ) as

$$CF_k(S,\phi) = \bigoplus_{p\in\mathrm{Crit}(f)\cap S,\, \mu(p)=k} \mathbb{Z}_2 \langle p \rangle \tag{1.2.6}$$

and equip it with the *Floer differential*

$$d\colon CF_{k+1}(S,\phi) \to CF_k(S,\phi), \quad d\langle p \rangle = \sum_{q\in\mathrm{Crit}(f)\cap S,\, \mu(q)=k} \#_2 \hat{M}_S(p,q). \tag{1.2.7}$$

Note that so far we have neither proved that $d^2 = 0$ nor used the fact that S is *isolated* ϕ–invariant. Before we address these related issues, let us look at a few simple examples.

Example 1.13. (1) Let (f,ξ) be a Morse pair and $p \in \mathrm{Crit}(f)$. Then $\{p\}$ is a compact isolated ϕ-invariant. The Floer complex $CF_\bullet(\{p\},\phi)$ is concentrated in degree $\mu(p)$ and has trivial differential. In particular, we have $d^2 = 0$ and thus

$$HF_*(\{p\},\phi) = \begin{cases} \mathbb{Z}_2 & \text{if } * = \mu(p) \\ 0 & \text{else} \end{cases} \tag{1.2.8}$$

Note that this is not the same as $H_*(\{p\};\mathbb{Z}_2)$ for $\mu(p) > 0$. So Floer complex of (S,ϕ) does not necessarily compute the homology of S.

(2) Let $\gamma\colon \mathbb{R} \to M$ be an integral curve of $-\xi$ in $M(p,q)$ with $p,q \in \mathrm{Crit}(f)$. Then $T_\gamma = \gamma(\mathbb{R}) \cup \{p,q\}$ is compact and ϕ–invariant. The Floer complex $CF_\bullet(T_\gamma,\phi)$ is generated by $\langle p \rangle, \langle q \rangle$ in degrees $\mu(p) > \mu(q)$. If $\mu(p) \geq \mu(q) + 2$, then the Floer differential vanishes. If $\mu(p) = \mu(q) = 1$, then $d\langle p \rangle = \langle q \rangle$. In both cases, we have $d^2 = 0$ and

$$HF_*(T_\gamma,\phi) = \begin{cases} \mathbb{Z}_2 & \text{if } \mu(p_+) \geq \mu(p_-) + 2 \text{ and } * = \mu(p_\pm) \\ 0 & \text{else.} \end{cases} \tag{1.2.9}$$

However, T_γ is always homeomorphic to $[0,1]$.

(3) Now consider the "heart shaped 2–sphere" in $\mathbb{R}^3$, that is, a deformed 2–sphere on which the height function has a unique minimum q, two maxima p, p', and a saddle point s. We can arrange that the negative gradient of the height function is a Morse gradient. Let γ be the unique trajectory of the downward gradient flow in $M(p,s)$ and let δ be one of the trajectories in $M(s,q)$. Then $T_\gamma \cup T_{\gamma'}$ is compact and ϕ-invariant, but not ϕ-isolated. Indeed, every neighborhood of $T_\gamma \cup T_{\gamma'}$ contains complete trajectories in $M(p,q)$. The Floer complex of $(T_\gamma \cup T_{\gamma'},\phi)$ is generated by p,s,q and with differential

$$d\langle p \rangle = \langle r \rangle, \quad d\langle r \rangle = \langle q \rangle, \quad \text{and} \quad d\langle q \rangle = 0. \tag{1.2.10}$$

In particular, we have $d^2\langle p \rangle = \langle q \rangle \neq 0$.

These examples beg the following questions: Lecture 3, 18.4.23

Q1: When is the Floer complex $CF_\bullet(S,\phi)$ actually a chain complex?

Q2: In that case, what is $HF_*(S,\phi)$? Is it the mod 2 homology of some space?

The following result is due to Floer [Flo89, Theorem 1] and builds on the work of Conley [Con78] which we will say more about in the next section.

Theorem 1.14 (Conley [Con78], Floer [Flo89])**.** *Let (f,ξ) be a Morse–Smale pair on a smooth n–manifold M and ϕ the flow generated by $-\xi$. Moreover, let $S \subset M$ a compact isolated ϕ–invariant set and $U \subset M$ a ϕ-isolating neighborhood of S.*

(i) There exists a compact ϕ-isolating neighborhood $N \subset U$ for S and a compact subset $E \subset N \setminus S$ such that

(1) If $x \in N$ and $\phi_t(x) \notin N$, then $\phi_s(x) \in E$ for some $s \in [0,t]$.

(2) If $x \in A$ and $\phi_t(x) \notin A$ for $t > 0$, then $\phi_t(x) \notin N$.

(ii) The Floer differential on $CF_\bullet(S,\phi)$ satisfies $d^2 = 0$ and there is an isomorphism

$$HF_*(S,\phi) = H_*(N,E;\mathbb{Z}_2). \tag{1.2.11}$$

The first statement is due to Conley and builds the foundation of his *index theory* – which, by the way, has nothing to do with index theory of elliptic operators. The second statement is due to Floer and the proof uses several features of Conley's theory. We discuss Floer's proof in Section 1.3.2.

1.3 Flows and Conley index theory

1.3.1 Conley index theory for isolated invariant sets

In order to prepare for the proof of Floer's theorem, it is helpful to work in a more general setting. The standard reference for this material is Salamon's article [Sal85].

Flows. We first abstract from the notion of flows of vector fields.

Definition 1.15 (Flows)**.** A *(global) flow* on a topological space X is a continuous right $\mathbb{R}$–action, that is, a continuous map

$$\phi\colon X \times \mathbb{R} \to X, \quad \phi(x,t) = \phi_t(x) = x_\phi(t) = x(0), \tag{1.3.1}$$

satisfying the *flow properties*

$$\phi_0(x) = x \qquad \text{equivalenty} \quad x(0) = x \tag{1.3.2}$$

$$\phi_{s+t}(x) = \phi_t\phi_s(x) \qquad \text{equivalenty} \quad x(s+t) = x(s)(t) \tag{1.3.3}$$

The curves $t \mapsto x(t) = \phi_t(x)$ are called *integral curves* and their images $x(\mathbb{R})$ *trajectories* or *orbits*. More generally, a *local flow* is a map $\phi\colon U \to M$ defined on a connected open neighborhood $U \subset X \times \mathbb{R}$ of $X \times \{0\}$ such that (1.3.2) and (1.3.3) are satisfied whenever both sides are defined.

Remark 1.16 (Smooth flows). Recall that if ξ is a vector field on a smooth manifold M, then the initial value problem

$$\partial_t\phi^\xi(x,t) = \xi(\phi^\xi(x,t)), \quad \phi^\xi(x,0) = x \tag{1.3.4}$$

determines a local flow $\phi^\xi \colon U^\xi \to M$ in the above sense and the map ϕ^ξ is smooth. Conversely, every smooth local flow ϕ on M determines a vector field on M by

$$\xi^\phi(x) = \partial_t \phi(x, 0) = \phi_* \frac{\partial}{\partial t}\Big|_{x,0} \in T_x M. \tag{1.3.5}$$

The constructions are essentially inverse, except that ϕ^{ξ^ϕ} might have a larger domain than ϕ.

The notion of invariant, isolating, and isolated invariant sets in Definition 1.12 carry over verbatim to flows on arbitrary spaces.

Index pairs and the Conley index. The next definition is motivated by Theorem 1.14(i). Throughout, let X be a locally compact metrizable space to conform with [Sal85].

Definition 1.17 (Index pairs)**.** Let (X, ϕ) by a local flow and $S \subset X$ a compact isolated invariant set. An *index pair* for S is a pair (N, E) of compact subset $E \subset N \subset X$ such that

(i) $\overline{N \setminus E}$ is an isolating neighborhood for S and $E \cap S = \emptyset$.

(ii) If $x \in E$ and $\phi_t(x) \in N$ for all $s \in [0, T]$, then $\phi_t(x) \in E$ for all $t \in [0, T]$.

(iii) If $x \in N$ and $\phi_T(x) \notin N$ for some $T > 0$, then there exists a $t \in [0, T)$ such that $\phi_s(x) \in N$ for all $s \in [0, t]$ and $\phi_t(x) \in E$.

The set E is called an *exit set* for N, because every orbit which leaves N forward in time must go through E by (iii) and necessarily leaves N when it leaves E by (ii).

We leave it to the reader to check that the pair (N, E) in the statement of Theorem 1.14(ii) is an index pair. What follows is the foundational theorem of Conley index theory.

Theorem 1.18 (Existence and uniqueness of index pairs, c.f. [Sal85, Ch. 4])**.** *Let S be a compact isolated invariant set for a local flow (X, ϕ).*

(i) If $U \subset X$ is any neighborhood of S, then there exists an index pair (N, E) for S with $\mathrm{cl}(N \setminus E) \subset U$.

(ii) If (N', E') is another index pair for S, then the flow map ϕ singles out a natural homotopy class of based homotopy equivalences $N/E \xrightarrow{\simeq} N'/E'$.

This justifies the following defintion:

Definition 1.19 (The Conley index)**.** Let S be a compact isolated invariant set for a local flow (X, ϕ). The *Conley index* of S is the based homotopy type

$$C(S, \phi) = [N/E] \tag{1.3.6}$$

where (N, E) is any index pair for S. We allow ourselves the slight abuse of notation to write $C(S, \phi) = N/E$.

The statement of Theorem 1.14(ii) can be recast as

$$HF_*(S, \phi) \cong H_*(C(S, \phi); \mathbb{Z}_2). \tag{1.3.7}$$

The proof of Theorem 1.18 is somewhat technical and we postpone the discussion. For now, it is more beneficial to illustrate the definitions above with some examples.

Example 1.20. (1) Let $\xi_0(x, y) = (-x, y)$ be the vector field on $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, the local model for Morse gradients. Recall that $-\xi$ generates the global flow

$$\phi \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad , \phi_t(x, y) = (e^t x, e^{-t} y). \tag{1.3.8}$$

The origin $\{0\}$ is an isolated invariant set for ϕ. One readily checks that each pair of the form

$$\big(D^k_\varepsilon \times D^{n-k}_\varepsilon, \partial D^k_\varepsilon \times D^{n-k}_\varepsilon\big), \quad \varepsilon > 0. \tag{1.3.9}$$

is an index pair for $\{0\}$. For the Conley index, we find

$$C(\{0\}, \phi^{-\xi_0}) = D^k_\varepsilon \times D^{n-k}_\varepsilon / D^k_\varepsilon \times D^{n-k}_\varepsilon \simeq D^k/\partial D^k \simeq S^k. \tag{1.3.10}$$

More generally, for a Morse pair (f, ξ) on a smooth manifold M and we can transplant the discussion above via Morse charts to each $p \in \mathrm{Crit}(f)$ with the result that

$$C(\{p\}, \phi^{-\xi}) \simeq S^{\mu(p)}. \tag{1.3.11}$$

It is in this sense that the Conley index refines the Morse index. Recall from Example 1.13(1) that

$$HF_*(\{p\}, \phi) \cong \tilde{H}_*(S^{\mu(p)}; \mathbb{Z}_2), \tag{1.3.12}$$

which is in line with Theorem 1.14(ii).

(2) Let (f, ξ) be a Morse–pair on a *closed* n–manifold M and $\phi = \phi^{-\xi}$. Then M itself is trivially a compact isolated invariant set and $(M, \emptyset)$ is an index pair (the only one in this case). The Conley index is $C(M, \phi) = M/\emptyset = M_+$, the based homotopy type of M with a disjoint base point added. By Theorem 1.11, we have

$$HF_*(M, \phi) \cong H_*(M; \mathbb{Z}_2) \cong \tilde{H}(M_+; \mathbb{Z}_2). \tag{1.3.13}$$

Again, this matches Theorem 1.14(ii).

(3) Let (f, ξ) and ϕ be as above and $a < b$ two regular values of f. Let $M^b_a = f^{-1}([a, b])$ and $M^b = f^{-1}((-\infty, b])$. The critical points in $M^b_a = f^{-1}([a, b])$ together with all trajectories between them form a compact isolated invariant set $S \subset M$. Two obvious index pairs are given by

$$(M^b_a, f^{-1}(a)) \quad \text{and} \quad (M^b, M^a). \tag{1.3.14}$$

Clearly, $M^b_a/f^{-1}(a)$ and M^b/M^a are homeomorphic.

(4) As a special case of the last example, consider the tilted torus in $\mathbb{R}^2$ with the downward gradient flow of the height function. Take S to be the set of index 1 critical points. Various choices of index pairs (see blackboard) are possible, and all give $C(S, \phi) \simeq S^1 \vee S^1$.

Lemma 1.21. *Let S_1 and S_2 be disjoint compact isolated invariant sets for a local flow (X, ϕ). Then $S \amalg T$ is compact isolated invariant with*

$$C(S_1 \amalg S_2, \phi) = C(S_1, \phi) \vee C(S_2, \phi). \tag{1.3.15}$$

Proof. ▶ Let U_i be an isolating neighborhood of S_i.

▶ We may assume that U_1 and U_2 are disjoint by shrinking them.

▶ In that case, $U_1 \amalg U_1$ is an isolating neighborhood for $S_1 \amalg S_2$.

▶ Use Theorem 1.18(i) to find index pairs (N_i, E_i) for S_i with $\mathrm{cl}(N_i \setminus E_i) \subset U_i$.

▶ If the U_i were chosen sufficiently small, then the N_i will be disjoint.

▶ In that case $(N_1 \amalg N_2, E_1 \amalg E_2)$ is an index pair for $S_1 \amalg S_2$ with

$$(N_1 \amalg N_2)/(E_1 \amalg E_2) \approx N_1/E_1 \vee N_2/E_2. \tag{1.3.16}$$

▶ The claim now follows from Theorem 1.18(ii). □

Lecture 4, 25.4.23

Construction of index pairs. We sketch a proof of Theorem 1.18(i) based [Sal85, Ch. 4.1] and [Con78, Ch. 4.1]. We refer to these sources for any omitted details.

Let (X, ϕ) be a local flow on a locally compact metrizable space. Local compactness yields that every compact isolated invariant set has a compact isolating neighborhood.[2] Suppose that we are given the following data:

- a compact isolated invariant set $S \subset X$,
- a compact isolating neighborhood $N_0 \subset X$ for S, and
- an arbitrary neighborhood $U \subset X$ of S.

Our goal is to construct an index pair (N, E) such that $\mathrm{cl}(N \setminus E) \subset U$. For brevity, we use the notation $x \cdot t = \phi_t(x)$. The main characters of this story are the sets

$$S^{\pm} = \left\{ x \in N_0 \,\middle|\, x \cdot \mathbb{R}^{\pm} \subset N_0 \right\} \tag{1.3.17}$$

and the construction that assigns to arbitrary subset $Z \subset Y \subset X$ the set

$$P(Z, Y) = \{ y \in Y \,|\, \exists z \in Z,\, t \geq 0 \text{ with } z \cdot [0, t] \subset Y \text{ and } y = z \cdot t \}\,. \tag{1.3.18}$$

We make two observations. First, we have

$$S = \mathrm{Inv}(N_0) = S^+ \cap S^-. \tag{1.3.19}$$

Second, the set $P(Y, Z)$ is *positively invariant in* Y in the sense that $y \in P(Z, Y)$ and $y \cdot [0, t] \subset Y$ imply $y \cdot [0, t] \subset P(Z, Y)$. In fact, it is the smallest subset of Y with this property that contains Z. The construction of index pairs has three main steps, each of which establishes some form of compactness:

(1) The simplest task is to show that the sets $S^{\pm}$ are compact (c.f. [Sal85, Lemma 3.7]). Assuming this, we can choose open neighborhoods $U^{\pm}$ of $S^{\pm}$ such that

$$\mathrm{cl}(U^+ \cap U^-) \subset U \cap \mathrm{int}(N_0). \tag{1.3.20}$$

(2) The first difficulty is to prove that the set $P(N_0 \backslash U^+, N_0)$ is closed and therefore compact (c.f. [Sal85, Lemma 4.2(i)]).

(3) The second difficulty is to locate a compact neighborhood N^- of S^- inside U^- that is positively invariant in N_0 (c.f. [Sal85, Lemma 4.2(ii)]).

From here onward, it is rather straight forward to prove that

$$N = N^- \cup P(N_0 \setminus U^+, N_0), \quad E = P(N_0 \setminus U^+, N_0) \tag{1.3.21}$$

constitutes an index pair for S with $\mathrm{cl}(N \setminus E) \subset U^+ \cap U^- \subset U \cap \mathrm{int}(N_0)$. This proves Theorem 1.18(i).

It is instructive to go through the construction for the flow $\phi_t(x, y) = (e^t x, e^{-t} y)$ on $\mathbb{R}^2$ and $S = \{0\}$ with different choices of N_0 and $U^{\pm}$. (An example will be discussed in class on the blackboard.) One should come to the conclusion that the exit set E in the above construction tends to be rather large.

[2]**Caution!** In [Sal85] and [Con78], compactness is included in the definition of isolating neighbordhoods.

Index pairs with special properties. There are other, more refined constructions which produce index pairs with additional properties. For example, one can always find index pairs (N, E) such that the inclusion $E \subset N$ is a cofibration (c.f. [Sal85, Ch. 5.1 & Prop. 2.4]). This is desirable from the perspective of homotopy theory. Among other things, it gives isomorphisms

$$h(N, E) \cong \tilde{h}(N/E) := h(N/E, *) \tag{1.3.22}$$

where h is any functor defined on pairs of spaces satisfying the usual homotopy invariance and excision axioms (c.f. [tD08, Prop. 10.4.5]).

It should be no surprise that one can do much better for smooth flows. The following result was proved by Conley and Easton [CE71].

Theorem 1.22 (Isolating blocks)**.** *Let M be a smooth manifold, ϕ a smooth local flow on M, and $S \subset M$ a compact isolated invariant set. For any neighborhood U of S there exist a compact submanifold with boundary $B \subset U$ with the following properties:*

(i) B is an isolating neighborhood for S.

(ii) The sets $\partial^{\pm} B = \{x \in \partial B \,|\, \exists \epsilon > 0 : x \cdot (\pm(0, \varepsilon)) = \emptyset\}$ are compact submanifolds of ∂B with common boundary $\partial^+ B = \partial^- B$ (possibly empty) tangent to the ϕ-trajectories.

In particular, B has the same dimension as M and $(B, \partial^- B)$ is an index pair for S.

The sets B produced in the above theorem are usually called *isolating blocks B*. The index pairs $(B, \partial^- B)$ are prototypical for the general definition. Going back to Example 1.20(3), we can now recognize the set $M_a^b = f^{-1}([a, b])$ as an isolating block with boundary decomposition $\partial^- M_a^b = f^{-1}(a)$ and $\partial^+ M_a^b = f^{-1}(b)$.

In a way, the general definition of index pairs crystallizes certain key properties of the pairs $(B, \partial^- B)$. In practice, many applications of Conley index theory boils down to finding and organizing appropriate index pairs to gain insight into a given situation. The general setup provides a very flexible theory.

Flow induces maps between index pairs. We now address the uniqueness part of Theorem 1.18. Recall that the goal is to show that the Conley index $C(S, \phi) = [N/E]$ is independent of the choice of index pair. We sketch the elegant argument in [Sal85, Ch. 4.2]. The idea is to exploit the following observation.

Lemma 1.23 (c.f. [Sal85, Lemma 4.6])**.** *Let K be a compact isolating neighborhood for S and U any neighborhood. Then there exists a $t > 0$ such that $x \cdot [-t, t] \subset N$ implies $x \in U$.*

Put differently, the longer a trajectory $x \cdot [-t, t]$ is defined in a given compact isolating neighborhood, the closer the point x must be to S.

Now let us consider not two, but three index pairs

$$(N, E), \quad (N', E'), \quad (N'', E'') \tag{1.3.23}$$

for the same compact isolated invariant set S. Using Lemma 1.23 we can find $T \geq 0$ such that the following implications hold for $t \geq T$:

$$x \cdot [-t, t] \subset N \setminus E \implies x \in N' \setminus E' \tag{1.3.24}$$

$$x \cdot [-t, t] \subset N' \setminus E' \implies x \in N \setminus E \tag{1.3.25}$$

We can then define a *flow induced map* $f_t \colon N/E \to N'/E'$ as

$$f_t([x]) = \begin{cases} [x \cdot 3t] & \text{if } x \cdot [0, 2t] \subset N \setminus E \text{ and } x \cdot [t, 3t] \subset N' \setminus E' \\ [E'] & \text{else.} \end{cases} \tag{1.3.26}$$

Now it takes some work to prove that $(t,[x]) \mapsto f_t([x])$ is continuous (see [Sal85, Lemma 4.7]). Clearly, the homotopy class of f_t is independent of $t \geq T$ and each f_t is homotopic to f_T.

Similarly, we get flow induced maps

$$f'_t\colon N'/E' \to N''/E'' \quad \text{and} \quad f''_t\colon N/E \to N''/E'' \tag{1.3.27}$$

for $t \geq T' \geq 0$, respectively $t \geq T'' \geq 0$. The reason that these maps are defined as they they are closed under composition in the sense that for $t \geq \max\{T, T', T''\}$ we have

$$f'_t \circ f_t = f''_{2t}. \tag{1.3.28}$$

Now, for $(N'', E'') = (N, E)$ we can take $T' = T$ and $T'' = 0$. In that case, f''_{2t} is homotopic to $f''_0 = \mathrm{id}\colon N/E \to N/E$. Repeating the same argument with (N, E) and (N', E') switched, we conclude that f_t and f'_t are inverse homotopy equivalences. This establishes Theorem 1.18(ii).

1.3.2 The Conley index and Floer homology

Lecture 5, 2.5.23

We now return to Theorem 1.14(ii) which we restate for convenience.

Theorem 1.24 (Floer [Flo89])**.** *Let (f, ξ) be a Morse–Smale pair on a smooth n–manifold M and $S \subset M$ a compact isolated invariant set for the local flow $\phi = \phi^{-\xi}$. Then the Floer differential in $CF_\bullet(S, \phi)$ satisfies $d^2 = 0$ and there is an isomorphism*

$$HF_*(S, \phi) = H_*(CF_\bullet(S, \phi)) \cong H_*(C(S, \phi); \mathbb{Z}_2). \tag{1.3.29}$$

In other words, the (mod 2) Floer complex of $CF_\bullet(S, \phi)$ computes (mod 2) homology of the Conley index $C(S, \phi)$.

We are still not quite ready for the proof yet. We need two more definitions and one more theorem.

Definition 1.25 (Limit sets)**.** Let (X, ϕ) be a local flow. The *α- and ω-limit sets* of a point $x \in X$ are defined as

$$\alpha(x) = \{a \in X \mid a = \lim(x \cdot t_n) \text{ for some } t_n \to -\infty\} \tag{1.3.30}$$
$$\omega(x) = \{w \in X \mid w = \lim(x \cdot t_n) \text{ for some } t_n \to \infty\}. \tag{1.3.31}$$

The limit sets consist of those points to which the flow trajectory through x gets arbitrarily close forward or backward in time. For a Morse pair (f, ξ) on a manifold M and ϕ generated by $-\xi$ we the limit sets are just the limit points of trajectories. In particular, for $p \in \mathrm{Crit}(f)$ we can write

$$W^u(p) = \{x \in M \mid \alpha(x) = \{p\}\} \quad \text{and} \quad W^s(p) = \{x \in M \mid \omega(x) = \{p\}\}. \tag{1.3.32}$$

However, in general the limit sets may be empty or contain more that than one point, in extreme cases they might even be the entire space. (I discussed examples on the blackboard.)

Definition 1.26 (Morse decompositions)**.** Let (X, ϕ) be a local flow and $S \subset X$ compact isolated invariant. A *Morse decomposition* of S is a collection of disjoint isolated invariant subsets $S_1, \ldots, S_n \subset S$ such that for all $x \in S \setminus \cup_i S_i$ we have

$$\alpha(x) \subset S_i \quad \text{and} \quad \omega(x) \subset S_j \quad \text{for some} \quad i < j. \tag{1.3.33}$$

It should be apparent that the sets S_i play the role of critical points in Morse theory. However, the new definition is much more flexible. The following lemma is proved in [Sal85, Corollary 4.4] and relates the Conley indices of the isolated invariant sets in a Morse decomposition.

Lemma 1.27 (Morse filtrations)**.** *Let S be a compact isolated invariant set for a local flow (X, ϕ) with X locally compact Hausdorff and $S_1, \dots, S_r \subset S$ a Morse descomposition of S. If (N, E) is an index pair for S, then there exists a filtration by compact subsets*

$$E = N_0 \subset N_1 \subset \cdots \subset N_r = N \tag{1.3.34}$$

such that (N_k, N_{k-1}) is an index pair for S_k.

The sets $\{N_i\}$ are called a *Morse filtration* of (N, E) compatible with the Morse decomposition $\{S_i\}$. As we will now see, Morse filtrations can be used for inductive arguments much like CW decompositions.

Proof of Theorem 1.24 (sketch). There are two main steps.
Step 1: Exploiting a Morse filtration.

- For $k = 0, \dots, n$ consider $S_k = \{p \in \mathrm{Crit}(f) \cap S \,|\, \mu(p) = k\}$. This is a Morse decomposition of S.

- The Conley index of S_k can be identified as

$$C(S_k, \phi) \simeq \bigvee_{p \in S_k} S^k \tag{1.3.35}$$

  where have used $C(\{p\}, \phi) \simeq S^{\mu(p)}$ and $C(A \amalg B, \phi) = C(A, \phi) \vee C(B, \phi)$ for disjoint compact isolated invariant sets A, B.

- Let (N, E) be an index pair for S and $E = N_{-1} \subset \cdots \subset N_n \subset N$ a compatible Morse filtration as in Lemma 1.27.

- Then (N_k, N_{k-1}) is an index pair for S_k and we have canonical isomorphisms

$$H_*(N_k, N_{k-1}; \mathbb{Z}_2) \cong \tilde{H}_*(C(S_k, \phi); \mathbb{Z}_2) \cong \begin{cases} CF_k(S, \phi), & * = k \\ 0, & * \neq k. \end{cases} \tag{1.3.36}$$

  Note the similarity with CW filtrations and cellular homology.

- As for CW filtrations, we obtain a chain complex $D_\bullet$ with chain groups

$$D_k = H_k(N_k, N_{k-1}; \mathbb{Z}_2) \cong CF_k(S, \phi) \tag{1.3.37}$$

  and differentials $\partial \colon D_{k+1} \to D_k$ given by the connecting maps of the triples (N_{k+1}, N_k, N_{k-1}).

- Now the same argument used in the identification of cellular homology (see [tD08, Ch. 12.2], for example) shows that

$$H_*(D_\bullet) \cong H_*(N, E; \mathbb{Z}_2) \cong H_*(C(S, \phi); \mathbb{Z}_2). \tag{1.3.38}$$

Step 2: It remains to identify $\partial \colon D_{k+1} \to D_k$ with the Floer differential.

- Let $S_{k+1} \& S_k$ be the union of S_{k+1}, S_k and all trajectories between them. This is a compact isolated invariant set with Morse decomposition $\{S_k, S_{k+1}\}$ and Morse filtration $N_{k-1} \subset N_k \subset N_k$.

- It clearly suffices to prove the claim for $S_{k+1} \& S_k$ for each k, since the differentials d_{k+1} and ∂_{k+1} in $CF_\bullet(S, \phi)$ and $D_\bullet$ are determine in $S_{k+1} \& S_k$ and $N_{k-1} \subset N_k \subset N_k$, respectively.

- For $p \in S_{k+1}$ and $q \in S_k$ we obtain another Morse decomposition $\{S_k \backslash \{q\}, \{q\}, \{p\}, S_{k+1} \backslash \{p\}\}$ of $S_{k+1} \& S_k$.

- A double induction using a refined Morse filtration reduced the general problem further to the case $S_{k+1} = \{p\}$ and $S_k = \{q\}$.
- In that case, we have $N_{k+1}/N_k \simeq S^{k+1}$ and $N_k/N_{k-1} \simeq S^k$.
- It remains to show that $\partial\colon D_{k+1} \cong \tilde{H}_{k+1}(S^{k+1}; \mathbb{Z}_2) \to \tilde{H}_k(S^k; \mathbb{Z}_2) \cong D_k$ is multiplication by $\#_2\hat{M}(p,q)$ using the canonical identification of both groups with $\mathbb{Z}_2$.
- The argument is essentially the same as the identification of the Floer differential in Theorem 1.11. Instead of a self-indexing Morse function, one has to choose a suitable index pair. See [Flo89, p. 214 f.] for more and [BH04, p. 213 ff.] for even more details. □

1.4 Equivariant generalizations

One major advantage of Conley index theory is that is very easy to take symmetries into account. The study of spaces with symmetry is the subject of *equivariant topology.* We begin by reviewing some basic definitions. Standard references with an emphasis on equivariant algebraic topology are [tD87] and [May96]. The current state of the art in equivariant stable homotopy theory is laid out in [Sch18].

1.4.1 Equivariant topology: basis definitions.

Symmetries are modeled by continuous group actions. To avoid point set topological pathologies, we assume that all spaces are Hausdorff.

Definition 1.28 (G–spaces and maps)**.** Let G be a Hausdorff topological group.

(a) A *G–space* is a Hausdorff space X with a continuous left G–action denoted by $(g, x) \mapsto gx$.

(b) A *G–map* $f\colon X \to Y$ between two G–spaces is a continuous map such that $f(gx) = gf(x)$ for all $g \in G$.

Many notions from ordinary, non-equivariant topology carry over to the equivariant setting by simply "putting a G everywhere". However, others do not and it does take some time to develop an intuition for where the problems lurk. This can become very subtle, as demonstrated by Frank Adams' famous rant in [Ada84, §6], which everyone should read if only for entertainment.

For example, G–manifolds, G–homeomorphisms, G–diffeomorphisms, G–homotopies, and G–homotopy equivalences are defined in the obvious way and behave as expected. The first small surprise is the realization that base points should not be arbitrary.

Definition 1.29 (Fixed points)**.** Let X be a G–space.

(a) If X is a G–space, then the set of *G–fixed points* is $X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$.

(b) A G–space X together with a G–fixed base point $x_0 \in X^G$ is called a *based G–space.*

Restricting to G–fixed base points ensures that the constant map to the base point is always a G–map, as it should be. From here on, one can define based versions of G–maps, G–homotopies, etc as expected.

The outcome is that there are reasonably behaved categories of G–spaces and based G–spaces. Let us now take a closer look at the objects.

Definition 1.30. Let X be a G–space and $x \in X$.

(a) The subspace $Gx = \{gx \in X \mid g \in G\} \subset X$ is called the *G–orbit* of x. The quotient X/G is called the *orbit space.*[3]

[3]Strictly speaking, one should arguably write $G\backslash X$ for left actions and X/G for right actions. However, this slight abuse of notation rarely causes confusion.

(b) The subgroup $G_x = \{g \in G \mid gx = x\}$ is called the *stabilizer* or *isotropy subgroup* of x.

The G–orbits are the smallest G–invariant subsets of X. They should be thought of as the analogues of points in ordinary topology. It is clear from the definition that G_x is a *closed subgroup* of G, that is, simultaneously a subgroup and a closed subspace. The orbits and stabilizers are related by the canonical G–map

$$G \to X, \quad g \mapsto gx, \quad x \in X. \tag{1.4.1}$$

with G acting on itself from the left. By construction of G_x, this descends to a G–map

$$o_x\colon G/G_x \to Gx, \quad o_x(gG_x) = gx \tag{1.4.2}$$

where G_x acts on G from the right to form the orbit space, while the left action of G on itself descends to G/G_x. This is always a continuous injection. An example where o_x fails to be an embedding is any $\mathbb{R}$–action on the torus whose orbits have irrational slope. However, there are obvious conditions that ensure that o_x is an embedding, for example if G/G_x is compact and X is Hausdorff. We record the following outcome of this discussion for the important special case that G is a compact Lie group.

Lemma 1.31. *Let G be a compact Lie group, X a Hausdorff G–space, and $x \in X$. Then the canonical map $G \to X$, $g \mapsto gx$, is a G–map and factors through a G–homeomorphism*

$$o_x\colon G/G_x \xrightarrow{\approx} Gx, \quad gG_x \mapsto gx \tag{1.4.3}$$

The orbit spaces G/H where $H \subset G$ is a closed subgroup are called *homogeneous spaces.* They should be thought of models for "points" in equivariant topology. From this perspective, one of the major differences in equivariant topology is that there are several types of "points" which can have complicated internal structures and interactions. As an example of a complicated "point", consider $G = O(n)$ and $H = O(k) \times O(n-k)$. The corresponding homogeneous space is

$$G_k(\mathbb{R}^n) \cong O(n)/O(k) \times O(n-k), \tag{1.4.4}$$

the Grassmannian of k–planes in $\mathbb{R}^n$ with its obvious $O(n)$–action.

1.4.2 G–flows and equivariant Conley index theory.

The good news is that most of Conley index theory generalizes to the equivariant setting by "putting a G everywhere" – at least when G is a compact Lie group, which we implicitly assume from now on.

Definition 1.32 (G–flows)**.** Let X be a G–space. A (local) flow ϕ on X is called a *(local) G–flow* if

$$(gx) \cdot t = g(x \cdot t) \quad \text{for all } g \in G \text{ and } t \in J_x. \tag{1.4.5}$$

All the basic definitions and theorems in Conley index theory go through for (local) G–flows by requiring all sets involved in the definitions or constructions to be G–invariant. In particular, this applies to invariant sets, isolating neighborhoods, index pairs, and all the sets involved in the construction of index pairs. The flow induced maps are then automatically G homotopy equivalences. In summary, we arrive at the following:

Theorem 1.33 (The G–Conley index)**.** *Let (X, ϕ) be a G–flow with G a compact Lie group and X locally compact and metrizable.*

(i) Every G–invariant compact isolated invariant set $S \subset X$ admits a G–index pair (N, E).

(ii) For any other G–index pair (N', E') there is a flow induced based G–homotopy equivalence $N/E \to N'/E'$.

In particular, there is a well-defined G–Conley index

$$C^G(S,\phi) = [N/E] \tag{1.4.6}$$

which comes in the form of a based G–homotopy type.

The compactness of G is certainly needed for the current approach to work. There are other approaches (see [Ryb87] of [Ben87], for example), but is not clear whether or not they can be applied to Seiberg–Witten theory on 3–manifolds, which is where we are ultimately headed.

1.4.3 Equivariant Floer homology?

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In contrast, it is not at all obvious how one should generalize the construction of Floer complexes and Floer homology. Here is a list of problems:

(1) On a G–manifold M one should study smooth functions $f\colon M \to \mathbb{R}$ that are G–invariant. The problems start with the simple observation that if p is a critical point such an f, then so is gp for every $g \in G$. In other words, $\mathrm{Crit}(f)$ is G–invariant and therefore a union of G–orbits which are generally submanifolds of positive dimensions.

(2) As a consequence, in G–equivariant Morse theory one has to allow $\mathrm{Crit}(f)$ to be a disjoint union of *critical G–orbits* on which the Hessian is non-degenerate in normal directions. The analogue of the Morse lemma involves tubular neighborhoods modeled on vector bundles of the form $G \times_H V \to G/H$ where $H \subset G$ is a closed subgroup and V is an orthogonal H–representation[4]. The latter splits as $V = V^- \oplus V^+$ and the local Morse model is the function $[g; v, w] \mapsto -|v|^2 + |w|^2$ on $G \times_H V$. More details can be found in [Was69].

(3) Similarly, Morse gradients are G–invariant in the sense that $g_*\xi(x) = \xi(gx)$. The stable and unstable manifolds $W^{s/u}(C)$ of a critical orbit C of type $(H; V^-, V^+)$ are immersed copies of the bundles $G \times_H V^\pm$. They are also G–invariant and so are the moduli spaces $M(C,D) = W^u(C) \cap W^s(D)$ of flow trajectories of $-\xi$ from C to another critical orbit D.

(4) The next subtlety is an equivariant version of the Smale condition. In contrast to the non-equivariant setting, where the Smale condition can always be achieved by a simple transversality argument based on Sard's theorem, there are generally obstructions to achieving equivariant transversality (see [Pet74]). In particular, one cannot take for granted that the intersections $W^u(C) \cap W^s(D) = M(C,D)$ can always be made transverse by modifying ξ through G–invariant Morse gradients. However, there are lucky accidents where the Smale condition is miraculously satisfied and the moduli spaces $M(C,D)$ are smooth G–submanifolds. However, they have no reason to be 0–dimensional, in general.

(5) All of the above indicates that it is far from obvious how one could define G–analogues of Floer complexes. However, we would probably expect that any reasonable Floer complex would compute some reasonable invariant of a G–Conley index $C^G(S,\phi)$, most likely some form of "equivariant homology". This begs the question: What does "equivariant homology" even mean?

While there is no clear cut way out of these problems, there are some special situation where certain equivariant homology or cohomology theories are computable by Floer theoretic methods. We briefly discuss these in the next section.

[4] An orthogonal G–representation is a real inner product space on which G acts by orthogonal transformations.

1.4.4 Borel homology and cohomology theories

We briefly review the definition and some properties of Borel homology and cohomology theories. We refer to [tD87, Ch. III] and [Hsi75, Ch. III] for more details. Ordinary, non-equivariant homology and cohomology theories are particular well-behaved on CW complexes. Here is an equivariant analogue:

Definition 1.34 (G–CW–complexes)**.** Let G be a compact Lie group.

(a) The product $G/H \times D^k$ with $H \subset G$ a closed subgroup, $k \geq 0$, and G acting trivially on D^k is called a *k–dimensional G–cell of type G/H*.

(b) A pair of G–spaces (X, A) is called a *relative G–CW–complex* if there is a filtration $A = X_{-1} \subset X_0 \subset \cdots \subset X = \cup X_i$ such that X_k is obtained from X_{k-1} by attaching a collection of G–cells $\amalg_i(G/H_i \times D^k)$ along a G–map $\amalg_i(G/H_i \times S^{k-1}) \to X_{k-1}$.

We want to discuss a tool that allows to translate equivariant problems into non-equivariant ones in order to make use of non-equivariant algebraic topology.

Universal G–spaces. Let G be a compact Lie group. Recall that a G–action on a space P is called *free* if $G_x = \{1\}$ for all $x \in X$ (or, equivalently, $X^g = \emptyset$ for all $1 \neq g \in G$). The orbit map $P \to P/G$ is then a principal G–bundle (modulo converting left to right actions).

The notion of a *universal G–space*, usually denoted by EG, has two different technical implementations. Both versions require the following properties:

(a) G acts freely on EG (i.e. $G_x = \{1\}$ for all $x \in EG$).

(b) EG is non-equivariantly contractible.

In addition, one of the two technical conditions is included:

(c) EG is a G–CW–complex.

(c') EG is a numerable G–space.

Here *numerable* means that the orbit map $EG \to EG/G$ is a principal G–bundle which is locally trivial over on open cover of EG/G which supports a partition of unity. One can show that (c) implies (c'), but we shall not worry about these details. In either case, the orbit space and map

$$BG = EG/G \quad \text{and} \quad EG \to BG \tag{1.4.7}$$

are called a *classifying space* and a *universal fibration* for G.

Proposition 1.35 (Universal G–sapces)**.** *Let G be a compact Lie group.*

(i) There exists a universal G–space EG.

(ii) For every free G–space X which is either a G–CW–complex or numerable there is a G–map $X \to EG$ and any two such G–maps are G–homotopic.

In particular, EG is unique up to G–homotopy equivalence, BG is unique up to homotopy equivalence, and $EG \to BG$ is unique up to isomorphism of principal G–bundles.

Example 1.36 (The circle group)**.** In the case of the unit circle group $\mathbb{T} \subset \mathbb{C}$ there is a standard construction of a universal $\mathbb{T}$–space.[5] Note that $\mathbb{T}$ acts freely by scalar multiplication on the unit sphere $S(\mathbb{C}^n) \subset \mathbb{C}^n$ for $n \geq 1$. It is a standard fact that the colimit

$$E\mathbb{T} = S(\mathbb{C}^\infty) = \operatorname*{colim}_{n\to\infty} S(\mathbb{C}^n) \tag{1.4.8}$$

[5] The notation $\mathbb{T}$ is commonly used in equivariant algebraic topology. Other common names are S^1 or U_1.

along the inclusions $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$, $x \mapsto (x, 0)$, is non-equivariantly a contractible CW complex. Clearly, the $\mathbb{T}$–actions on $S(\mathbb{C}^n)$ extend to a free $\mathbb{T}$–action on $E\mathbb{T}$, making it a universal $\mathbb{T}$–space. The corresponding classifying space is

$$B\mathbb{T} = S(\mathbb{C}^\infty)/\mathbb{T} \cong \operatorname*{colim}_{n\to\infty} S(\mathbb{C}^n)/\mathbb{T} \cong \operatorname*{colim}_{n\to\infty} \mathbb{C}\mathrm{P}^{n-1} = \mathbb{C}\mathrm{P}^\infty \tag{1.4.9}$$

and the orbit map $E\mathbb{T} \to B\mathbb{T}$ corresponds to the unit sphere bundle of the tautological line bundle over $\mathbb{C}\mathrm{P}^\infty$.

The Borel construction. Fix a compact Lie group G and a universal G–space EG. Given an arbitrary G–space X, we define its *Borel construction* as the orbit space

$$X_{hG} = EG \times_G X = (EG \times X)/G \tag{1.4.10}$$

with respect to the diagonal action. Thinking of EG as a principal G–bundle, it is clear that the map

$$p_X \colon X_{hG} \to BG, \quad [e; x] \mapsto [e] \tag{1.4.11}$$

makes X_{hG} the total space of an associated fiber bundle with typical fiber X (considered as a space without G–action). The Borel construction is functorial in the sense that a G–map $f \colon X \to Y$ induces a bundle map

$$f_{hG} = \mathrm{id} \times_G f \colon X_{hG} \to Y_{gH}, \quad [e; x] \mapsto [e; f(x)]. \tag{1.4.12}$$

The idea is that the bundles structure of X_{hG} reflects properties of the G–action. Here's a first illustration:

Lemma 1.37. *If G acts trivially on X, then there is a canonical homeomorphism*

$$X_{hG} \approx BG \times X, \quad [e; x] \mapsto ([e], x) \tag{1.4.13}$$

which trivializes the bundle map p_X.

Proof. The maps $[e; x] \mapsto ([e], x)$ and $([e], x) \to [e; x]$ are both well-defined, because $gx = x$ for all $g \in G$ and $x \in X$. They are clearly mutually inverse and easily proved to be continuous (using local sections of p_X for the second map). □

On the other extreme, we can also say something for free actions.

Lemma 1.38. *Let X be a free G–CW–complex. Then the map*

$$q_X \colon X_{hG} \to X/G, \quad [e; x] \mapsto [x] \tag{1.4.14}$$

is a homotopy equivalence.

Proof. Thinking of $X \mapsto X/G$ as a principal G–bundle, we can view q_X as an associated fiber bundle with contractible fiber EG. In particular, q_X induces isomorphisms on all homotopy groups and the assumptions guarantee that X_{hG} and X/G are CW complexes. The claim now follows from Whitehead's theorem. □

Borel homology and cohomology. Passing through the Borel construction, we can obtain G–homotopy invariants of a G–space X from non-equivariant homotopy invariants of X_{hG}. As an example, we have the *Borel homology and cohomology* groups

$$H^G_*(X) = H_*(X_{hG}) \quad \text{and} \quad H^*_G(X) = H^*(X_{hG}) \tag{1.4.15}$$

where H_* and H^* denotes singular homology and cohomology with coefficients in a commutative ring with unit. Given a G–subspace $A \subset X$ we can consider A_{hG} as a subspace of X_{hG} and define

$$H^G_*(X, A) = H_*(X_{hG}, A_{hG}) \quad \text{and} \quad H^*_G(X, A) = H^*(X_{hG}, A_{hG}). \tag{1.4.16}$$

By the functoriality of the Borel construction, a G–map $f\colon (X,A)\to(Y,B)$ induces maps

$$f_*\colon H^G_*(X)\to H^G_*(Y) \quad\text{and}\quad f^*\colon H^*_G(Y)\to H^*_G(Y).$$

The functors H^G_* and H^G_* are easily seen to satisfy G–equivariant versions of homotopy invariance, excision, and exactness, all of which follow from the corresponding properties of H_* and H^*. In view of the philosophy that G–orbits are analogues of points ordinary topology, we should be interested in the values on homogeneous spaces G/H.

Lemma 1.39. *For every closed subgroup $H\subset G$ there is a canonical isomorphism*

$$H^*_G(G/H)\cong H^*(BH).$$

Proof. We have a sequence of homeomorphisms

$$(G/H)_{hG} = EG\times_G (G/H)\approx (EG\times_G G)/H\approx EG/H.$$

Since EG also serves as a universal H–space, we get a homotopy equivalence $EG/H\simeq BH$ which is well-defined up to homotopy. □

In particular, for $H=G$ we get

$$H^*_G(\mathrm{pt})\cong H^*_G(G/G)\cong H^*(BG).$$

Specializing further to $G=\mathbb{T}$ we find

$$H^*_{\mathbb{T}}(\mathrm{pt})\cong H^*(B\mathbb{T})\cong H^*(\mathbb{C}\mathrm{P}^\infty)\cong R[u]$$

where R is the coefficient ring and $u\in H^2(\mathbb{C}\mathrm{P}^\infty)$ is the usual generator. In particularly, $H^*_{\mathbb{T}}(\mathrm{pt})$ is non-zero in infinitely many degrees and this is typically also the case for $H^*_G(\mathrm{pt})$.

Another core feature of the Borel theories is that $H^*_G(X)$ and $H^*(X)$ are naturally modules over the R algebra $H^*_G(\mathrm{pt})\cong H^*(BG)$. The module structures originate from the cup and cap products in ordinary homology combined with the bundle projection $p_X\colon X_{hG}\to *_{hG}\cong BG$:

$$H^*(BG)\otimes H^*_G(X_{hG})\to H^*_G(X_{hG}),\quad \xi\otimes x\mapsto (p_X^*\xi)\cup x \tag{1.4.17}$$

$$H^*(BG)\otimes H^G_*(X_{hG})\to H^G_*(X_{hG}),\quad \xi\otimes x\mapsto (p_X^*\xi)\cap x \tag{1.4.18}$$

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Relation to Floer homology. It turns out that Borel homology and cohomology are accessible to Floer theory, at least in some special situations. We only mention two instances, one of which will be picked up later in the discussion of monopole Floer homology.

(1) One result in this direction is proved in [AB95]. Assuming the Smale condition holds for a G–Morse pair (f,ξ) on a G–manifold M, there is a version of Morse cohomology that computes $H^*_G(M;\mathbb{R})$. The construction involves moduli spaces of trajectories and differential forms, and hence only apply to real coefficients.

(2) The story simplifies for the circle group $G=\mathbb{T}$. A Floer theoretic approach to this setting, tailor-made for applications to Seiberg–Witten theory, was developed by Kronheimer and Mrowka [KM07, Ch. 2.4–2.6]. As we will see, Seiberg–Witten theory on 3–manifolds will eventually bring us into this setting. Moreover, we will not have to worry about arbitrary $\mathbb{T}$–actions, but only those that are free on $X\setminus X^{\mathbb{T}}$. We will get back to this in due time.

1.5 A survey of Floer theory in infinite dimensions

As mentioned before, ideas inspired by Morse theory and Floer homology are often useful, even in circumstances where they do not make literal sense. These applications often involve functions on or equations in function spaces, which can often be viewed as infinite dimensional manifolds. We will mention a few examples momentarily, but it is important to realize beforehand that there are serious obstacles that have to be overcome

Problems in infinite dimensions:

(1) Infinite dimensional manifolds, modeled on infinite dimensional vector spaces (e.g. Banach, Hilbert, Fréchet,...) are never locally compact. This amplifies the compactness problems that were already present in the finite dimensional theory. This affects both the moduli spaces in Floer theory and the very foundation of Conley index theory.

(2) While there is a reasonable generalization of Morse theory to Hilbert manifolds, the Hessian of a random Morse function will usually have infinitely many positive and negative eigenvalues. In these situations there is no (absolute) Morse index.

(3) The existence and uniqueness theorems for ordinary differential equations are usually not applicable. The "flow equations" often become non-linear partial differential equations.

Instances of infinite dimensional Morse and Floer homology: Nevertheless, using Morse and Floer theory as a guiding principles has led to many insights. We mention a few examples.

(1) The theory of geodesics on a Riemannian manifolds can be based on the *energy functional*

$$E(\gamma) = \int_{[} 0,1]|\dot{\gamma}(t)|^2 dt \tag{1.5.1}$$

defined on a suitable space of paths $\gamma\colon [0,1] \to \mathbb{R}$. Pretending that E is a Morse function led Bott to his proof of his famous periodicity theorem. This is treated in detail in Milnor's book "Morse theory" [Mil63].

(2) Another examples is the *symplectic action functional*

$$a(u) = \int_D^2 u^*\omega \tag{1.5.2}$$

where (M, ω) is a symplectic manifold, $L, L' \subset M$ are Lagrangian subspaces, and $u\colon D^2 \to M$ is a map with $u(0) \in L \cap L'$, thought of as a point in the universal cover of the space of paths in M from L to L' based at a point in $L \cap L'$. Floer managed to define Floer chain complexes generated by the intersection points, enabling him to solve the *Arnold conjecture*. See [Flo89, Ch. 4] and the references therein.

(3) After his success with the Arnold conjecture, Floer applied has methods successfully to the *Chern–Simons functional*

$$cs(A) = \frac{1}{8\pi^2} \int_Y \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

where A is a connection on a trivial SU_2 bundle over a closed 4–manifold Y. Details can be found in [Don02]

(4) Finally, we mention the *Chern–Simons–Dirac functional*

$$\mathcal{L}(a, \phi) = \frac{1}{2} \langle a, *da \rangle_{L^2} + \frac{1}{2} \langle \phi, D_A \rangle_{L^2} + \frac{1}{2} \left\langle *F_{A_0^t}, a \right\rangle_{L^2}$$

for a pair of a spinor ϕ and a spin^c connection $A = A_0 + a$ on a 3–manifold Y equipped with a spin^c structure. This is the basis of monopole Floer homology that we will discuss in the remaining course.

Chapter 2

The Seiberg–Witten Equations

We momentarily leave Floer homology behind and embark on a digression on Seiberg–Witten theory. The *Seiberg–Witten equations* are a system of non-linear partial differential equations defined on 4–dimensional manifolds that are equipped with a *spin^c structure.* They take the form

$$\frac{1}{2}F^+_{A^t} = \rho^{-1}(\phi\phi^*)_0 \qquad D_A\phi = 0 \tag{2.0.1}$$

where ϕ is a *spinor* and A is a *spin^c connection.* These equations originally arose in physics and were introduced to mathematics by Witten [Wit94]. However, they did have a mathematical precursor in the *Yang–Mills equations*

$$F^+_A = \frac{1}{2}(F_A + *F_A) = 0 \tag{2.0.2}$$

for a connection A on a principal SU_2 bundle over a 4-manifold which also originate from physics. These equations were studied with spectacular success by Donaldson [Don83, DK90]. The ultimate goal of this course is to prove a generalization of the following result:

Theorem 2.1 (Donaldson [Don83])**.** *Let X be a closed, oriented, topological 4–manifold with definite intersection form Q_X. If X admits a smooth structure, then Q_X is diagonalizable over the integers.*

The power of this result becomes apparent in the light of another major theorem of Freedman.

Theorem 2.2 (Freedman [Fre82])**.** *Every unimodular symmetric bilinear form over $\mathbb{Z}$ arises as the intersection form Q_X of a simply connected, oriented, topological 4–manifold X. Moreover, all such X are classified up to orientation preserving homeomorphism by the isometry class of Q_X and the Kirby–Siebenmann invariant $ks(X) \in \mathbb{Z}_2$.*

A surprising offshoot of these results is the existence of an "exotic $\mathbb{R}^4$", that is, a smooth 4–manifold that is homeomorphic but not diffeomorphic to $\mathbb{R}^4$. In other words, $\mathbb{R}^4$ supports smooth structures that are not diffeomorphic to the standard one. In contrast, it was known that the smooth structure on $\mathbb{R}^n$ for $n \neq 4$ is unique up to diffeomorphism.

Theorem 2.3 (Taubes [Tau84])**.** *There are uncountably many smooth 4–manifolds that are homeomorphic but not diffeomorphic to $\mathbb{R}^4$.*

The Seiberg–Witten equations are not particularly self-explanatory and it is our first to learn how to read them correctly. We assume some previous exposure to Riemannian geometry, the theory Clifford algebras, spin$^{(c)}$ structures, Dirac operators, etc. Our main references are [LM89] and the first chapter of [KM07]. The eternally unpublished book draft [Sal99] is also recommended.

2.1 Spin^c structures and spinor bundles.

The first ingredient needed to write down the Seiberg–Witten equations is a *spin^c structure*. These can be described in many different ways and we choose the one that is most convenient for our present purposes (c.f [KM07, Sal99]). We fix the following notation and conventions:

- All manifolds are assumed to be smooth, oriented, and carry Riemannian metrics.
- All vector bundles implicitly carry bundle metrics.
- M will denote an arbitrary manifold of dimension n.
- Y will always be 3–manifold, in later sections closed.
- X will always be a 4–manifold, later either closed or compact with $\partial X = Y$.

Without further ado, here is our working definition:

Definition 2.4 (Spinor bundles and spin^c structures)**.**
Let M be an oriented Riemannian n–manifold with $n = 2k$ or $2k + 1$.

(a) A *(complex) spinor bundle* on M is a pair (S, ρ) where S is a Hermitian vector bundle of rank 2^k together with a bundle map $\rho\colon T^*M \to \mathrm{End}_{\mathbb{C}}(S)$ such that

$$\rho(a)^2 = -|a|^2 \,\mathrm{id}_E \quad \text{and} \quad \rho(a)^* = -\rho(a) \tag{2.1.1}$$

for all $a \in T^*M$. In addition, if $n = 2k + 1$ is odd, we require that

$$\rho(e_1) \cdots \rho(e_n) = -i^{k+1}\,\mathrm{id}_S \tag{2.1.2}$$

for every oriented orthonormal basis $e_1, \dots, e_n \in T^*_x M$, $x \in M$. The map ρ is called *Clifford multiplication* and is usually dropped from the notation.

(b) An *isomorphism of spinor bundle* (S, ρ) and (S', ρ') is a unitary vector bundle isomorphism $U\colon S \xrightarrow{\cong} S'$ which is *Clifford linear* in the sense that $U \circ \rho(a) = \rho'(a) \circ U$ for all $a \in T^*M$.

(c) A *spin^c structure* on M is an isomorphism class of spinor bundles. We write $\mathrm{Spin}^c(M)$ for the set of spin^c structures.

We note that if $e_1, \dots, e_n \in T^*_M$ is an orthonormal basis, then

$$\rho(e_i)^2 = -\,\mathrm{id}_S \quad \text{and} \quad \rho(e_i)\rho(e_j) = -\rho(e_j)\rho(e_i) \quad (i \neq j). \tag{2.1.3}$$

This follows from inserting e_i and $e_i + e_j$ into the first equation in (2.1.1).

Remark 2.5 (Relation to Clifford algebras). Those familiar with the theory of Clifford algebras will realize that the first condition in (2.1.1) implies that ρ extends to a fiberwise action of the complex Clifford algebra bundle $\mathbb{C}\mathrm{l}(M)$ on E. The dimension assumption make the fibers E_x, $x \in M$, irreducible as a $\mathbb{C}\mathrm{l}(M)_x$–modules, and (2.1.2) fixes one of the two isomorphism classes of irreducible $\mathbb{C}\mathrm{l}(M)_x$–modules. The relevant details are discussed in [LM89, Ch. I.1–5]. For those unfamiliar with Clifford algebras, it suffices to know that $\mathbb{C}\mathrm{l}(M)$ is isomorphic as vector bundle to the complex exterior algebra

$$\Lambda^*_{\mathbb{C}} M = \Lambda^* T^*M \otimes \mathbb{C}. \tag{2.1.4}$$

We can extend Clifford multiplication to a map

$$\rho\colon \Lambda^*_{\mathbb{C}} M \to \mathrm{End}_{\mathbb{C}}(E) \tag{2.1.5}$$

by requiring for $\alpha, \beta \in \Lambda^*_{\mathbb{C}} M$ that

$$\rho(\alpha \wedge \beta) = \frac{1}{2}\big(\rho(\alpha)\rho(\beta) + (-1)^{|\alpha||\beta|}\rho(\beta)\rho(\alpha)\big). \tag{2.1.6}$$

The condition (2.1.2) for odd $n = 2k+1$ then becomes $\rho(\mathrm{vol}_M) = -i^{k+1}\,\mathrm{id}_S$. In particular, for $n = 3$ we get $\rho(\mathrm{vol}_M) = \mathrm{id}_S$ which agrees with the conventions in [KM07, .] In even dimensions $n = 2k$, a direct computation shows that $\rho(\mathrm{vol}_M)^2 = (-1)^k\,\mathrm{id}_S$. We will get back to this shortly. The following will frequently be useful. Lecture 8, 23.5.23

Lemma 2.6. *Let (S, ρ) be a spinor bundle over M. If $T \in \Gamma(\mathrm{End}_{\mathbb{C}}(S))$ is Clifford linear, then $T\phi = f\phi$ for some $f \in C^\infty(M, \mathbb{C})$.*

Proof. As noted, the fiber S_x over $x \in M$ is an irreducible module over the $\mathbb{C}$–algebra $\mathbb{C}\mathrm{l}(T^*_x M)$. A version of Schur's lemma states that every $\mathbb{C}\mathrm{l}(T^*_x M)$–linear endomorphism of S_x is given by multiplication with a complex number. □

Existence and classification of spin^c strucutures. Assuming that one spin^c structure on M exists, the classification of all others is fairly easy.

Proposition 2.7 (Classification of spin^c strucutres)**.** *Let (S, ρ) be a spinor bundle on M and L a Hermitian line bundle. Then $(S \otimes L, \rho \otimes \mathrm{id})$ is also a spinor bundle. The construction descends to free and transitive action of $H^2(M; \mathbb{Z})$ on the* $\mathrm{Spin}^c(M)$

$$\mathrm{Spin}^c(M) \times H^2(M; \mathbb{Z}) \to \mathrm{Spin}^c(M), \quad \big([S, \rho], c\big) \mapsto [S \otimes L, \rho \otimes \mathrm{id}]$$

where L is a Hermitian line bundle with $c_1(L) = c$.

Proof. We sketch the proof and refer to [KM07, Prop. 1.1.1] for further details.

- The verification that $(S \otimes L, \rho \otimes \mathrm{id})$ is a spinor bundle is trivial.
- Conversely, if (S, ρ) and (S', ρ') are spinor bundles, one can show that

  $$L = \{T \in \mathrm{Hom}_{\mathbb{C}}(S, S') \,|\, T\rho(a) = \rho'(a)T \text{ for all } a \in T^*M\}$$

  is a rank 1 sub-bundle of $\mathrm{Hom}_{\mathbb{C}}(S, S')$, henceforth referred to as the *difference line bundle*.
- It is easy to see that (S, ρ) and (S', ρ') are isomorphic iff the *difference line bundle* L is trivial.
- Moreover, the difference line bundle of (S', ρ') and $(S \otimes L, \rho \otimes \mathrm{id})$ turns out to be $L \otimes L^*$ which is canonically trivialized by the section corresponding to id_L under the canonical isomorphism $\mathrm{End}_{\mathbb{C}}(L) \cong L \otimes L^*$.
- Lastly, it is well known that Hermitian line bundles form a group under the tensor product which is isomorphic to $H^2(M; \mathbb{Z})$ via c_1. □

The existence of spin^c structures is more subtle. Here is the general result.

Proposition 2.8 (Existence of spin^c structures, c.f. [LM89, Corollary D.5])**.** *An oriented Riemannian manifold M admits a spin^c structure if and only if $w_2(M) \in H^2(M; \mathbb{Z}_2)$ is the mod 2 reduction of a class in $H^2(X; \mathbb{Z})$.*

Theorem 2.9 (Spin^c structures in dimensions ≤ 4)**.** *All oriented Riemannian manifolds of dimension ≤ 4 admit spin^c structures.*

Proof.
- If $n \leq 1$ or $n = 2$ and M is non-compact or $\partial M \neq \emptyset$, we have $H^2(M; \mathbb{Z}_2) = 0$ and the condition on $w_2(M)$ in Proposition 2.8 is trivially satisfied.

- For $n = 2$ and M closed, the reduction map $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z}_2)$ is surjective by the Bockstein sequence (see [Hat02, Ch. 3E]).
- For $n = 3$ it is known that every oriented 3–manifold is parallelizable, that is, TM is trivial so that $w_2(M) = 0$. This interesting fact can be proved using obstruction theory and the fact that M has the homotopy type of a CW complex of dimension ≤ 3.
- For $n = 4$ the condition on $w_2(M)$ can be verified using the Bockstein sequence, the universal coefficient theorem, the Wu formula, and some general facts about Abelian groups (see [GS99, Proposition 5.7.4 and Remark 5.7.5]).

□

The chiral splitting in even dimensions. Now suppose that M has even dimension $n = 2k$. We have already noted that $\rho(\mathrm{vol}_M)^2 = (-1)^k \,\mathrm{id}_S$. In order to deal with the sign on the right hand side, it is convenient to introduce the *chirality operator*

$$\alpha_M = \rho(i^k \,\mathrm{vol}_M) = i^k \rho(e_1)\cdots\rho(e_n). \tag{2.1.7}$$

Again by direct computations one can easily verify the following properties:

Lemma 2.10. *For $n = 2k$ even, the chirality operator α_M satisfies*

$$\alpha_M^2 = \mathrm{id}_S \quad \text{and} \quad \alpha_M^* = \alpha_M. \tag{2.1.8}$$

Moreover, for all $a \in T^*M$ we have

$$\alpha_M \rho(a) = -\rho(a)\alpha_M. \tag{2.1.9}$$

As a consequence of (2.1.8), we get a decomposition of S into ± 1 eigenbundles of α_M

$$S = S^+ \oplus S^-, \quad S^\pm = \ker(\alpha_M \mp \mathrm{id}_S) = (\mathrm{id}_S \pm \alpha_M)S \tag{2.1.10}$$

and (2.1.9) shows that Clifford multiplication with $0 \neq a \in T_x^*M$ gives an isomorphism

$$\rho(a)\colon S_x^\pm \xrightarrow{\cong} S_x^\mp. \tag{2.1.11}$$

In particular, S^+ and S^- have the same rank 2^{k-1}. Note that the isomorphisms in (2.1.11) are only defined in a single fiber. In fact, the bundles S^+ and S^- are generally not isomorphic. Generally, if $\omega \in \Lambda_{\mathbb{C}}^{\mathrm{ev}} M$ is a form of even degree, then $\rho(\omega)S^\pm \subset S^\pm$ while for $\omega \in \Lambda_{\mathbb{C}}^{\mathrm{odd}} M$ we have $\rho(\omega)S^\pm \subset S^\mp$.

Spin^c strucutues via principal bundles. Another common description of spin^c structure uses principal Spin$_c$ bundles. As indicated in Remark 2.5, the Clifford algebra $\mathbb{C}\mathrm{l}_n$ of $\mathbb{R}^n$ has a unique irreducible complex representation (up to isomorphism) for which the obvious analogue of (2.1.2) is satisfied. The dimension of any such representation Δ_n can be computed as 2^k where $n = 2k$ or $2k+1$. Inside $\mathbb{C}\mathrm{l}_n$ we find the multiplicative subgroup Spin_n^c which is generated by products $z(v \cdot w)$ with $v, w \in \mathbb{R}^n$ and $z \in \mathbb{C}$ with $|v| = |w| = |z| = 1$. The group Spin_n^c has an obvious representation on Δ_n and a more subtle one on $\mathbb{R}^n$. We denote these representations by

$$\sigma\colon \mathrm{Spin}_n^c \to \mathrm{End}_{\mathbb{C}}(\Delta_n) \quad \text{and} \quad \alpha\colon \mathrm{Spin}_n^c \to SO_n. \tag{2.1.12}$$

Definition 2.11. A *principal spin^c structure* on M as a pair (P, τ) consisting of

- a principal Spin_n^c–bundle P, and
- an isomorphism $\tau\colon P \times_\alpha \mathbb{R}^n \cong T^*M$ that preserves metrics and orientations.

An *isomorphism of principal spin^c structures* (P, τ) and (P', τ') is a Spin^c_n–equivariant diffeomorphism $\varphi\colon P \xrightarrow{\cong} P'$ such that $\tau' \circ (\varphi \times_\alpha \mathrm{id}_{\mathbb{R}^n}) = \tau$.

From a principal spin^c structure (P, τ) we obtain a spinor bundle in the sense of Definition 2.4 by $S = P \times_\sigma \Delta_n$ with Clifford multiplication induced by the Cl_n–action on Δ_n. Isomorphic choices of (P, τ) and Δ_n give isomorphic results for (S, ρ).

Conversely, given a spinor bundle (S, ρ) we can construct a principal Spin^c_n–bundle P as the set of pairs (u, v) consisting of isomorphisms $u\colon \mathbb{R}^n \xrightarrow{\cong} T^*_x M$ and $v\colon \Delta_n \xrightarrow{\cong} S_x$ with $x \in X$, preserving all orientations and inner products, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n \otimes \Delta_n & \xrightarrow{\rho_n} & \Delta_n \\ {\scriptstyle u\otimes v}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle v} \\ T^*_x M \otimes S_x & \xrightarrow{\rho} & S_x. \end{array}$$

A right action of Spin^c_n on P is given by $(u, v)a = \big(u \circ \alpha(a), v \circ \sigma(a)\big)$ and there is a canonical topology that makes P a principal Spin^c_n bundle over M. Moreover, we have canonical isomorphisms

$$\tau\colon P \times_\alpha \mathbb{R}^n \xrightarrow{\cong} T^*M, \qquad [u, v; a] \mapsto u(a), \tag{2.1.13}$$

$$\varphi\colon P \times_\sigma \Delta_n \xrightarrow{\cong} S, \qquad [u, v; \phi] \mapsto v(\phi). \tag{2.1.14}$$

Again, changing (S, ρ) up to isomorphism gives isomorphic (P, τ). Everything is set up such that the two constructions are mutually inverse up to isomorphism. We can thus equivalently think of spin^c structures as isomorphism classes of spinor bundles or principal spin^c structures.

Models in dimensions 3 and 4. First, suppose that (S, ρ) is a spinor bundle over a 3–manifold Y. In this case, S has complex rank 2. Given an oriented orthonormal basis $e_1, e_2, e_3 \in T^*_y Y$ at a point $y \in Y$ one can find an orthonormal basis of the corresponding fiber S_y such that Clifford multiplication is represented by the matrices $\rho(e_j) = \sigma_j$ where

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{2.1.15}$$

Note that these form of basis for the real vector space $\mathfrak{su}_2$ of trace-free, skew-adjoint complex 2-by-2 matrices; this is the Lie algebra of the special unitary group SU_2.

Now let (S, ρ) be a spinor bundle over a 4–manifold X. Then S has rank 4 while $S^\pm$ each have rank 2. If $e_0, e_1, e_2, e_3 \in T^*_x X$ is an oriented orthonormal basis, we can find an orthonormal basis for S such that

$$\rho(e_0) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \quad \text{and} \quad \rho(e_j) = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad (j = 1, 2, 3). \tag{2.1.16}$$

where I_2 is the 2-by-2 identity matrix. Again, these matrices are trace-free.

The determinant line bundle. Let $\mathfrak{s}$ be a spin^c structure on M represented by a spinor bundle (S, ρ). There is a canonical line bundle associated to this data which is known as the *determinant line bundle* and denoted by $\det(S)$. Since we are only interested in dimensions 3 and 4, we can get away with the following ad hoc definition:

$$\det(S) = \begin{cases} \Lambda^2_{\mathbb{C}} S, & n = 3 \\ \Lambda^2_{\mathbb{C}} S^+, & n = 4 \end{cases} \tag{2.1.17}$$

In general, the most common construction of $\det(S)$ involves the principal spin^c structure derived from (S, ρ) and the group homomorphism

$$\delta\colon \mathrm{Spin}^c_n \to U_1, \quad \delta(z \cdot v_1 \cdots v_k) = z^2. \tag{2.1.18}$$

If P_ρ is the principal Spin^c_n–bundle derived from (S, E), we can define

$$\det(S) = P_\rho \times_\delta \mathbb{C}. \tag{2.1.19}$$

The reason for the name is that there is another group homomorphism $\lambda\colon U_n \to \mathrm{Spin}^c_{2n}$ such that the composition $U_n \xrightarrow{\lambda} \mathrm{Spin}^c_{2n} \xrightarrow{\delta} U_1$ is the complex determinant map.

Proposition 2.12. *(i) Every almost complex manifold (M, J) has a canonical spin^c structure $\mathfrak{s}_J$ and $\det(\mathfrak{s}_J) \cong \Lambda^{\mathrm{top}}_{\mathbb{C}} TM$.*

(ii) The first Chern class $c_1(\mathfrak{s}) = c_1(\det(\mathfrak{s})) \in H^2(M;\mathbb{Z})$ reduced mod 2 to the Stiefel–Whitney class $w_2(M) \in H^2(M;\mathbb{Z}_2)$.

(iii) For $n = 3$ or 4 we have $c_1(\mathfrak{s}) = c_1(S)$ and $c_1(\mathfrak{s}) = c_1(S^+)$, respectively.

2.2 The quadratic term

Lecture 9, 6.6.23

We are still in the process of learning to read the Seiberg–Witten equations on a 4–manifold X:

$$\frac{1}{2} F^+_{A^t} = \rho^{-1}(\phi\phi^*)_0 \qquad D^+_A \phi = 0.$$

At this point, we can understand two symbols ρ and ϕ:

- ρ is the Clifford multiplication on some spinor bundle (S, ρ), and
- $\phi \in \Gamma(S^+)$ is a section of the positive spinor bundle.

We next tackle the combined expression $\rho^{-1}(\phi\phi^*)_0$ which is called *the quadratic term* in the Seiberg–Witten equations. It helps to put this into a broader context.

Splitting complex endomorphism bundles. Let E be a complex vector bundle over M of rank r. We can decompose the endomorphism bundle as

$$\mathrm{End}_{\mathbb{C}}(E) = \mathbb{C}\,\mathrm{id}_E \oplus \mathfrak{su}(E) \oplus i\mathfrak{su}(E) \tag{2.2.1}$$

where $\mathfrak{su}(E)$ (resp. $i\mathfrak{su}(E)$) denotes the fiberwise trace-free and skew-adjoint (resp. self-adjoint) endomorphisms. The projections onto the summands can be described explicitly as follows. We first introduce the *trace-less part* of $A \in \mathrm{End}_{\mathbb{C}}(E)$

$$A_0 = A - \frac{1}{r}\,\mathrm{tr}_{\mathbb{C}}(A)\,\mathrm{id}_E \tag{2.2.2}$$

whose name is justified by $\mathrm{tr}_{\mathbb{C}}(A_0) = 0$ which follows from $\mathrm{tr}_{\mathbb{C}}\,\mathrm{id}_E = r$. We can then write

$$A = \underbrace{\frac{1}{r}\,\mathrm{tr}_{\mathbb{C}}(A)\,\mathrm{id}_E}_{\in \mathbb{C}\,\mathrm{id}_E} + \underbrace{\frac{1}{2}(A_0 - A_0^*)}_{\in \mathfrak{su}(E)} + \underbrace{\frac{1}{2}(A_0 + A_0^*)}_{\in i\mathfrak{su}(E)}. \tag{2.2.3}$$

From spinors to endomorphisms. Now let (S,ρ) be a spinor bundle over M. Given spinors $\phi,\psi \in \Gamma(S)$ we can form an endomorphism

$$\psi\phi^* \in \Gamma(\mathrm{End}_{\mathbb{C}}(S)), \quad \psi\phi^*(\kappa) = \langle \phi, \kappa\rangle\, \phi.$$

Our convention is that Hermitian scalar products are complex linear in the second entry and conjugate linear in the first.

Lemma 2.13. *Let (S,ρ) be a spinor bundle over M. The map*

$$\Gamma(S)\times\Gamma(S) \to \Gamma(\mathrm{End}_{\mathbb{C}}(S)), \quad (\psi,\phi)\to \psi\phi^*$$

is complex linear in ϕ and complex anti-linear in ψ. Moreover, we have

$$(\psi\phi^*)^* = \phi\psi^* \quad \textit{and} \quad \mathrm{tr}_{\mathbb{C}}(\psi\phi^*) = \langle \phi,\psi\rangle\,.$$

In particular, $\phi\phi^*$ is self-adjoint for every $\phi\in\Gamma(S)$.

Proof. The linearity properties are obvious. The trace can be computed as

$$\mathrm{tr}_{\mathbb{C}}(\phi\psi^*) = \sum_i \langle s_i, \phi\psi^*(s_i)\rangle = \sum_i \langle \langle s_i,\psi\rangle\, s_i, \phi\rangle = \langle \psi,\phi\rangle\,.$$

where s_i is a local frame of S. The adjoint is identified as follows:

$$\langle s_i, \phi\psi^*(s_j)\rangle = \langle s_i, \langle \psi, s_j\rangle\,\phi\rangle = \langle \langle \phi s_i\rangle\,\psi, s_j\rangle = \langle \psi\phi^*(s_i), s_j\rangle\,. \qquad \square$$

From endomorphisms to forms. We can now parse the expression $(\phi\phi^*)_0 \in \mathfrak{su}(S)$ and it remains to understand how to apply ρ^{-1} to $(\phi\phi^*)_0$. To that end, we have the following general statement:

Proposition 2.14. *Let (S,ρ) be a spinor bundle over a manifold M.*

(i) The map $\rho\colon \Lambda^*_{\mathbb{C}}M \to \mathrm{End}_{\mathbb{C}}(S)$ is surjective and gives isomorphisms

$$\begin{aligned} \rho\colon \Lambda^*_{\mathbb{C}}M &\to \mathrm{End}_{\mathbb{C}}(S) \quad \textit{for } n \textit{ even, and}\\ \rho\colon \Lambda^{\le k}_{\mathbb{C}}M &\to \mathrm{End}_{\mathbb{C}}(S) \quad \textit{for } n=2k+1 \textit{ odd.}\end{aligned}$$

(ii) For $\omega\in\Lambda^k_{\mathbb{C}}M$ we have $\rho(\omega)^* = (-1)^{\frac{k(k+1)}{2}}\rho(\bar\omega)$.

Proof. The surjectivity of ρ as well as the injectivity for n even follow from the isomorphism $\Lambda^*_{\mathbb{C}}M \cong \mathbb{C}\mathrm{l}(M)$ and the classification of Clifford algebras and their irreducible modules (c.f. [LM89, Chs. I.4&5]).

The formula in (ii) can be proved pointwise using an orthonormal basis $e_1,\dots,e_n\in T^*_xM$. For $e_I = e_{i_1}\wedge\cdots\wedge e_{i_k}$ we find

$$\begin{aligned}\rho(e_I)^* &= \rho(e_{i_1}\wedge\cdots\wedge e_{i_k})^*\\ &= \big(\rho(e_{i_1})\cdots\rho(e_{i_k})\big)^*\\ &= (-1)^k\rho(e_{i_k})\cdots\rho(e_{i_1})\\ &= (-1)^k(-1)^{k(k-1)/2}\rho(e_{i_1})\cdots\rho(e_{i_k})\\ &= (-1)^{k(k+1)/2}\rho(e_I)\end{aligned}$$

Lastly, for $n=2k+1$ we know that $\rho(e_I)\rho(*e_I) = \rho(\mathrm{vol}) = i^{k+1}$ and $\rho(e_I)^2 = (-1)^{k(k-1)/2}$. It follows that for $\omega\in\Lambda^p_CM$

$$\rho(*\omega) = i^{m(p)}\rho(\omega) \tag{2.2.4}$$

for some integer $m(k,p)$ determined by k and p. This together with the classification theorems for Clifford algebras give the remaining isomorphism in (i). $\square$

For $n = 3$ we can draw the following conclusion:

Corollary 2.15. *If (S, ρ) is a spinor bundle over a 3–manifold Y, then Clifford multiplication gives rise to isomorphisms*

$$\rho\colon T^*Y \xrightarrow{\cong} \mathfrak{su}(S) \quad \textit{and} \quad \rho\colon iT^*Y \xrightarrow{\cong} i\mathfrak{su}(S). \tag{2.2.5}$$

In particular, for every $\phi \in \Gamma(S)$ we obtain an imaginary valued 1–form

$$\rho^{-1}(\phi\phi^*)_0 \in i\Omega^1(Y). \tag{2.2.6}$$

Proof. Exercise. □

If $n = 2k$ is even, the chiral splitting $S = S^+ \oplus S^-$ gives another decomposition

$$\mathrm{End}_{\mathbb{C}}(S) \cong \mathrm{End}_{\mathbb{C}}(S^+) \oplus \mathrm{End}_{\mathbb{C}}(S^-) \oplus \mathrm{Hom}_{\mathbb{C}}(S^+, S^-) \oplus \mathrm{Hom}_{\mathbb{C}}(S^-, S^+). \tag{2.2.7}$$

For $\phi \in \Gamma(S^+)$ we can consider $\phi\phi^*$ as an element of $\mathrm{End}_{\mathbb{C}}(S^+)$ and by Lemma 2.13 we find

$$(\phi\phi^*)_0 = \phi\phi^* - \frac{|\phi|^2}{\mathrm{rk}(S^+)}\,\mathrm{id}_{S^+} = \phi\phi^* - \frac{|\phi|^2}{2^{k-1}}\,\mathrm{id}_{S^+} \in i\mathfrak{su}(S_+)$$

Lastly, for $n = 4$ the Hodge operator gives a self-adjoint map $*\colon \Lambda^2 M \to \Lambda^2 M$ with $*^2 = 1$. This gives a splitting

$$\Lambda^2 M = \Lambda^2_+ M \oplus \Lambda^2_- M, \quad \Lambda^2_\pm M = \ker(* \mp \mathrm{id})$$

into *self-dual* and *anti-self-dual* 2–forms.

Corollary 2.16. *If (S, ρ) is a spinor bundle over a 4–manifold X, then Clifford multiplication gives rise to isomorphisms*

$$\rho\colon \Lambda^2_\pm X \xrightarrow{\cong} \mathfrak{su}(S^\pm) \quad \textit{and} \quad \rho\colon i\Lambda^2_\pm X \xrightarrow{\cong} i\mathfrak{su}(S^\pm). \tag{2.2.8}$$

In particular, for $\phi \in \Gamma(S^+)$ we obtain a self-dual imaginary valued 2–form

$$\rho^{-1}(\phi\phi^*)_0 \in i\Omega^2_+(M).$$

Proof. Exercise. □

2.3 Spin^c connections and Dirac operators

Here are the Seiberg–Witten equations once more:

$$\frac{1}{2}F^+_{A^t} = \rho^{-1}(\phi\phi^*)_0 \qquad D^+_A\phi = 0.$$

Having completely understood the quadratic term $\rho^{-1}(\phi\phi^*)_0$, we now tackle the symbols involving A. Most of these make sense in a more general context:

- ▶ A, A^t, F_A, and D_A are defined for arbitrary spin^c manifolds.
- ▶ D^+_A makes sense in all even dimensions.
- ▶ $F^+_{A^t}$ is special to 4–manifolds (essentially, because $n - 2 = 2$ implies $n = 4$).

2.3.1 Spin^c connections

Let (S, ρ) be a spinor bundle on M. We denote connections on S by A and think of them in terms of the covariant derivative $\nabla^A \colon \Gamma(S) \to \Gamma(T^*M \otimes S)$. We implicitly require all connections to be *Hermitian*, that is, we have

$$d\,\langle \phi, \psi \rangle = \left\langle \nabla^A \phi, \psi \right\rangle + \left\langle \phi, \nabla^A \psi \right\rangle \quad (\phi, \psi \in \Gamma(S)). \tag{2.3.1}$$

Recall that the difference of two Hermitian connections is a 1–form with values in the vector bundle $\mathfrak{u}(S)$ of skew-adjoint endomorphisms of S.

Definition 2.17. A connection A on S is called a *spin^c connection* (or *Clifford connection*) if it is compatible with the Clifford multiplication in the sense that

$$\nabla^A(\rho(a)\phi) = \rho(a)\nabla^A\phi + \rho(\nabla^{LC}a)\phi \tag{2.3.2}$$

for all $\phi \in \Gamma(S)$ and $a \in \Omega^1(X)$. The superscript in ∇^{LC} indicates the Levi–Civita connection on T^*M. We write $\mathcal{A}(S)$ for the set of spin^c connections of S.

Lemma 2.18. *$\mathcal{A}(S)$ is an affine space modeled on the real vector space $i\Omega^1(M)$. More precisely:*

(i) For $A \in \mathcal{A}(S)$ and $a \in i\Omega^1(M)$ we obtain $A + a \in \mathcal{A}(S)$ defined by

$$\nabla^{A+a}\phi = \nabla^A\phi + a \otimes \phi. \tag{2.3.3}$$

(ii) If we fix one spin^c connection $A_0 \in \mathcal{A}(S)$, then we have $A = A_0 + a$ for a uniquely determined $a \in i\Omega^1(M)$.

Proof. In order to prove (i) we have to verify (2.3.2) for ∇^{A+a}. Let $v \in TX$.

$$\begin{aligned}
\nabla_v^{A+a}(\rho(\alpha)\phi) &= \nabla_v^A(\rho(\alpha)\phi) + a(v)\rho(\alpha)\phi \\
&= \rho(\alpha)\nabla_v^A\phi + \rho(\nabla_v^{LC}\alpha)\phi + a(v)\rho(\alpha)\phi \\
&= \rho(\alpha)\nabla_v^A\phi + \rho(\nabla_v^{LC}\alpha)\phi + \rho(\alpha)a(v)\phi \\
&= \rho(\alpha)(\nabla_v^A\phi + a(v)\phi) + \rho(\nabla_v^{LC}\alpha)\phi \\
&= \rho(\alpha)\nabla_v^{A+a}\phi + \rho(\nabla_v^{LC}\alpha)\phi
\end{aligned}$$

Now, given arbitrary $A_0, A \in \mathcal{A}(S)$, we from the general theory of Hermitian connections that $A = A_0 + \tilde{a}$ for a unique 1–form $\tilde{a} \in \Omega^1(M; \mathfrak{u}(S))$. The condition (2.3.2) implies that $\tilde{a}$ is pointwise Clifford linear and thus given by multiplication with a complex number by Lemma 2.6. Since a is pointwise skew-adjoint, that complex number must be purely imaginary. It follows that $\tilde{a} = a \otimes \mathrm{id}_S$ with $a \in i\Omega^1(M)$. □

Remark 2.19. We have allowed ourselves a small abuse of notation. Strictly speaking, one should write $A = A_0 + a \otimes \mathrm{id}_S$ to emphasize the $\mathfrak{u}(S)$–valued nature of the 1–form measuring the difference between A and A_0.

The following is an easy consequence of (2.3.2).

Lemma 2.20. *If $n = \dim(M) = 2k$ is even, then ∇^A preserves the splitting $S = S^+ \oplus S^-$ and thus induces connections on $S^\pm$.*

Recall from (2.1.19) and (2.1.17) that (S, ρ) has an associated determinant line bundle $\det(S)$ which takes the following form in dimensions 3 and 4:

$$\det(S) = \begin{cases} \Lambda^2_{\mathbb{C}} S, & n = 3 \\ \Lambda^2_{\mathbb{C}} S^+, & n = 4 \end{cases}$$

This explains the symbol A^t in the Seiberg–Witten equations.

Definition 2.21. Given $A \in \mathcal{A}(S)$ we write A^t for the induced connection on $\det(S)$.

2.3.2 Curvature

Let us momentarily think of $\Gamma(S)$ and $\Gamma(T^*M \otimes S)$ as the spaces of 0– and 1–forms on M with values in S. Combining $A \in \mathcal{A}(S)$ with the Levi–Civita connection on T^*M we can extent ∇^A to maps

$$d^A \colon \Omega^k(M;S) \to \Omega^{k+1}(M;S) \tag{2.3.4}$$

where $\Omega^k(M;S) = \Gamma(\Lambda^k T^*M \otimes S)$. This gives a sequence

$$\Omega^0(M;S) \xrightarrow{d^A} \Omega^1(M;S) \xrightarrow{d^A} \Omega^2(M;S) \xrightarrow{d^A} \cdots \tag{2.3.5}$$

which resembles the de Rham complex. However, we will typically have $(d^A)^2 \neq 0$. The failure of (2.3.5) to be a complex is measured by the *curvature* of A which is a 2–form

$$F_A \in \Omega^2(M;\mathfrak{u}(S)) \tag{2.3.6}$$

which is determined by the equation

$$((d^A)^2\phi)(v,w) = F_A(v,w)\phi \in \Gamma(S) \tag{2.3.7}$$

where $\phi \in \Gamma(S)$ and $v,w \in \Gamma(TM)$.

The interaction of curvature and the affine structure of $\mathcal{A}(S)$ is easily understood. If $A = A_0 + a$ with $A_0 \in \mathcal{A}(S)$ fixed and $a \in i\Omega^1(M)$, standard arguments with connections show that

$$F_A = F_{A_0} + da \otimes \mathrm{id}_S \in \Omega^2(M;\mathfrak{u}(S)). \tag{2.3.8}$$

One can also compare the induced connections on $\det(S)$. Since the endomorphisms bundles of line bundles are canonically trivialized by the identity map, we have a canonical isomorphism

$$\mathfrak{u}(\det(S)) \cong M \times \mathfrak{u}_1 \cong M \times i\mathbb{R}. \tag{2.3.9}$$

In particular, we can consider the curvature forms of A^t and A_0^t as imaginary valued 2–forms via the resulting isomorphisms

$$F_{A^t}, F_{A_0^t} \in \Omega^2(M;\mathfrak{u}(\det(S))) \cong \Omega^2(M;i\mathbb{R}) \cong i\Omega^2(M). \tag{2.3.10}$$

Combining (2.3.8) with the explicit description of $\det(S)$ for $n = 3$ or 4, we arrive at the following conclusion.

Lemma 2.22. *Let $n = \dim(M) = 3$ or 4. Using the identifications in (2.3.10) we have*

$$F_{A^t} = F_{A_0^t} + 2da \in i\Omega^2(M) \tag{2.3.11}$$

The factor of 2 is caused by the second exterior powers in (2.1.17) which, in turn, appear because S for $n = 3$ and S^+ for $n = 4$ have rank 2. There is a more general formula which takes the form $F_{A^t} = F_{A_0^t} + c_n\, da$ where c_n is a constant depending on n, but the precise value of c_n shall not concern outside dimensions 3 and 4.

Lastly, if $\dim(X) = 4$, we can form the *self-dual part* of F_{A^t} and note that

$$F^+_{A^t} = \frac{1}{2}\big(F_{A^t} + *F_{A^t}\big) = F^+_{A_0^t} + 2d^+a \in i\Omega^2_+(X) \tag{2.3.12}$$

where $d^+ = \frac{1}{2}(d + *d) \colon \Omega^1(X) \to \Omega^2_+(X)$.

Remembering Corollary 2.16, we find that $\rho(F^+_{A^t})$ is a self-adjoint endomorphism of S^+. At this point, we have finally managed to decode the equation $F_{A^t} = \rho^{-1}(\phi\phi)_0$ which couples a spin^c connection $\mathcal{A}(S)$ to a positive spinor $\phi \in \Gamma(S^+)$.

2.3.3 Dirac operators

It remains to decipher the equation $D_A^+\phi = 0$. The last missing piece of the puzzle are the *Dirac operators* associated to spin^c connections. Again, these are defined for arbitrary spin^c manifolds.

Definition 2.23. Let (S,ρ) be a spinor bundle over M. Every spin^c connection $A \in \mathcal{A}(S)$ has an associated *(full) Dirac operator* which is defined as the composition

$$D_A\colon \Gamma(S) \xrightarrow{\nabla^A} \Gamma(T^*M \otimes S) \xrightarrow{\rho} \Gamma(S). \tag{2.3.13}$$

If $n = \dim(M) = 2k$ is even, D_A restricts to the *chiral Dirac operators*

$$D_A^\pm\colon \Gamma(S^\pm) \to \Gamma(S^\mp). \tag{2.3.14}$$

We can also express D_A in terms of a local frame $e_1, \dots, e_n$ for TM by the formula

$$D_A(\phi)(x) = \sum_{i=1}^n \rho(e_i^\flat)\nabla_{e_i}^A \phi(x) \tag{2.3.15}$$

where $e_i^\flat = \langle e_i, \cdots\rangle$ is the dual frame for T^*M; the same formula holds for $D_A^\pm\phi$ with $\phi \in \Gamma(S^\pm)$.

We can now read the Seiberg–Witten equations. In order to study them further, we will need to know some properties of D_A.

Lemma 2.24. *Let $A \in \mathcal{A}(S)$ be a spin^c connection.*

(i) For $\phi \in \Gamma(S)$ and $f \in C^\infty(M;\mathbb{C})$ we have $D_A(f\phi) = fD_A\phi + \rho(df)\phi$.

(ii) If $A = A_0 + a$ with $a \in i\Omega^1(M)$, then $D_A\phi = D_{A_0} + \rho(a)\phi$.

(iii) For $\phi, \psi \in \Gamma(S)$ we have $\langle \phi, D_A\psi\rangle - \langle D_A\phi, \psi\rangle = d^*\left\langle \rho(\cdot)^\flat\phi, \psi\right\rangle_{\mathbb{C}}$ where d^* is the codifferential.

Proof. The proofs of (i) and (ii) are straight forward from the definitions. For (iii) we use (2.3.15) and compute

$$\langle \phi, D_A\psi\rangle - \langle D_A\phi, \psi\rangle = \sum_i \left(\left\langle \phi, \rho(e_i^\flat)\nabla_{e_i}^A\psi \right\rangle - \left\langle \rho(e_i^\flat)\nabla_{e_i}^A\phi, \psi \right\rangle \right) = \cdots$$

$$\cdots = -\sum_i e_i \lrcorner\, \nabla_{e_i}^A \left\langle \rho(\cdot)^\flat\phi, \psi \right\rangle = d^* \left\langle \rho(\cdot)^\flat\phi, \psi \right\rangle. \quad \square$$

Corollary 2.25. *The Dirac operator D_A is a first order elliptic differential operator.*

Proof. It is clear that D_A is a first order differential operator. Its principal symbol σ_{D_A} can thus be computed using Lemma 2.24(i) as

$$\sigma_{D_A}(df)\phi = i(D(f\phi) - fD(\phi)) = i\rho(df)\phi. \tag{2.3.16}$$

where $f \in C^\infty(M;\mathbb{C})$ and $\phi \in \Gamma(S)$. We conclude that $\sigma_{D_A} = i\rho\colon T^*M \to \mathrm{End}_{\mathbb{C}}(S)$. The ellipticity of D_A follows, since $\mu(a)$ is invertible for $0 \neq a \in T^*M$. $\square$

Corollary 2.26. *If M is closed, then D_A is (formally) self-adjoint with respect to the L^2 inner product on $\Gamma(S)$. If $n = \dim(M) = 2k$ is even, then $(D_A^+)^* = D_A^-$.*

Proof. According to Lemma 2.24(iii) we have

$$\int_D \langle D_A\phi, \psi\rangle_{\mathbb{C}} - \langle \phi, D_A\psi\rangle \,\mathrm{vol}_M = \cdots = \int_M d^* \left\langle \rho(\cdot)^\flat\phi, \psi \right\rangle_{\mathbb{C}} \mathrm{vol}_M\,. \tag{2.3.17}$$

The integral on the right hand side vanishes. More generally, for all $a \in \Omega^1(M;\mathbb{C})$ we have $\int_M d^*a \,\mathrm{vol}_M = 0$. This proves the self-adjointness of D_A which immediately implies $(D_A^+)^* = D_A^-$. $\square$

It is a general fact that elliptic differential operators on closed manifolds are *Fredholm operators*, which means that they have finite dimensional kernels and cokernels. The difference of dimensions is called the *index*. In the case of the full Dirac operators D_A, the self-adjointness implies

$$\operatorname{ind}(D_A) = \dim\ker(D_A) - \dim \underbrace{\operatorname{coker}(D_A)}_{\cong \ker(D_A^*)} = 0$$

However, for $n = 2k$ even the chiral Dirac operator D_A^+ typically has non-zero index. Specializing to $n = 4$, the Atiyah–Singer index theorem gives the following topological formula (c.f. [LM89, Theorems III.13.8 and D.15]).

Theorem 2.27 (Atiyah–Singer)**.** *Let (S,ρ) be a spinor bundle over a closed, oriented 4–manifold X. Then $D_A^+ : \Gamma(S^+) \to \Gamma(S^-)$ is a Fredholm operator with index*

$$\operatorname{ind}(D_A^+) = \frac{1}{8}\big(c_1^2(S^+)[X] - \sigma(X)\big). \tag{2.3.18}$$

2.4 The Seiberg–Witten equations on 4–manifolds

Lecture 10, 13.6.23

Let us now focus our attention to a 4–manifold X. In addition to the standard assumptions, we take X to be closed and connected and fix a spin^c structure $\mathfrak{s} = [S,\rho]$ on X, whose existence is guaranteed by Proposition 2.8. In our framework, the spin^c structure is realized by a spinor bundle (S,ρ). For brevity, we henceforth write

$$q(\phi) = \rho^{-1}(\phi\phi^*)_0 \quad \text{and} \quad q(\phi,\psi) = \rho^{-1}(\psi\phi^*)_0. \tag{2.4.1}$$

We set out to study the *Seiberg–Witten equations*[1] for pairs (A,ϕ) consisting of a spin^c connection $A \in \mathcal{A}(S)$ and a positive spinor $\phi \in \Gamma(S^+)$

$$\tfrac{1}{2}F_{A^t}^+ = q(\phi) \qquad D_A^+\phi = 0. \tag{2.4.2}$$

We refer to $\frac{1}{2}F_{A^t}^+ = \rho^{-1}(\phi\phi^*)_0$ as the *monopole equation*, and to $D_A^+\phi = 0$ as the *Dirac equation*. The pair (A,ϕ) is called a *(Seiberg–Witten) configuration*. Solutions (A,ϕ) of (2.4.2) are called *monopoles*.

2.4.1 The monopole maps

As topologists, we like to think of spaces of solutions to an equations as zero sets of maps. This leads us to consider the *Seiberg–Witten map* (or *monopole map*)

$$\begin{gathered}\mathfrak{F} : \mathcal{A}(S) \times \Gamma(S^+) \to i\Omega_+^2(X) \oplus \Gamma(S^-)\\ \mathfrak{F}(A,\phi) = \big(\tfrac{1}{2}F_{A^t}^+ - q(\phi),\, D_A^+\phi\big).\end{gathered} \tag{2.4.3}$$

By fixing a spin^c connection $A_0 \in \mathcal{A}(S)$ for reference, we can use the affine structure of $\mathcal{A}(S)$ to convert $\mathfrak{F}$ into a map of vector spaces

$$\begin{gathered}\mathfrak{F}_0 : i\Omega^1(X) \oplus \Gamma(S^+) \to i\Omega_+^2(X) \oplus \Gamma(S^-)\\ \mathfrak{F}_0(a,\phi) = \big(d^+a - q(\phi) + \tfrac{1}{2}F_0^+,\, D^+\phi + \rho(a)\phi\big) = \mathfrak{F}(A_0 + a,\phi).\end{gathered} \tag{2.4.4}$$

where we have used the abbreviations $F_0 = F_{A_0^t}$ and $D = D_{A_0}$ and Lemmas 2.22 and 2.24(ii) to rewrite $F_{(A_0+a)^t}$ and D_{A_0+a}. We call $\mathfrak{F}_0$ as the *based monopole map* at $A_0 \in \mathcal{A}(S)$.

[1] The factor $\frac{1}{2}$ in the monopole equation was originally missing in the previous lectures. It has been added to stay compatible with the conventions in [KM07].

Whether one works with $\mathfrak{F}$ or $\mathfrak{F}_0$ is largely a matter of taste. The benefit of working with $\mathfrak{F}_0$ is the linear structure of the source, which comes at the price of having to make a non-canonical choice of A_0. We usually prefer to work with $\mathfrak{F}_0$, since it makes analytical features more transparent.

For brevity, we denote the sources and targets of $\mathfrak{F}$ and $\mathfrak{F}_0$ by

$$\begin{array}{c} \mathcal{C}(X,\mathfrak{s}) := \mathcal{A}(S)\times\Gamma(S^+) \xrightarrow{\ \mathfrak{F}\ } \\ \qquad\qquad\qquad\qquad i\Omega^2_+(X)\oplus\Gamma(S^-) =: \mathcal{D}(X,\mathfrak{s}) \\ \mathcal{C}_0(X,\mathfrak{s}) := i\Omega^1(X)\oplus\Gamma(S^+) \xrightarrow{\ \mathfrak{F}_0\ } \end{array} \tag{2.4.5}$$

The description of $\mathfrak{F}_0(a,\phi)$ makes it clear that $\mathfrak{F}_0$ can be written as a sum $\mathfrak{F}_0 = L + Q$ of a linear operator L and a quadratic map Q. More precisely,

$$\mathfrak{F}_0(a,\phi) = \underbrace{\left(d^+a, D^+\phi\right)}_{=:L(a,\phi)} + \underbrace{\left(\tfrac{1}{2}F_0^+ - q(\phi), \rho(a)\phi\right)}_{=:Q(a,\phi)}. \tag{2.4.6}$$

Some important structural features of $\mathfrak{F}_0$ are apparent:

(1) The source and target of $\mathfrak{F}_0$ are the sections of *mixed* vector bundles $iT^*X \oplus S^+$ and $i\Lambda^2_+X \oplus S^-$ which each have a real and a complex summand.

(2) L is an $\mathbb{R}$–linear first order differential operator.

(3) The second component D^+ of $L(a,\phi)$ is $\mathbb{C}$–linear and elliptic.

(4) The first component d^+ of $L(a,\phi)$ is *not* elliptic! We'll get back to this point.

(5) The second component of $Q(a,\phi)$ is bilinear in (a,ϕ).

(6) The first component of $Q(a,\phi)$ is *affine quadratic* in ϕ, that is, it is the sum of a constant term $\frac{1}{2}F_0^+$ and a quadratic term satisfying $-q(\lambda\phi) = -|\lambda|^2 q(\phi)$.

Since $\mathfrak{F}_0$ is clearly non-linear, the zero set $\mathfrak{F}_0^{-1}(0)$ is has no reason to be a linear space. However, $\mathfrak{F}_0$ is clearly a smooth map in a suitable sense and we might hope to exhibit $\mathfrak{F}_0^{-1}(q)$ as a type of manifold, at least if $q \in \mathcal{D}$ is a regular value of sorts. If we were really lucky, we could derive some non-trivial information from $\mathfrak{F}_0(q)$ which does not depend on the particular choice of q – much like the (mod 2) degree of a smooth map $f\colon S^n \to S^n$ can be computed by counting points in $f^{-1}(q)$ for any regular value $q \in S^n$.

Unfortunately, the infinite dimensional nature of the situation makes this a bit cumbersome. The problem is that spaces of smooth sections with their C^∞ topology are Fréchet spaces, a class of topological vector spaces that is strictly larger than Banach spaces, for which most of the analysis known from the finite dimensional context breaks down. For instance, the inverse function theorem (invertible derivative implies local diffeomorphism) is no longer available in its usual form, neither is the regular value theorem, nor are the existence and uniqueness theorem for ordinary differential equations.

However, this does not make analysis Fréchet spaces entirely impossible. There are weaker versions of the inverse function theorem in Fréchet spaces which can be used to prove interesting things such as the Nash–Moser embedding theorem. Hamilton's article [Ham82] is a good reference for these things. But it turns out that there is another way out for us.

2.4.2 A glimpse at the functional analytic setup

A common way out of the problems is to work with L^2 Sobolev spaces, a class of Hilbert spaces which Interacts particularly well with elliptic operators.

Sobolev spaces. To keep things simple, we take M to be closed and oriented. If $E \to M$ is any real or complex vector bundle equipped with a bundle metric, then we have a *real L^2–inner* product

$$(\phi, \psi) = \int_M \operatorname{Re} \langle \phi, \phi \rangle \operatorname{vol}_M \qquad (\phi, \psi \in \Gamma(E))$$

where the real part is obvious irrelevant in the real case. In the complex case, we write $(\phi, \psi)_{\mathbb{C}} = \int_M \langle \phi, \phi \rangle \operatorname{vol}_M$ for the Hermitian inner product. The L^2 *norm* is defined by

$$\|\phi\|_0^2 = (\phi, \phi) = \int_M |\phi|^2 \operatorname{vol}_M$$

Note the use of different brackets to distinguish L^2 and point-wise inner products and norms. If ∇ is any connection on E, we define the *Sobolev norms*

$$\|\phi\|_k^2 = \sum_{i=0}^{k} \left\|\nabla^k \phi\right\|_0^2 \tag{2.4.7}$$

The completion of $\Gamma(E)$ with respect to $\| \; \|_k$ is the *Sobolev space* $L_k^2(E)$. The Sobolev spaces are Banach spaces, in fact, they are Hilbert spaces. It is well known from the theory of partial differential equations, that they are particularly well-suited to study elliptic differential operators.

Sobolev completion. Now let (S, ρ) be a spinor bundle over a closed, oriented 4–manifold X representing $\mathfrak{s} \in \operatorname{Spin}_c(X)$. We fix a (smooth) spin^c connection $A_0 \in \mathcal{A}(S)$ and an integer $k \geq 3$. Recall that $\mathcal{C}_0(X, \mathfrak{s})$ and $\mathcal{D}(X, \mathfrak{s})$ are spaces of sections of vector bundles. We consider their Sobolev completions

$$\begin{aligned} \mathcal{C}_0^{(k)}(X, \mathfrak{s}) &= L_k^2(iT^*X \oplus S^+) \\ \mathcal{D}^{(k)}(X, \mathfrak{s}) &= L_k^2(i\Lambda_+^2 X \oplus S^-) \end{aligned} \tag{2.4.8}$$

Proposition 2.28. *For $k \geq 3$, the based monopole map extends to a continuous map*

$$\mathfrak{F}_0 \colon \mathcal{C}_0^{(k+1)}(X, \mathfrak{s}) \to \mathcal{D}^{(k)}(X, \mathfrak{s}). \tag{2.4.9}$$

This is a smooth map of Hilbert spaces. The derivative at $(a, \phi) \in C_0^{(k+1)}(X, \mathfrak{s})$ is given by

$$\begin{aligned} &d\mathfrak{F}_0(a, \phi) \colon \mathcal{C}_0^{(k+1)}(X, \mathfrak{s}) \to \mathcal{D}^{(k)}(X, \mathfrak{s}) \\ &d\mathfrak{F}_0(a, \phi)(b, \psi) = \big(d^+ b, D^+ \psi\big) + \big(-q(\phi, \psi) - q(\psi, \phi),\, \rho(a)\psi + \rho(b)\phi\big) \end{aligned} \tag{2.4.10}$$

Proof. The continuous extension is provided by the mapping properties of differential operators on Sobolev spaces and the Sobolev multiplication theorem. By continuity, it suffices to compute $d\mathfrak{F}(a, \phi)(b, \psi)$ for smooth configurations. Recall that $\mathfrak{F}_0 = L + Q$. The linear part L does not cause any trouble and we get

$$d\mathfrak{F}_0(a, \phi)(b, \psi) = \tfrac{d}{dt}\big|_{t=0} \mathfrak{F}_0(a + tb, \phi + t\psi) = L(b, \psi) + dQ(a, \phi)(b, \psi) \tag{2.4.11}$$

For the quadratic part, we find

$$\begin{aligned} dQ(a, \phi)(b, \psi) &= \tfrac{d}{dt}\big|_{t=0} \big(\tfrac{1}{2} F_0^+ - q(\phi + t\psi, \phi + t\psi),\, \rho(a + tb)(\phi + t\psi)\big) \\ &= \big(q(\phi, \psi) - q(\psi, \phi),\, \rho(a)\psi + \rho(b)\phi\big). \end{aligned}$$

□

Note that $d\mathfrak{F}_0(a, \phi) = L + dQ(a, \phi)$ is a linear first order differential operator with the same principal symbol as L (since $dQ(a, \phi)$ has order zero).

Hilbert manifolds and Fredholm maps. Passing to L^2 Sobolev completions makes puts us into an analytic setting that is a close to the finite dimensional situation as possible. We briefly review the definitions and theorems that are most relevant to Seiberg–Witten theory.

Definition 2.29 (Hilbert manifolds)**.** A *Hilbert manifold* $\mathcal{M}$ is a second countable Hausdorff space which is locally homeomorphic to open subsets of a separable Hilbert space.

Since all separable Hilbert spaces of infinite dimensions are isomorphic, the definition is unambigious. Moreover, the basic theorem of calculus work in Hilbert spaces and we can define smooth structures and smooth maps as in the finite dimensional setting. As in finite dimensions, we assume that all Hilbert manifolds implicitly carry a smooth structure. Tangent spaces and tangent bundles can be defined either in terms of charts or using (germs of) smooth curves. Each tangent spaces is isomorphic to the model Hilbert space, but not canonically so.

Definition 2.30 (Fredholm maps)**.** A smooth map $\mathcal{F}\colon \mathcal{M} \to \mathcal{N}$ between Hilbert manifolds is called a *Fredholm map* if its derivative

$$d\mathcal{F}(p)\colon T_p\mathcal{M} \to T_{\mathcal{F}(p)}\mathcal{N} \tag{2.4.12}$$

is a Fredholm operator for each $p \in \mathcal{M}$, that is, $d\mathcal{F}(p)$ has closed range and finite dimensional kernel and cokernel.

Critical points and regular values are defined just as in finite dimensions. We have the following version of the regular value theorem.

Theorem 2.31 (Regular value theorem)**.** *Let $\mathcal{F}\colon \mathcal{M} \to \mathcal{N}$ be a smooth map between connected Hilbert manifolds. If $q \in \mathcal{N}$ is a regular value, then $\mathcal{F}^{-1}(q)$ is a smooth Hilbert submanifold of $\mathcal{M}$. Its tangent spaces are canonically identified as*

$$T_p\mathcal{F}^{-1}(q) \cong \ker d\mathcal{F}(p), \quad p \in \mathcal{F}^{-1}(q).$$

If $\mathcal{F}$ is a Fredholm map, then $\mathcal{F}^{-1}(q)$ has finite dimension

$$\dim \mathcal{F}^{-1}(q) = \operatorname{ind}_{\mathbb{R}} d\mathcal{F}(p). \tag{2.4.13}$$

For Fredholm maps, there is also a version of Sard's theorem. Recall that a *Baire set* is a set that can be written as the countable intersection of dense open subsets. It is known that every Hilbert manifold has the *Baire property* that every Baire set is dense.

Theorem 2.32 (Sard–Smale theorem)**.** *Let $\mathcal{F}\colon \mathcal{M} \to \mathcal{N}$ be a Fredholm map between Hilbert manifolds. Then the set of regular values is a Baire set and, in particular, dense in $\mathcal{N}$.*

2.4.3 The gauge group action

The Seiberg–Witten equations have a large symmetry group. This is feature, not a bug, and will eventually lead us back to a $\mathbb{T}$–equivariant topology.

The gauge group. For the moment, we consider a spinor bundle (S, ρ) over a general n–manifold M again. The natural symmetry group of (S, ρ) is the multiplicative sub-group

$$U_\rho(S) = \{U\colon S \to S \,|\, U \text{ is unitary and Clifford linear}\} \subset \Gamma(\mathrm{End}_{\mathbb{C}}(S)). \tag{2.4.14}$$

As it turns out, $U_\rho(S)$ is independent of (S, ρ). Indeed, we know from Lemma 2.6 that every Clifford linear $U \in \mathrm{End}_{\mathbb{C}}(S)$ is given by multiplication with a complex valued function $u\colon M \to \mathbb{C}$, and if U is also unitary, we must have $u\colon M \to \mathbb{T}$. We can therefore identify $U_\rho(S)$ with the group

$$\mathcal{G}(M) = C^\infty(M, \mathbb{T}). \tag{2.4.15}$$

We refer to $\mathcal{G}(M)$ as the *gauge group* of M. The point-wise multiplication and inversion are continuous in the C^∞ topology which makes $\mathcal{G}(X)$ a Fréchet Lie group with Lie algebra

$$\operatorname{Lie}\mathcal{G}(X) \cong \Omega^0(M; i\mathbb{R}). \tag{2.4.16}$$

The gauge group $\mathcal{G}(M)$ has canonical actions on $\Gamma(S)$ (and $\Gamma(S^\pm)$ for n even) by fiberwise scalar multiplication, and also on $\mathcal{A}(S)$ by conjugating covariant derivatives with the action on $\Gamma(S)$ (see Exercise 8.1). The action of $u \in \mathcal{G}(M)$ on $A \in \mathcal{A}(S)$ can be understood rather explicitly in terms of the affine structure of $\mathcal{A}(S)$ as

$$uA = A - u^{-1}du \in \mathcal{A}(S). \tag{2.4.17}$$

Here we think of $\mathcal{G}(M)$ as a subset of $\Omega^0(M;\mathbb{C})$ to form $du \in \Omega^1(M;\mathbb{C})$ and u^{-1} indicates point-wise inversion in $\mathbb{T}$. To justify (2.4.17), we have to argue that $u^{-1}du \in i\Omega^1(M)$ which follows from the computation

$$\overline{u^{-1}du} = \overline{u^{-1}}d\bar{u} = ud(u^{-1}) = -uu^{-2}du = -u^{-1}du. \tag{2.4.18}$$

For later reference, we note that a similar calculation shows that $u^{-1}du$ is always closed:

$$d(u^{-1}du) = d(u^{-1}) \wedge du = -u^{-2}du \wedge du = 0. \tag{2.4.19}$$

Lastly, we let $\mathcal{G}(M)$ act on forms $\omega \in \Omega^*(M;\mathbb{C})$ of mixed degree via

$$u \cdot \omega = \omega - u^{-1}du. \tag{2.4.20}$$

Note that the action is trivial on $\Omega^k(M;\mathbb{C})$ for $k \neq 1$. Lecture 11, 20.6.23

We will need to understand the action of $\mathcal{G}(M)$ on $\mathcal{A}(S) \times \Gamma(S)$ in some more detail.

Lemma 2.33. *Let (S,ρ) be a spinor bundle over M.*

(i) The $\mathcal{G}(M)$–action on $\Gamma(S)$ is free away from $0 \in \Gamma(S)$ which is a fixed point. The action is $\mathbb{C}$–linear and unitary with respect to the Hermitian L^2 inner product.

(ii) The $\mathcal{G}(M)$–action on $\mathcal{A}(S)$ has constant stabilizers

$$\mathcal{G}(M)_{(A,\phi)} = \mathcal{G}^c(M)$$

where $\mathcal{G}^c(M) \subset \mathcal{G}(M)$ is the subgroup of locally constant maps $M \to \mathbb{T}$. In particular, if M is connected, then $\mathcal{G}(M)_{(A,\phi)} \cong \mathbb{T}$.

Proof. For (ii) note that $u\colon M \to \mathbb{T}$ is locally constant iff $du = 0$ iff $u^{-1}du = 0$. If M is connected, then any such u is constant. (i) is obvious. □

The following terminology is commonly used in the literature on Seiberg–Witten theory (and, more generally, gauge theory).

Definition 2.34 (Reducible/irreducible)**.** A configuration $(A,\phi) \in \mathcal{A}(S) \times \Gamma(S)$ is called *irreducible* if $\phi \neq 0$. Configurations of the form $(A,0)$ are called *reducible.*

As an immediate consequence of Lemma 2.33, we get:

Corollary 2.35. *The diagonal $\mathcal{G}(M)$ action on $\mathcal{A}(S) \times \Gamma(S)$ is free away from the reducible configurations $(A,0)$ each of which has stabilizer $\mathcal{G}^c(M)$.*

Remark 2.36. For technical reasons, it is also necessary to introduce Sobolev completions of the gauge group $\mathcal{G}(M)$. By the Sobolev embedding and multiplication theorems, for $2(k+1) > n$ the Sobolev space $L^2_{k+1}(M,\mathbb{C})$ consists of continuous functions and is a Banach algebra with respect to pointwise multiplication. We define

$$\mathcal{G}^{(k+1)}(M) = \left\{ u \in L^2_{k+1}(M,\mathbb{C}) \;\middle|\; |u(x)| = 1\, \forall x \in M \right\}$$

and note that this is a Hilbert Lie group which acts smoothly on $L^2_k(\Lambda^*_{\mathbb{C}}M)$ and $L^2_k(S)$.

The monopole maps and the gauge group action. In the 4-dimensional setting, the gauge groups acts on the sources and targets of the monopole maps.

Lemma 2.37. *Let (S, ρ) be a spinor bundle over a 4–manifold X.*

(i) The monopole maps $\mathfrak{F}\colon \mathcal{C}(X, \mathfrak{s}) \to \mathcal{D}(X, \mathfrak{s})$ is $\mathcal{G}(X)$–equivariant.

(ii) The preimages $\mathfrak{F}^{-1}(\eta, 0)$ with $\eta \in i\Omega^2_+(X)$ are $\mathcal{G}(X)$–invariant.

The same statements hold for $\mathfrak{F}_0\colon \mathcal{C}_0(X, \mathfrak{s}) \to \mathcal{D}(X, \mathfrak{s})$ and $\mathfrak{F}_0^{-1}(\eta, 0)$ and its Sobolev completions $\mathcal{C}_0^{(k+1)}(X, \mathfrak{s}) \to \mathcal{D}^{(k)}(X, \mathfrak{s})$ with the action of $\mathcal{G}^{(k+2)}(X)$.

Proof. We focus on $\mathfrak{F}$, since the arguments for $\mathfrak{F}_0$ are analogous. We first note that (ii) follows from (i) and the observation that $(\eta, 0) \in \mathcal{D}(X, \mathfrak{s})$ is $\mathcal{G}(X)$–fixed. For (i) we have to show that

$$\mathfrak{F}(uA, u\phi) = u\mathfrak{F}(A, \phi) = \big(\tfrac{1}{2}F_{A^t} - q(\phi),\, uD_A\phi\big) \tag{2.4.21}$$

We first note that

$$(u\phi)(u\phi)^* = u\bar{u}(\phi\phi^*) = \phi\phi^* \tag{2.4.22}$$

which implies $q(u\phi) = q(\phi)$. Next, recall that $d^+ = P^+d$ where $P^+ = \frac{1}{2}(\mathrm{id} + *)$. Now (2.4.18) gives

$$d^+(u^{-1}du) = \frac{1}{2}(* + \mathrm{id})d(u^{-1}du) = 0. \tag{2.4.23}$$

From (2.4.17) and (2.3.12) we get

$$F^+_{uA^t} = F_{A^t} - 2d^+(u^{-1}du) = F^+_{A^t} \tag{2.4.24}$$

and thus

$$\mathfrak{F}(uA, u\phi) = \big(\tfrac{1}{2}F_{A^t} - q(\phi),\, D_{uA}(u\phi)\big). \tag{2.4.25}$$

Finally, using Lemma 2.24 we get

$$D_{uA}(u\phi) = uD_{uA}\phi + \rho(du)\phi = u(D_A\phi - \rho(u^{-1}du)\phi) + \rho(du)\phi = uD_A\phi. \qquad \square$$

Loosely following [KM07, Def. 1.3.1], we define

$$N_\eta(X, \mathfrak{s}) := \mathfrak{F}^{-1}(2\eta, 0)/\mathcal{G}(X) \subset \mathcal{C}(X, \mathfrak{s})/\mathcal{G}(X) =: \mathcal{B}(X, \mathfrak{s}) \tag{2.4.26}$$

and refer to $N_\eta(X, \mathfrak{s})$ as the *monopole moduli space* with *perturbation* $\eta \in i\Omega^2_+(X)$. The following theorem summarizes the most important properties of these spaces.

Theorem 2.38 (c.f. [KM07, Theorem 1.4.4])**.** *Let X be a closed 4–manifold with $b_2^+(X) \geq 1$. There is a dense set of forms $\eta \in i\Omega^2_+(X)$ for which $N_\eta(X, \mathfrak{s})$ is a compact, orientable manifold without boundary of dimension*

$$\begin{aligned} \dim N_\eta(X, \mathfrak{s}) &= (b_1(X) - b_2^+(X) - 1) + 2\,\mathrm{ind}_{\mathbb{C}}(D_A^+) \\ &= \frac{1}{4}\big(c_1^2(S^+)[X] - 2\chi(X) - 3\sigma(X)\big) \end{aligned} \tag{2.4.27}$$

We will not prove the entire result, but only indicate how it comes together.

Coulomb gauge fixing. As mentioned earlier, the linear part of the Seiberg–Witten equations is not elliptic. This can be remedied with the help of the gauge group. Since the arguments are not specific to dimension 4, we consider a general manifold M which we assume to be closed.

Definition 2.39. Let (S, ρ) be a spinor bundle over M and $\mathcal{A}_0 \in \mathcal{A}(S)$. We say that $A = A_0 + a \in \mathcal{A}(S)$ is in *Coulomb gauge* with respect to A_0 if it satisfies the *Coulomb condition* $d^*a = 0$.

Lemma 2.40. *Let (S,ρ) be a spinor bundle over a closed manifold M and $A_0 \in \mathcal{A}(S)$ a fixed spin^c connection. For every $A = A_0 + a \in \mathcal{A}(S)$ we can find $u \in \mathcal{G}(M)$ such that*

$$d^*(a - u^{-1}du) = 0. \tag{2.4.28}$$

In other words, every spin^c connection can be put into Coulomb gauge with respect to A_0.

Proof. We try to find u of the form $u = e^f$ for some $f \in i\Omega^0(M)$. We compute

$$u^{-1}du = e^{-f}de^f = e^{-f}e^f df = df \tag{2.4.29}$$

and note that the equation (2.4.28) becomes

$$\Delta f = d^*df = d^*a. \tag{2.4.30}$$

This is a special case of the Poisson equation which can be solved using the Hodge decomposition. □

Clearly, if A is already in Coulomb gauge with respect to A_0, then $uA = A - u^{-1}du$ is in Coulomb gauge if and only if $u \in \mathcal{G}(M)$ satisfies $d^*(u^{-1}du) = 0$. In this case, we call u *harmonic* and define the *harmonic gauge group* as

$$\mathcal{G}^h(M) = \left\{ u \in \mathcal{G}(M) \,\middle|\, d^*(u^{-1}du) = 0 \right\}. \tag{2.4.31}$$

2.4.4 The Seiberg–Witten–Coulomb system

Now let (S,ρ) be a spinor bundle over a closed 4–manifold X again. We say that a configuration $(A,\phi) \in \mathcal{C}(X,\mathfrak{s})$ is in Coulomb gauge with respect to $A_0 \in \mathcal{A}(S)$ if $A = A_0 + a$ with $d^*a = 0$, that is, if A is in Coulomb gauge. According to Lemma 2.40, we can find a gauge transformation of the form $u = e^f$ such that $(uA, u\phi)$ is in Coulomb gauge. Since the Seiberg–Witten equations are gauge invariant by Lemma 2.37, every gauge equivalence class of monopoles has representatives which solve the *Seiberg–Witten–Coulomb system*

$$d^+a - q(\phi) + \tfrac{1}{2}F_0^+ = 0 \qquad D_A\phi = 0 \qquad d^*a = 0. \tag{2.4.32}$$

where the first equation is just the monopole equation $\frac{1}{2}F_{A^t}^+ = q(\phi)$ rewritten in terms of a. Adding the Coulomb condition $d^*a = 0$ effectively reduces the symmetry of the equations from the infinite dimensional gauge group $\mathcal{G}(X)$ to the finite dimensional harmonic gauge group $\mathcal{G}^h(X)$. In addition, it also takes care of the failure of d^+ to be elliptic.

Lemma 2.41. *The operator $d^* + d^+ \colon \Omega^1(X) \to \Omega^0(X) \oplus \Omega^2_+(X)$ is elliptic. If X is closed, then $d^* + d^+$ is Fredholm with index*

$$\operatorname{ind}_{\mathbb{R}}(d^* + d^+) = b_1(X) - b_2^+(X) - b_0(X). \tag{2.4.33}$$

Proof. (1) The symbol of $d^* + d^+$ is readily computed as

$$\sigma_{d^*+d^+}(\xi)a = -\xi^\sharp \lrcorner a + P^+(\xi \wedge a) = P^+(\xi \wedge a) - \langle a, \xi\rangle \tag{2.4.34}$$

Since $\Lambda^1 X = T^*X$ and $\Lambda^0 X \oplus \Lambda^2_+ X$ both have rank 4, it suffices that $\sigma_{d^*+d^+}(\xi)$ is injective for $\xi \neq 0$.

Suppose that $a, \xi \in T_x^*X$ are both non-zero with $\langle a, \xi\rangle = 0$. We may assume that $|a| = |\xi| = 1$ and extend a, ξ to an orthonormal basis of T_x^*X. Then $P^+(\xi \wedge a)$ is part of an orthonormal basis of $\Lambda^2_+ T_x^*X$ and, in particular, non-zero. In particular, we $\sigma_{d^*+d^+}(\xi)a \neq 0$.

(2) The kernel of $d^* + d^+$ can be determined explicitly. We have

$$(d^* + d^+)a = 0 \quad \Leftrightarrow \quad d^+a = 0,\, d^*a = 0 \quad \Leftrightarrow \quad da = 0,\, d^*a = 0 \tag{2.4.35}$$

The first equivalence is obvious. For the second, note that $d^+a = 0$ trivially implies

$$0 = 2d^*d^+a = d^*(da + *da) = d^*da + *d*^2\, da = d^*da. \tag{2.4.36}$$

Since M is closed, d^* is the L^2 adjoint of d and we get

$$0 = (a, d^*da)_0 = \|da\|_0\,. \tag{2.4.37}$$

Now the Hodge and de Rham theorems give

$$\ker(d^* + d^+) = \mathcal{H}^1(X) \cong H^1(X;\mathbb{R}).$$

where $\mathcal{H}^k(X)$ is the space of harmonic k–forms.

(3) The cokernel of $d^* + d^+$ is isomorphic to the kernel of the adjoint $(d^* + d^+)^* = (d^+)^* + d$. We claim that

$$(d^* + d^+)^* = d^* + d\colon \Omega^2_+(X) \oplus \Omega^0(X) \to \Omega^1(X). \tag{2.4.38}$$

Obviously, we have $d^{**} = d$ and $(d^+)^* = d^*$ follows from the identity Now, for $\eta \in \Omega^2_+(X)$ and $a \in \Omega^1(X)$ we have

$$(d^+a, \eta) = (da, \eta) = (a, d^*\eta), \quad a \in \Omega^1(X),\, \eta \in \Omega^2_+(X). \tag{2.4.39}$$

If we add $f \in \Omega^0(X)$ to the mix, we get $(d^* + d^+)(\eta, f) = d^*\eta + df$ and, since d^2, the summands are orthogonal and we get

$$d^*\eta + df = 0 \quad \Leftrightarrow \quad d^*\eta = 0,\, df = 0. \tag{2.4.40}$$

Lastly, for $\eta \in \Omega^2_+(X)$ we have $d^*\eta = 0$ iff $d\eta = 0$. Altogether, we find

$$\operatorname{coker}(d^* + d^+) \cong \mathcal{H}^2_+(X) \oplus \mathcal{H}^0(X) \tag{2.4.41}$$

Where $\mathcal{H}^2_+(X)$ is space of self-dual harmonic 2-forms. Again, Hodge-de Rham theory shows that $\mathcal{H}^2_+(X)$ has dimension $b_2^+(X)$.

□

Lecture 12, 27.6.23

The L^2 orthogonal complement of the constant functions in $\Omega^0(X)$ consists of those functions that integrate to zero on each component of X. Denote this space by $\Omega^0_0(X)$. The Hodge decomposition gives another description:

$$\Omega^0_0(X) = d^*\Omega^1(X). \tag{2.4.42}$$

Now $d^* + d^+$ naturally maps into $\Omega^0_0(X) \oplus \Omega^2_+(X)$. Replacing $\Omega^0(X)$ with $\Omega^0_0(X)$ in the codomain of $d^* + d^+$ removes $\mathcal{H}^0(X)$ from the kernel of the adjoint. The result is a Fredholm operator

$$d^*_0 + d^+\colon \Omega^1(X) \to \Omega^0_0(X) \oplus \Omega^2_+(X), \quad a \mapsto (d^*a, d^+a). \tag{2.4.43}$$

whose index is given by

$$\operatorname{ind}_{\mathbb{R}}(d^*_0 + d^+) = b_1(X) - b_2^+(X). \tag{2.4.44}$$

We now proceed as with the standard Seiberg–Witten equations and consider the map

$$\begin{aligned} \widetilde{\mathfrak{F}}_0\colon \underbrace{i\Omega^1(X) \oplus \Gamma(S^+)}_{\mathcal{C}_0(X,\mathfrak{s})} &\to \underbrace{i\Omega^0_0(X) \oplus \Omega^2_+(X) \oplus \Gamma(S^-)}_{=:\widetilde{\mathcal{D}}(X,\mathfrak{s})} \\ \widetilde{\mathfrak{F}}_0(a, \phi) &= \big(d^*a, d^+a - q(\phi) + \tfrac{1}{2}F_0^t, D\phi + \rho(a)\phi\big). \end{aligned} \tag{2.4.45}$$

Here we choose $i\Omega^0_0(X)$ over $i\Omega^0(X)$ in order to give $\widetilde{\mathfrak{F}}_0$ a chance to have regular values. Indeed, the derivative is given by

$$d\widetilde{\mathfrak{F}}_0(a,\phi)(b,\psi) = \big(d^*b, d\mathfrak{F}_0(s,\phi)(b,c)\big). \tag{2.4.46}$$

If we worked with $i\Omega^0(X)$, the first component could never be surjective.

Proposition 2.42. *For every integer $k \geq 3$ the map $\widetilde{\mathfrak{F}}_0$ extends continuously to a smooth Fredholm map*

$$\widetilde{\mathfrak{F}}_0 \colon \mathcal{C}_0^{(k+1)}(X,\mathfrak{s}) \to \widetilde{\mathcal{D}}^{(k)}(X,\mathfrak{s}). \tag{2.4.47}$$

If $q = (f,\eta,\psi) \in \widetilde{\mathcal{D}}(X,\mathfrak{s})$ is a regular value, then $\widetilde{\mathfrak{F}}_0^{-1}(q)$ is a smooth manifold of finite dimension

$$\begin{aligned} \dim \widetilde{\mathfrak{F}}_0^{-1}(q) &= \operatorname{ind}_{\mathbb{R}}(d_0^* + d^+) + 2\operatorname{ind}_{\mathbb{C}}(D^+) \\ &= b_1(X) - b_2^+(X) + \frac{1}{4}\big(c_1(S^+)^2[X] - \sigma(X)\big) \\ &= \frac{1}{4}\big(c_1(S^+)^2[X] - 2\chi(X) - 3\sigma(X)\big) + b_0(X) \end{aligned} \tag{2.4.48}$$

Proof. The continuous extension is obvious. Note that $d\widetilde{\mathfrak{F}}_0(a,\phi)$ has the same principal symbol as $\widetilde{L}(a,\phi) = (d^*a_0, d^+a, D\phi)$ which is elliptic and therefore Fredholm. According to Theorem 2.31, $\widetilde{\mathfrak{F}}_0^{-1}(q)$ is a smooth manifold of dimension

$$\begin{aligned} \dim \widetilde{\mathfrak{F}}_0^{-1}(q) &= \operatorname{ind}_{\mathbb{R}}(\widetilde{L}) = \operatorname{ind}_{\mathbb{R}}(d_0^* + d^+) + 2\operatorname{ind}_{\mathbb{C}}(D^+) \\ &= b_1(X) - b_2^+(X) + \frac{1}{4}\big(c_1(S^+)^2[X] - \sigma(X)\big). \end{aligned} \tag{2.4.49}$$

The last equality follows from Theorem 2.27 and Lemma 2.41. Rearranging the terms using

$$\begin{aligned} \chi(X) &= b_2^+(X) + b_2^-(X) - 2b_1(X) + 2b_0(X) \\ \sigma(X) &= b_2^+(X) - b_2^-(X) \end{aligned} \tag{2.4.50}$$

gives the desired formula. □

Given $\eta \in L^2_k(i\Lambda^2_+)$, we obtain a $\mathcal{G}^h(X)$–invariant subspace

$$\widetilde{N}_\eta^{(k)}(X,\mathfrak{s}) = \widetilde{\mathfrak{F}}_0^{-1}(0,2\eta,0) = \left\{(a,\phi) \in \widetilde{\mathcal{C}}_0^{(k+1)}(X,\mathfrak{s}) \,\middle|\, \widetilde{\mathfrak{F}}_0(a,\phi) = (0,\eta,0)\right\}. \tag{2.4.51}$$

We want to compare these spaces with the moduli spaces $N_\eta(X,\mathfrak{s})$ defined in (5.4.1) for smooth η. The first thing to note is that the apparent dependence on k is not really there.

Theorem 2.43 (Regularity)**.** *If $\eta \in i\Omega^2_+(X)$ is a smooth form, then $\widetilde{N}_\eta^{(k)}(X,\mathfrak{s})$ is independent of $k \geq 3$ and consists of smooth configurations. In that case we simply write $\widetilde{N}_\eta(X,\mathfrak{s})$.*

Proof (sketch). This essentially follows from the ellipticity of the operators $d^* + d^+$ and D^+ by a technique known as "elliptic bootstrapping". The basic idea is to write the defining equations for $\widetilde{N}_\eta(X,\mathfrak{s})$ as

$$(d^* + d^+)a = \big(0, -\tfrac{1}{2}F_0^t + 2\eta + q(\phi)\big) \tag{2.4.52}$$

$$D^+\phi = -\rho(a)\phi. \tag{2.4.53}$$

The ellitptic regularity theorem says if u is a weak (distributional) solution of $Pu = v$ where P is a linear elliptic differential operator of order ℓ over a closed manifold and $v \in L^2_k$, then u is an $L^2_{k+\ell}$ section.

We can use this to argue inductively that

$$(a,\phi) \in L^2_{k+1} \quad \Longrightarrow \quad (a,\phi) \in L^2_{k+2} \quad \forall k \geq 3. \tag{2.4.54}$$

Indeed, the Sobolev multiplication theorem gives $\rho(a)\phi \in L^2_{k+1}$ and elliptic regularity for D^+ implies $\phi \in L^2_{k+2}$. Another application of the Sobolev multipliation theorem gives $q(\phi) \in L^2_{k+2}$ and elliptic regularity for $d^* + d^+$ shows $a \in L^2_{k+2}$. Repeating this argument indefinitely we can conclude $(a,\phi) \in C^\infty$ using the Sobolev embedding theorem.

This shows that the inclusion $\widetilde{N}^{(k+1)}_\eta(X,\mathfrak{s}) \hookrightarrow \widetilde{N}^{(k)}_\eta(X,\mathfrak{s})$ is a continuous bijection for $k \geq 3$. The continuity of the inverse follows from the Rellich lemma which states that that the inclusion $L^2_{k+2} \hookrightarrow L^2_{k+1}$ is a compact map. □

While Theorem 2.43 is concerned with the regularity of elements of $\widetilde{N}_\eta(X,\mathfrak{s})$, we next address the regularity of $\widetilde{N}_\eta(X,\mathfrak{s})$ as a space. This issue is often referred to as *transversality* in this context.

Definition 2.44 (Regular perturbations)**.** We say that $\eta \in i\Omega^2_+(X)$ is *regular* if $(0, 2\eta, 0)$ is a regular value of $\widetilde{\mathfrak{F}}_0 \colon \mathcal{C}^{(k+1)}_0(X,\mathfrak{s}) \to \widetilde{\mathcal{D}}^{(k)}(X,\mathfrak{s})$ for all $k \geq 3$.

Recall that a *Baire set* is a set that can be written as the countable intersection of dense open subsets and that every Fréchet space, such as $i\Omega^+_2(X)$, has the *Baire property* that every Baire set is dense.

Theorem 2.45 (Transversality)**.** *The set of regular $\eta \in i\Omega^2_+(X)$ is a Baire set and, in particular, dense in $i\Omega^2_+(X)$. If η is regular, then $\widetilde{N}_\eta(X,\mathfrak{s})$ is a finite dimensional smooth manifold on which $\mathcal{G}^h(X)$ acts smoothly. The dimension is given by (2.4.49).*

Proof (sketch). The manifold properties of $\widetilde{N}_\eta(X,\mathfrak{s})$ and smoothness of the $\mathcal{G}^h(X)$–action are immediate from the definitions and Proposition 2.42. The abundance of regular $\eta \in i\Omega^2_+(X)$ essentially follows from the Sard–Smale theorem (Theorem 2.32), with the caveat that we are looking for regular values that live in a subspace of infinite codimension. We outline the proof given in [Sal99, Chs. 7.2 & 8.4].

(1) The first step is to show that zero is a regular value of the map

$$\mathcal{C}^{(k+1)}_0(X,\mathfrak{s}) \to iL^2_k\Omega^0_0(X) \oplus L^2_k(S^-), \quad (a,\phi) \mapsto (d^*a, D^+\phi + \rho(a)\phi) \tag{2.4.55}$$

for $k \geq 3$ where $L^2_k\Omega^0_0(X)$ is the L^2_k completion of $\Omega^0_0(X)$.

(2) The zero set $\mathcal{Z}$ of (2.4.55) is then a smooth Hilbert submanifold of $\mathcal{C}^{(k+1)}_0(X,\mathfrak{s})$ and

$$\mathcal{Z} \to L^2_k(i\Lambda^2_+X), \quad (a,\phi) \mapsto d^+a - q(\phi) + \tfrac{1}{2}F^+_0 \tag{2.4.56}$$

is easily shown to be a Fredholm map. The preimage of $2\eta \in iL^2_k(\Lambda^2_+X)$ coincides with $N^{(k)}_\eta(X,\mathfrak{s})$ and 2η is a regular value of (2.4.56) if and only if $(0,2\eta,0)$ is a regular value of the relevant completion of $\widetilde{\mathfrak{F}}_0$.

(3) The Sard–Smale Theorem 2.32 gives a Baire set of regular values in $iL^2_k(\Lambda^2_+X)$ for each $k \geq 3$. The intersection in $iL^2(\Lambda^2_+X)$ is contained in $i\Omega^2_+(X)$ by the Sobolev embedding theorem and consists of regular elements (that is, regular values for all Sobolev completions). One can further show that it is dense in the C^∞ topology and also in $iL^2_k(\Lambda^2_+X)$ for all $k \geq 3$. □

Lastly, we address the regularity of the orbit space $\widetilde{N}_\eta(X,\mathfrak{s})/\mathcal{G}^h(X)$ and its relation to $N_\eta(X,\mathfrak{s})$. The latter is rather straight forward.

Lemma 2.46. *For regular $\eta \in i\Omega^2_+(X)$ there is a canonical homeomorphism*

$$\widetilde{N}_\eta(X,\mathfrak{s})/\mathcal{G}^h(X) \xrightarrow{\approx} N_\eta(X,\mathfrak{s}) \tag{2.4.57}$$

induced by the embedding $\widetilde{\mathfrak{F}}_0^{-1}(0,2\eta,0) \hookrightarrow \mathfrak{F}_0^{-1}(2\eta,0)$ that sends (a,ϕ) to $(A+a_0,\phi)$.

The lemma exhibits $N_\eta(X,\mathfrak{s})$ as the quotient of a finite dimensional smooth manifold by a smooth $\mathcal{G}^h(X)$–action. If the action was free and proper, this would give $N_\eta(X,\mathfrak{s})$ a natural smooth manifold structure for which (2.4.57) is a diffeomorphism. Properness follows from more general compactness theorems (c.f. [KM07, Theorem 5.2.1]).

Theorem 2.47 (Properness)**.** *The $\mathcal{G}^h(X)$–action on $\widetilde{N}_\eta(X,\mathfrak{s})$ is proper.*

However, we know from Lemma 2.33 that the action is only free away from the reducible configurations $(a,0)$ which have stabilizer $\mathcal{G}^c(X)$. As it turns out, it is possible to avoid reducible configurations in reasonably many situations.

Lemma 2.48 (Avoiding reducibles)**.** *If $b_2^+(X) \geq 1$, then the set of regular $\eta \in i\Omega^2_+(X)$ for which $\widetilde{N}_\eta(X,\mathfrak{s})$ does not contain reducible configurations is dense in $i\Omega^2_+(X)$. In that case, $N_\eta(X,\mathfrak{s})$ is an orientable smooth manifold of dimension*

$$\begin{aligned}\dim N_\eta(X,\mathfrak{s}) &= \dim \widetilde{N}_\eta(X,\mathfrak{s}) - \dim \mathcal{G}^h(X) \\ &= \frac{1}{4}\big(c_1(S^+)^2[X] - 2\chi(X) - 3\sigma(X)\big).\end{aligned} \tag{2.4.58}$$

Proof. (1) The reducible elements $(a,0) \in \widetilde{N}_\eta(X,\mathfrak{s})$ are the solutions of the equation

$$d^*a = 0, \quad \frac{1}{2}F_0^+ + d^+a = 2\eta. \tag{2.4.59}$$

Put differently, $N_\eta(X,\mathfrak{s})$ contains reducible elements iff $\eta = \frac{1}{4}F_0^+ + \frac{1}{2}d^+a$.

(2) Hodge theory shows that the set of η for which $N_\eta(X,\mathfrak{s})$ contains reducibles is an affine subspace of codimension $b_2^+(X)$.

(3) If $b_2^+(X) \geq 1$, then the complement is open and dense and its intersection with the set of regular perturbations is a Baire set.

□

Remark 2.49 (Orientability). One can also show that $\widetilde{N}_\eta(X,\mathfrak{s})$ is orientable for regular η. One can show that orientations correspond to orientations of the vector space $\mathcal{H}^1(X)\oplus\mathcal{H}^2_+(X)$ (c.f. [Sal99, Propisition 7.20]). Moreover, $\mathcal{G}^h(X)$ acts by orientation preserving diffeomorphisms so that $N_\eta(X,\mathfrak{s})$ is also orientable in case it is free of reducibles.

At this point, we should remind ourselves that we were hoping to find topological information in the about the pair $(X,\mathfrak{s})$ in the spaces $N_\eta(X,\mathfrak{s})$. A priori, these spaces depend explicitly on the choice of η and implicitly on the Riemannian metric g on X and the reference connection A_0. Let $\gamma = (g_t,\eta_t,A_t)_{t\in[0,1]}$ be a smooth path of Riemannian metrics g_t together with perturbations $\eta_t \in i\Omega^2_+(X,g_t)$ and spin^c connections on $A_t \in \mathcal{A}(S,\rho_t)$. Note that the notion of self-duality changes along the path of metric, and so does the Clifford multiplication on S and thus the entire Seiberg–Witten map. We consider the *parameterized moduli space*

$$\begin{aligned}\widetilde{W}_\gamma(X,\mathfrak{s}) &= \Big\{(a,\phi,t) \in \mathcal{C}_0(X,\mathfrak{s}) \times [0,1] \,\Big|\, \widetilde{\mathfrak{F}}_0(a,\phi) = (0,2\eta_t,0)\Big\} \\ &= \bigcup_{t\in[0,1]} \widetilde{N}_t(X,\mathfrak{s}) \times \{t\} \subset \mathcal{C}_0(X,\mathfrak{s}) \times [0,1]\end{aligned} \tag{2.4.60}$$

where $\widetilde{N}_t(X,\mathfrak{s})$ is the extended moduli space for the triple (g_t,η_t,A_t). Similarly, let $N_t(X,\mathfrak{s}) \subset \mathcal{B}(X,\mathfrak{s})$ be the moduli space for the pair (g_t,η_t)

Theorem 2.50 (Cobordism)**.** *There is a Baire set of paths γ such that $\widetilde{W}_\gamma(X,\mathfrak{s})$ is a smooth $\mathcal{G}^h(X)$–manifold with boundary*

$$\partial\widetilde{W}_\gamma(X,\mathfrak{s}) \cong \widetilde{N}_1(X,\mathfrak{s}) \amalg \widetilde{N}_0(X,\mathfrak{s}). \tag{2.4.61}$$

The orbit space $W_\gamma(X,\mathfrak{s}) = \widetilde{W}_\gamma(X,\mathfrak{s})/\mathcal{G}^h(X)$ is compact and for $b_2^+(X) \geq 2$ there is a dense set of pairs γ for which $\widetilde{W}_\gamma(X,\mathfrak{s})$ is free of reducibles. In that case, the $W_\gamma(X,\mathfrak{s})$ is a cobordism from $N_0(X,\mathfrak{s})$ to $N_1(X,\mathfrak{s})$. Furthermore, once an orientation on $\mathcal{H}^1(X)\oplus\mathcal{H}^2_+(X)$ is fixed, the cobordisms $\widetilde{W}_\gamma(X,\mathfrak{s})$ and $W_\gamma(X,\mathfrak{s})$ have natural orientations.

2.4.5 Seiberg–Witten invariants of closed 4–manifolds

Lecture 13, 4.7.23

As before, let $(X,\mathfrak{s})$ be a closed spin^c 4–manifold. In addition to the implicit orientation and Riemannian metric on X, we also fix an orientation μ_X of the real vector space $\mathcal{H}^1(X)\oplus\mathcal{H}^2_+(X)$; this datum is usually called a *homology orientation* of X. We also assume that $b_2^+(X)\geq 2$. Recall that

$$\mathcal{C}^*(X,\mathfrak{s}) = \{(A,\phi)\in\mathcal{C}(X,\mathfrak{s}) \mid \Phi\neq 0\} \quad \text{and} \quad \mathcal{B}^*(X,\mathfrak{s}) = \mathcal{C}^*(X,\mathfrak{s})/\mathcal{G}(X) \tag{2.4.62}$$

denote the spaces of irreducible Seiberg–Witten configurations and gauge equivalence classes thereof. It follows from Theorems 2.45 and 2.50 that there is a well-defined homology class

$$[N_\eta(X,\mathfrak{s})] \in H_*\big(\mathcal{B}^*(X,\mathfrak{s});\mathbb{Z}\big) \tag{2.4.63}$$

where $\eta\in i\Omega^2_+(X)$ is any regular perturbation. In essence, this *is* the Seiberg–Witten invariant of $(X,\mathfrak{s})$. However, the following definition is more common:

Definition 2.51 (Seiberg–Witten invariants)**.** Let $(X,\mathfrak{s})$ be a closed spin^c 4–manifold with $b_2^+(X)$ equipped with homology orientation. The *Seiberg–Witten invariant* of $(X,\mathfrak{s})$ is the map

$$\mathfrak{m}(\cdot|X,\mathfrak{s})\colon H^*\big(\mathcal{B}^*(X,\mathfrak{s});\mathbb{Z}\big)\to\mathbb{Z}, \quad \mathfrak{m}(\xi|X,\mathfrak{s}) = \langle\xi,[N_\eta(X,\mathfrak{s})]\rangle \tag{2.4.64}$$

where $\langle\ ,\ \rangle$ denotes the Kronecker pairing and $\eta\in i\Omega^2_+(X)$ is any regular perturbation.

We know at least one element in $\mathcal{B}^*(X,\mathfrak{s})$, namely $1\in H^*\big(\mathcal{B}^*(X,\mathfrak{s});\mathbb{Z}\big)$. However, for $\mathfrak{m}(1|X,\mathfrak{s})$ to be non-zero, we need the dimension

$$\dim N_\eta(X,\mathfrak{s}) = \frac{1}{2}\big(c_1(S^+)^2[X] - 2\chi(X) - 3\sigma(X)\big) \tag{2.4.65}$$

to be zero. This is known to be the case precisely when the spin^c structure comes from an almost complex structure on X. In that case, $\mathfrak{m}(1|X,\mathfrak{s})$ is just the signed counts of points in the compact, oriented 0–manifold $N_\eta(X,\mathfrak{s})$. The higher cohomology of $\mathcal{B}^*(X,\mathfrak{s})$ can be understood as follows.

Proposition 2.52. *Let $(X,\mathfrak{s})$ be a closed, connected spin^c 4–manifold. Then there is a homotopy equivalence*

$$\mathcal{B}^*(X,\mathfrak{s}) \simeq \mathbb{CP}^\infty\times Pic(X) \tag{2.4.66}$$

where $Pic(X) = H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$. In particular, there is an isomorphism

$$H^*\big(\mathcal{B}^*(X,\mathfrak{s});\mathbb{Z}\big) \cong \mathbb{Z}[u]\otimes_{\mathbb{Z}}\Lambda^*H^1(X,\mathbb{Z}) \tag{2.4.67}$$

where $u\in H^2(\mathbb{CP}^\infty;\mathbb{Z})$ is the first Chern class of the tautological line bundle.

Proof. Fix $A_0 \in \mathcal{A}(S)$ and consider the subspace

$$\mathcal{S}^*(X, \mathfrak{s}) = \{(A_0 + a, \phi) \in \mathcal{C}(X, \mathfrak{s}) \,|\, d^* a = 0\, \phi \neq 0\} \subset \mathcal{C}^*(X, \mathfrak{s}). \tag{2.4.68}$$

Recall that $\mathcal{S}^*(X, \mathfrak{s})$ is preserved by the actions of $\mathcal{G}^h(X)$ and that the action is free by Lemma 2.33. According toLemma 2.40, the inclusion induces a homeomorphism

$$\mathcal{S}^*(X, \mathfrak{s})/\mathcal{G}^h(X) \cong \mathcal{C}^*(X, \mathfrak{s})/\mathcal{G}(X) = \mathcal{B}^*(X, \mathfrak{s}). \tag{2.4.69}$$

Next we fix a base point $x_0 \in X$ to split $\mathcal{G}^h(X)$ into a product

$$\mathcal{G}^h(X) = \mathbb{T} \times \mathcal{G}^h_*(X), \quad \mathcal{G}^h_*(X) = \left\{u \in \mathcal{G}^h(X) \,\middle|\, u(x_0) = 1\right\}. \tag{2.4.70}$$

One can show that every connected component of $\mathcal{G}(X)$ contains a unique element of $\mathcal{G}^h_*(X)$. Since $\mathbb{T} = S^1$ is an Eilenberg–Mac Lane space of type $K(\mathbb{Z}, 1)$, we have

$$\mathcal{G}^h_*(X) \cong \pi_0 \mathcal{G}(X) \cong H^1(X; \mathbb{Z}). \tag{2.4.71}$$

In particular, we have a an isomorphism of Lie groups

$$\mathcal{G}^h(X) \cong \mathbb{T} \cong H^1(X; \mathbb{Z}). \tag{2.4.72}$$

From this we can identify the classifying space of $\mathcal{G}^h(X)$ as

$$B\mathcal{G}^h(X) \cong B\mathbb{T} \times BH^1(X; Z) \cong \mathbb{C}\mathrm{P}^\infty \times Pic(X) \tag{2.4.73}$$

by noting that $Pic(X)$ is a classifying space for $H^1(X; \mathbb{Z})$.

Now, it is a curious fact of infinite dimensional topology that the inclusion $\Gamma(S^+)\backslash 0 \hookrightarrow \Gamma(S^+)$ is a homotopy equivalence with respect to to the C^∞–topology; in fact, this hold for every separable infinite dimensional Fréchet space (see [And69], for example). In particular, $\mathcal{S}^*(X, \mathfrak{s})$ is contractible. We would like to argue that $\mathcal{S}^*(X, \mathfrak{s}) \to \mathcal{B}^*(X, \mathfrak{s})$ is a universal $\mathcal{G}^h(X)$ –bundle, making $\mathcal{B}^*(X, \mathfrak{s})$ a classifying space for $\mathcal{G}^h(X)$ which is unique up to homotopy equivalence. While $\mathcal{S}^*(X, \mathfrak{s})$ is provably not a CW complex, the bundle $\mathcal{S}^*(X, \mathfrak{s}) \to \mathcal{B}^*(X, \mathfrak{s})$ is provably *numerable* and we can appeal to an analogous uniqueness statement for numerable bundles. □

The class in $H^2(\mathcal{B}^*(X, \mathfrak{s}); \mathbb{Z})$ that corresponds to $u \in H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ can be described more explicitly as the first Chern class of the principal $\mathbb{T} = U_1$–bundle

$$\mathcal{S}^*(X, \mathfrak{s})/\mathcal{G}^h_*(X) \mapsto \mathcal{S}^*(X, \mathfrak{s})/\mathcal{G}^h(X) \cong \mathcal{B}^*(X, \mathfrak{s}). \tag{2.4.74}$$

(1) The entirety of the invariants $\mathfrak{m}(\cdot|X, \mathfrak{s})$ as $\mathfrak{s}$ ranges over all the spin^c structures on X is a diffeomorphism invariant of triples $(X, \mathfrak{s}, \mu_X)$.

(2) The Seiberg–Witten invariants can be computed fairly explicitly for Kähler manifolds. In particular, they are not always trivial. The computation goes back to Witten's original article [Wit94], see also [Mor96, Ch. 7] for a textbook account.

(3) The Seiberg–Witten invariants are quite fragile. If $X = X_1 \# X_2$ with $b_2^+(X_1), b_2^+(X_2) \geq 1$, then the Seiberg–Witten invariants of X are known to vanish. In particular, taking the connected sum with $S^2 \times S^2$ always kills the Seiberg–Witten invariants. In contrast, the connected sums with $\overline{\mathbb{C}\mathrm{P}^2}$ retains non-triviality of Seiberg–Witten invariants.

(4) It is a long standing question whether the invariants $\mathfrak{m}(u^d|X, \mathfrak{s})$ can be non-zero for $d > 0$. The *simple type conjecture* states that these invariants should vanish for all closed 4–manifold with $b_2^+(X) \geq 2$.

In general, the invariants are notoriously hard to compute.

2.4.6 Stretching the neck

Let X be a closed oriented 4–manifold and suppose that we are given a decomposition

$$X = X_+ \cup X_- \tag{2.4.75}$$

into compact codimension zero submanifolds $X_\pm$ with common boundary

$$Y = \partial X_\pm = X_+ \cap X_-. \tag{2.4.76}$$

Is it possible to recover the Seiberg–Witten invariants of X from similar invariants associated to $X_\pm$ and Y? Recall that the Seiberg–Witten invariants of X are independent of the Riemannian metric used to define them. This suggests an idea to separate the information contained in the moduli spaces $N_\eta(X, \mathfrak{s})$ into information solely related to $X_\pm$ and Y. The idea is to make X cylindrical near Y, to stretch the cylinder to infinite length, and to try and keep track of the SW moduli spaces. Here the word cylinder needs to be interpreted in the following geometric sense.

Definition 2.53. Let (Y, g_Y) be a Riemannian manifold and $J \subset \mathbb{R}$ an interval. The product $J \times Y$ equipped with the *cylindrical metric* $dt^2 + g_Y$ is called a *metric cylinder* on Y of length $L = \sup J - \inf J$.

The neck stretching procedure. We orient Y as the boundary of X_+ and choose a metric g_0 on X which is cylindrical near Y in the sense that there is an orientation preserving, isometric embedding

$$\tau \colon ([-3, 3] \times Y, dt^2 + g_Y) \hookrightarrow (X, g_0) \tag{2.4.77}$$

where g_Y is a fixed metric on Y. We write νY for the image of τ and think of it as a *neck* for Y. For the stretching procedure let $\kappa\colon [-3, 3] \to [0, 1]$ be smooth function which is identically one in a neighborhood of $[-1, 1]$ and zero outside of $[-2, 2]$. We obtain a cutoff function on X with support in $\tau([-2, 2] \times Y)$ by

$$\rho\colon X \to [0, 1], \quad \rho(x) = \begin{cases} \kappa(t), & \text{if } x = \tau(t, y) \\ 0, & \text{else.} \end{cases} \tag{2.4.78}$$

Using this, we construct a family of Riemannian metrics

$$g_s = (1 - \rho)\, g + \rho\, \tau_*((1 + s)^2 dt^2 + g_s), \quad s \geq 0. \tag{2.4.79}$$

Geometrically, as the parameter s increases, the neck νY gets longer and longer. Indeed, the central part $\tau([-1, 1] \times Y)$ of the neck with the metric g_s is isometric to the to the metric cylinder $[-1-s, 1+s] \times Y$ of length $2(s+1)$. However, note that the underlying manifold X never changes.

The effect on Seiberg–Witten moduli spaces. We continue with the family of metrics $(g_s)_{s\geq 0}$ on X. As in the proof of the cobordism theorem, we choose families of spin^c connections A_s and perturbations $\eta_s \in i\Omega^2_+(X, g_s)$ and consider the parameterized moduli space

$$W_\gamma(X, \mathfrak{s}) = \bigcup_{s \geq 0} \{s\} \times N_s(X, \mathfrak{s}) \subset [0, \infty) \times \mathcal{B}(X, \mathfrak{s}) \tag{2.4.80}$$

where $\gamma = (g_s, A_s, \eta_s)_{s\geq 0}$ and $N_s(X, \mathfrak{s})$ is the moduli space for the triple γ_s. As before, one can show that suitable of choices of (A_s, η_s) this is a finite dimensional smooth manifold with boundary on which $\mathcal{G}^h(X)$ acts smoothly. But this time there is only one boundary component $N_0(X, \mathfrak{s})$ corresponding to the single boundary of $[0, \infty)$. It turns out that the limiting behavior of elements $x_s \in N_s(X, \mathfrak{s})$ as $s \to \infty$ can be understood sufficiently well to draw interesting conclusions.

To get an idea of how this works, we think of X as a disjoint union of $\mathring{X}_\pm = X_\pm \setminus Y$ and Y. Note that we can rescale the left part of the central neck as

$$\big([-1,0) \times Y, (1+s)^2 dt^2 + \gamma\big) \cong \big([-1,T) \times Y, dt^2 + \gamma\big). \tag{2.4.81}$$

In other words, $X^s_- = (\mathring{X}_-, g_s)$ has a cylindrical end of the form $[0,s) \times Y$. Similarly, $X^s_+ = (\mathring{X}_+, g_s)$ has a cylindrical end of the form $(-s,0] \times Y$.

At this point we lose the ambition to be precise and content ourselves with an heuristic outline of what can eventually be made rigorous:

(1) First of all, it is conceivable that the families of Riemannian manifolds $X^s_\pm$ have limits $X^\infty_\pm = (\mathring{X}_\pm, g_\infty)$ with infinite cylindrical ends of the form $\mathbb{R}_\pm \times Y$ where $\mathbb{R}_\pm = \pm[0,\infty)$.

(2) Assuming that A_s and η_s were chosen in a certain way, there are canonical limits $A^\pm_\infty$ and $\eta^\pm_\infty$ defined on $X^\infty_\pm$. That certain way means that A_s and η_s should derived from a pair (A_0, η_0) which is which is translation invariant on $\tau((-2,2) \times Y)$ in the hopefully obvious sense (that is also explained below).

(3) One can show then that every sequence $x_n \in N_{s_n}(X, \mathfrak{s})$ with $s_n \to \infty$ as $n \to \infty$ has limits $x^\pm_\infty \in N_\infty(X^\infty_\pm, \mathfrak{s})$.

(4) Moreover, there is a gauge invariant notion of *energy* for Seiberg–Witten configurations on $X^\infty_\pm$ and every finite energy monopole has a configuration on Y as asymptotic limit along the neck.

(5) Lastly, the limiting configurations $x^\pm_\infty$ turn out to have finite energy and the same asymptotic limit.

This suggests a strategy to define Seiberg–Witten invariants of $X_\pm$ using moduli spaces of finite energy monopoles on $X^\infty_\pm$. In doing so, one has to keep track of asymptotic limits. This road eventually led to the idea of Floer homology groups. Motivated by the above, we now focus on cylinders $J \times Y$.

2.4.7 The Seiberg–Witten equations on cylinders

Lecture 14, 11.7.23

Let Y be an oriented Riemannian 3–manifold. We want to study the Seiberg–Witten equations on the cylinder $Z = \mathbb{R} \times Y$. We write t for the $\mathbb{R}$–coordinate and $p\colon Z = \mathbb{R} \times Y \to Y$ for the projection onto Y. The tangent and cotangent bundles of Z is canonically split as

$$\begin{aligned} TZ &= \mathbb{R}\,\partial_t \oplus \ker(dp) \cong \mathbb{R}\,\partial_t \oplus p^*TY \quad \text{and} \\ T^*Z &= \mathbb{R}\,dt \oplus \ker(i_{\partial_t}) \cong \mathbb{R}\,dt \oplus p^*T^*Y. \end{aligned} \tag{2.4.82}$$

The splittings are orthogonal with respect to the cylindrical metric $g_Z = dt^2 + p^*g_Y$ and we orient orient Z using the volume form

$$\mathrm{vol}_Z = dt \wedge p^* \mathrm{vol}_Y\,. \tag{2.4.83}$$

To make sense of Seiberg–Witten equations on Z, we need a suitable spin^c structure that relates to the given one on Y. We start with a general remark about vector bundles over Z.

Bundles over the cylinder. Given any real or complex vector bundle $E \xrightarrow{\pi} Y$, we let

$$\widehat{E} = \mathbb{R} \times E \xrightarrow{\hat{\pi}} \mathbb{R} \times Y = Z, \quad \hat{\pi}(t,e) = \big(t, \pi(e)\big). \tag{2.4.84}$$

Note that $\widehat{E}$ canonically isomorphic to the pullback p^*E in the category of real or complex vector bundles. Since $\mathbb{R}$ is contractible, all vector bundles of Z are isomorphic to a bundle of this form. Concretely, if we write $i_t \colon Y \to Z$, $i_t(y) = (t, y)$ with fixed $t \in \mathbb{R}$, then for any vector bundle $F \to Z$ we have $F \cong p^* i_t^* F \cong \widehat{i_t^* F}$.

We can conveniently think of section of $\widehat{E}$ is smooth paths of sections of E which, in turn, we can think of as "time-dependent" sections of E. More precisely, given a map $\phi \colon \mathbb{R} \to \Gamma(S)$, we can form a section $\hat{\phi} \in \Gamma(\widehat{E})$ by

$$\hat{\phi} \colon Z \to \widehat{E}, \quad \hat{\phi}(t, y) = \big(t, \phi(t)(y)\big). \tag{2.4.85}$$

Conversely, every section $\Phi \colon Z \to \widehat{E}$ can be written as $\Phi(t, y) = \big(t, p\Phi(t, y)\big)$ and thus determines a path

$$\check{\Phi} \colon \mathbb{R} \to \Gamma(S), \quad \check{\Phi}(t)(y) = p\Phi(t, y). \tag{2.4.86}$$

Ignoring smoothness of sections and paths thereof, the assignments $\phi \mapsto \hat{\phi}$ and $\Phi \mapsto \check{\Phi}$ are easily seen to be mutually inverse isomorphisms of vector spaces. The maps send continuous sections of $\hat{E}$ to continuous path of continuous sections of E in the compact open topology by the adjunction $C(\mathbb{R}, C(Y, E)) \cong C(\mathbb{R} \times Y, E)$. Since that latter is a homeomorphism, we even get an isomorphism of tological vector spaces. With a little more work, the same statements hold for smooth sections with the obvious notion of smooth paths in Fréchet spaces.

Proposition 2.54 (Exponential adjunction for smooth sections, c.f. [KM97]). *Let $E \to Y$ be a real or complex vector bundles over a closed smooth manifold Y. Then the maps*

$$\Gamma(\widehat{E}) \;\underset{\phi \mapsto \hat{\phi}}{\overset{\Phi \mapsto \check{\Phi}}{\rightleftarrows}}\; C^\infty\big(\mathbb{R}, \Gamma(E)\big)$$

are mutually inverse isomorphisms of Fréchet spaces.

We henceforth identify sections of E with constant paths in $\Gamma(E)$. Note that the latter can also be characterized as those sections $\Phi \in \Gamma(\hat{E})$ that are *translation invariant* in the sense that $p\Phi(t, y)$ is independent of t. This can also be expressed as

$$\tau_s \Phi = \Phi \in \Gamma(E) \quad \text{where} \quad \tau_s \Phi(t, y) = \big(t, p\Phi(t + s, y)\big). \tag{2.4.87}$$

Every connection ∇ on E determines a connection $\hat{\nabla}$ on $\hat{E} \cong p^* E$ by pull-back. This relation between ∇ and $\hat{\nabla}$ is often written informally as

$$\hat{\nabla} = \frac{d}{dt} + \nabla. \tag{2.4.88}$$

Concretely, this means that with respect to the splitting $TZ = \mathbb{R}\partial_t \oplus \widehat{TZ}$ in (2.4.82) the covariant derivative of $\hat{\nabla}$ acts on a section of $\hat{E}$ given by a path $\phi \in C^\infty(\mathbb{R}, \Gamma(E))$ as

$$\hat{\nabla}_{\partial_t} \hat{\phi} = \widehat{\tfrac{d\phi}{dt}} \quad \text{and} \quad \hat{\nabla}_{\hat{v}} \hat{\phi} = \widehat{\nabla_v \phi} \quad \text{for} \quad v \in \Gamma(TY). \tag{2.4.89}$$

where $\dot{\phi} = \frac{d\phi}{dt}$ is the path derivative. The pullback connection $\hat{\nabla}$ is also translation invariant in the sense that

$$\tau_{-s} \nabla \tau_s \Phi = \nabla \Phi, \quad s \in \mathbb{R}, \tag{2.4.90}$$

and this condition characterizes pullback connections on $\hat{E}$.

Differential forms on the cylinder. Every differential form on $Z = \mathbb{R} \times Y$ can be uniquely written as a sum $\alpha = \beta + dt \wedge \gamma$ with $\partial_t \lrcorner \beta = 0$ and $\partial_t \lrcorner \gamma = 0$. The latter condition characterizes those forms on Z that can be written as a path of forms on Y. Indeed, the splitting for $T^* Z$ in (2.4.82) gives one for the exterior powers of its complexification

$$\Lambda^p_{\mathbb{C}} Z \cong p^*(\Lambda^p_{\mathbb{C}} Y \oplus \Lambda^{p-1}_{\mathbb{C}} Y) \cong \widehat{\Lambda^p_{\mathbb{C}} Y} \oplus \widehat{\Lambda^{p-1}_{\mathbb{C}} Y} \tag{2.4.91}$$

This gives a path interpretation of differential forms on Z.

Corollary 2.55. *Every* $\omega \in \Omega^p(Z;\mathbb{C})$ *can be uniquely written as*

$$\omega = \hat{\eta} + dt \wedge \hat{\chi}. \tag{2.4.92}$$

where $\eta \in C^\infty(\mathbb{R}, \Omega^p(Y;\mathbb{C}))$ *and* $\chi \in C^\infty(\mathbb{R}, \Omega^{p-1}(Y;\mathbb{C}))$*. For* $\lambda \in \Omega^p(Y;\mathbb{C})$ *the pulled back form* $p^*\lambda \in \Omega^p(Z;\mathbb{C})$ *corresponds to the constant paths* $\eta \equiv \lambda$ *and* $\chi \equiv 0$*.*

The de Rahm differential, the codifferential, and the Hodge operator on Z are related to their analogues on Y by the following formulas whose proofs we leave as an exercise.

Lemma 2.56. *Let* $\omega = \hat{\eta} + dt \wedge \hat{\chi} \in \Omega^p(Z)$ *with* $\eta \in C^\infty(\mathbb{R}, \Omega^p(Y;\mathbb{C}))$ *and* $\chi \in C^\infty(\mathbb{R}, \Omega^{p-1}(Y;\mathbb{C}))$*.*

$$*_Z\omega = \widehat{*_Y\chi} + (-1)^p dt \wedge \widehat{*_Y\eta} \tag{2.4.93}$$

$$d_Z\omega = \widehat{d_Y\eta} + dt \wedge (\widehat{\dot{\eta} - d_Y\chi}) \tag{2.4.94}$$

$$d_Z^*\omega = (\widehat{d_Y^*\eta - \dot{\chi}}) + dt \wedge \widehat{d_Y^*\chi} \tag{2.4.95}$$

From (2.4.93) applied to $\omega \in \Omega^2(X)$ we immediately see that

$$*_Z\omega = \omega \quad \Leftrightarrow \quad \eta = *_Y\chi. \tag{2.4.96}$$

This means that self-dual 2–forms on Z correspond to paths of 1–forms on Y. Concretely, we have a bijection

$$C^\infty(\mathbb{R}, \Omega^1(Y)) \xrightarrow{\cong} \Omega^2_+(Z), \quad b \mapsto \widehat{*_Y b} + dt \wedge \hat{b}. \tag{2.4.97}$$

Combining (2.4.93) and (2.4.94) for $a = \hat{b} + \hat{c}\, dt \in i\Omega^1(Z)$ we find

$$d_Z^+ a = \frac{1}{2}(*_Y(*_Y d_Y b + \dot{b} - d_Y c))\hat{} + \frac{1}{2} dt \wedge (*_Y d_Y b + \dot{b} - d_Y c)\hat{} \in i\Omega^2_+(M). \tag{2.4.98}$$

So $d_Z^+ a \in i\Omega^2_+(Z)$ corresponds to the path $\frac{1}{2}(\dot{b} + *_Y d_Y b - d_Y c)$ in $i\Omega^1(Y)$.

Spin^c structures on cylinders. There is a one-to-one correspondence between spin^c structure on Y and $Z = \mathbb{R} \times Y$. Recall that $p\colon Z \to Y$ is the projection onto Y. We also consider the embeddings $i_t\colon Y \hookrightarrow Z$, $y \mapsto (t, y)$ for $t \in \mathbb{R}$.

(1) If (S_Z, ρ_Z) is a spinor bundle over Z, then we obtain a spinor bundle for Y via

$$S_Y = i_0^* S_Z^+, \quad \rho_Y(a)\phi = -\rho_Z(dt)\rho_Z(p^*a)\phi. \tag{2.4.99}$$

The sign ensures the orientation condition (2.1.2) in Definition 2.4.

(2) Conversely, if (S_Y, ρ_Y) is a spinor bundle for Y, we obtain one for Z by taking

$$S_Z^\pm = \hat{S}_Y \quad \text{and} \quad S_Z = S_Z^+ \oplus S_Z^- = \hat{S}_Y \oplus \hat{S}_Y \tag{2.4.100}$$

with Clifford multiplication given by the block matrices

$$\rho_Z(dt) = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \quad \text{and} \quad \rho_Z(p^*a) = \begin{pmatrix} 0 & \rho_Y(a) \\ \rho_Y(a) & 0 \end{pmatrix} \quad \text{for } a \in T^*Y. \tag{2.4.101}$$

One can check that the first summand S_Z^+ is also the positive eigenspace of the chirality operator $\alpha_Z = \rho_Z(i^2 \operatorname{vol}_Z)$.

The verifications that both constructions define spinor bundles and are mutually inverse up to isomorphism are straight forward. As an aside, we point out that our conventions are the same as those in [KM07, §4.3 & §4.5], but one should be aware that different authors might set up the correspondence differently.

The quadratic terms on Y and Z. From now on we will assume that Z and Y carry spinor bundles that are related as in (2) above. Applying Proposition 2.54 to $S_Z^{\pm} = \hat{S}_Y$ we get a path description of spinors:

$$\Gamma(S_Z^{\pm}) \cong C^{\infty}(\mathbb{R}, \Gamma(S_Y)) \tag{2.4.102}$$

In particular, given a path $C^{\infty}(\mathbb{R}, \Gamma(S_Y))$ we can construct an endomorphism of S_Z^+ in two ways. On the one hand, we have the path of endomorphisms $\phi\phi^*$ of S_Y which can be viewed as a single endomorphism

$$\hat{\phi}\hat{\phi}^* = \widehat{\phi\phi^*} \in \mathrm{End}_{\mathbb{C}}(S_Z^+). \tag{2.4.103}$$

On the other hand, Corollary 2.15 gives a path $\rho_Y^{-1}(\phi\phi^*)_0$ in $i\Omega^1(Y)$ which, in turn, determines an element of $i\Omega^2_+(Z)$ via the isomorphism in (3.2.2). The latter is taken by ρ_Z to a self-adjoint, trace-free endomorphism of S_Z^+ (see Corollary 2.16). It should not be a big surprise that both constructions are related.

Lemma 2.57. *Let (S_Z, ρ_Z) be a spinor bundle on $Z = \mathbb{R} \times Y$ derived from a spinor bundle (S_Y, ρ_Y) on Y, and $\phi \in \mathbb{R} \to \Gamma(S)$ a path corresponding to $\hat{\phi} \in \Gamma(S_Z^+)$. Then*

$$\rho_Z^{-1}(\hat{\phi}\hat{\phi}^*)_0 = -\frac{1}{2}\big((*_Y \rho_Y^{-1}(\phi\phi^*)_0)\hat{} + dt \wedge (\rho_Y^{-1}(\phi\phi^*)_0)\hat{}\big). \tag{2.4.104}$$

Proof. By construction, we have

$$\rho_Z\big(dt \wedge (\rho_Y^{-1}(\phi\phi^*)_0)\hat{}\big) = \rho_Z(dt)\hat{\rho}_Y\big((\rho_Y^{-1}(\phi\phi^*)_0)\hat{}\big) = -(\hat{\phi}\hat{\phi}^*)_0 \tag{2.4.105}$$

where the minus sign is the action of $\rho_Z(dt)$ on S_Z^-. Similarly, we find

$$\rho_Z\big((*_Y \rho_Y^{-1}(\phi\phi^*)_0)\hat{}\big) = -(\hat{\phi}\hat{\phi}^*)_0 \tag{2.4.106}$$

where the minus sign comes from our orientation conventions for Clifford multiplication in odd dimensions, which yields $\rho_Y(*_Y\alpha) = -\rho_Y(\alpha)$ for all $\alpha \in \Omega^1(Y)$. ☐

Spin^c connections and Dirac operators on cylinders. Next, let us fix a spin^c connection $B_0 \in \mathcal{A}(S_Y)$ for reference and write $\hat{B}_0$ for the induced connection on $\hat{S}_Y \cong p^*S_Y$. The sum with itself gives a translation invariant spin^c connection

$$A_0 = \hat{B}_0 \oplus \hat{B}_0 \in \mathcal{A}(S_Z). \tag{2.4.107}$$

We take this A_0 as a base point for $\mathcal{A}(S_Z)$ and write any other spin^c connection on S_Z in the form

$$A = A_0 + a = A_0 + \hat{b} + \hat{c}dt. \tag{2.4.108}$$

where $a \in i\Omega^1(Z)$ corresponds to paths $b \in C^{\infty}(\mathbb{R}, i\Omega(Y))$ and $c \in C^{\infty}(\mathbb{R}, i\Omega^0(Y)) \cong iC^{\infty}(Z)$. Following [KM07, Def. 4.4.1], we note that the connection

$$\check{A} = A_0 + \hat{b} \in \mathcal{A}(S_Z) \tag{2.4.109}$$

given by the first two summands can be interpreted as a path of connections

$$B = B_0 + b \in C^{\infty}(\mathbb{R}, \mathcal{A}(S_Y)) \tag{2.4.110}$$

which is, in fact, independent of the choice of B_0. In general, $\check{A}$ does not determine A, since the information contained in c cannot be recovered from $\check{A}$. This discrepancy between spin^c connections on S_Z and paths thereof on S_Y can be fixed using the gauge group action.

Definition 2.58 (Temporal gauge)**.** A spin^c connection $A \in \mathcal{A}(S_Z)$ is in *temporal gauge* if it can be written as $A = A_0 + \hat{b}$ for some $B_0 \in \mathcal{A}(S)$ and $\hat{b} \in C^{\infty}(\mathbb{R}, i\Omega^1(Y))$.

Lemma 2.59 (Temporal gauge fixing)**.**

(i) For every $A \in \mathcal{A}(S_Z)$ there is a gauge transformation of the form $u = e^{if} \in \mathcal{G}(Z)$ such that uA is in temporal gauge.

(ii) Let $A \in \mathcal{A}(S_Z)$ be in temporal gauge and $u \in \mathcal{G}(Z)$. Then $uA = A - u^{-1}du$ is also in temporal gauge if and only if $\partial_t u = 0$, that is, $u(t, y) = u_0(y)$ for some $u_0 \in \mathcal{G}(Y)$.

Proof. (i) Write A as in (3.2.4). For $u = e^{if}$ we have

$$u^{-1}du = i\,df = i(\partial_t f\,dt + \check{d}f) \tag{2.4.111}$$

and thus

$$u^*A = A - (u^{-1}du) \otimes \mathrm{id} = A_0 + i(b - \check{d}f) \otimes \mathrm{id} + i(c - \partial_t f)\,dt \otimes \mathrm{id}\,. \tag{2.4.112}$$

Define $u = e^{if}$ with $f \in C^\infty(Z)$ given by

$$f(t, y) = \int_0^t c(s, y)\,ds. \tag{2.4.113}$$

Then $\partial_t f = c$ so that u^*A is in temporal gauge.

(ii) For arbitrary $u \in \mathcal{G}(Z)$ and $A \in \mathcal{A}(S_Z)$ in temporal gauge, we find

$$u^*A = A - (u^{-1}\check{d}u) \otimes \mathrm{id} - (u^{-1}\partial_t u) \otimes \mathrm{id}\,. \tag{2.4.114}$$

Since $u^{-1}\check{d}u \in i\Gamma(p^*T^*Y)$, the connection u^*A is in temporal gauge iff $\partial_t u = 0$. □

Combining the maps $C^\infty(\mathbb{R}, \mathcal{A}(S_Y)) \to \mathcal{A}(S_Z)$ and $\Gamma(S_Z^+) \cong C^\infty(\mathbb{R}, \Gamma(S_Y))$ with Lemma 3.2, we arrive at the following conclusion:

Corollary 2.60. *The map $C^\infty(\mathbb{R}, \mathcal{C}(Y)) \to \mathcal{C}(Z)$ induces a homeomorphism*

$$C^\infty\big(\mathbb{R}, \mathcal{C}(Y)/\mathcal{G}(Y)\big) \xrightarrow{\cong} \mathcal{C}(Z)/\mathcal{G}(Z) = \mathcal{B}(Z). \tag{2.4.115}$$

Remark 2.61. While conceptually convenient, the temporal gauge condition is not perfect. Unlike the Coulomb condition on closed manifolds, it does not reduce the Seiberg–Witten equations to an ellitpic system. The temporal gauge condition is also generally incompatible with the Coulomb condition $d_Z^* a = 0$ on the cylinder. However, there are tricks around this that will be discussed next semester.

Back to a general connection $A = \check{A} + \hat{c}\,dt$. We recall from (2.4.89) that $\nabla^{\hat{B}_0} = \frac{d}{dt} + \nabla^{B_0}$. Using this and the definition of ρ_Z gives

$$D_A^+\hat{\phi} = \big(\dot{\phi} + D_B\phi + c\phi\big)\hat{} = \big(\dot{\phi} + D\phi + \rho(b)\phi + c\phi\big)\hat{}. \tag{2.4.116}$$

We can also write this as

$$D_A^+ = \frac{d}{dt} + D_B + c. \tag{2.4.117}$$

Lastly, we note that we have an isomorphism of determinant line bundles

$$\det(\mathfrak{s}_Z) = \Lambda^2_{\mathbb{C}}(S_Z^+) = \Lambda^2_{\mathbb{C}}(\hat{S}_Y) \cong \widehat{\det(\mathfrak{s}_Y)} \tag{2.4.118}$$

and that the curvature of A_0^t is related to that of B_0 by

$$F_{A_0^t} = p^*F_{B_0^t} = \widehat{F_{B_0^t}}. \tag{2.4.119}$$

From this we can deduce that

$$\tfrac{1}{2}F_{A^t} = \tfrac{1}{2}F_{A_0^t} + d_Z(\hat{b} + \hat{c}\,dt) = \big(F_{B_0^t} + d_Y b\big)\hat{} + dt \wedge (\dot{b} - d_Y c)\hat{}. \tag{2.4.120}$$

The Seiberg–Witten equations as a gradient flow equation. Now let $(A, \Phi) \in \mathcal{C}(Z)$ be a Seiberg–Witten configuration. As in the previous section, we write $A = A_0 + \hat{b} + \hat{c}\,dt$ and $\Phi = \hat{\phi}$ with smooth paths b, c, and ϕ in $i\Omega^1(Y)$, $i \in$ The Seiberg–Witten equations for $A = A_0 + \hat{b} + \hat{c}\,dt$ take the form

$$\begin{aligned} D_A^+\Phi &= 0 & \dot{\phi} &= -(D_B\phi + c\phi) \\ \tfrac{1}{2}F_{A^t}^+ - \rho_Z^{-1}(\Phi\Phi^*)_0 &= 0 & \dot{b} &= -\big(*_Y\, d_Y b - dc + \rho_Y^{-1}(\phi\phi) + *_Y \tfrac{1}{2}F_{B_0^t}\big) \end{aligned}$$

If A happens to be in temporal gauge, then $c = 0$ and the equations simplify to

$$\begin{aligned} D_A^+\Phi &= 0 & \dot{\phi} &= -(D\phi + \rho(b)\phi) \\ \tfrac{1}{2}F_{A^t}^+ - \rho_Z^{-1}(\Phi\Phi^*)_0 &= 0 & \dot{b} &= -\big(*_Y\, d_Y b + \rho_Y^{-1}(\phi\phi) + *_Y \tfrac{1}{2}F_{B_0^t}\big) \end{aligned}$$

Note that the equations on the right hand side are formally a negative flow equation in the based configuration space $\mathcal{C}_0(Y)$. Th generator is the *Seiberg–Witten vector field*

$$\mathcal{X}\colon \mathcal{C}_0(Y) \to \mathcal{C}_0(Y), \quad \mathcal{X}(b,\phi) = \begin{pmatrix} *_Y d_Y b + \rho_Y^{-1}(\phi\phi) + *_Y \tfrac{1}{2}F_{B_0^t} \\ D\phi + \rho(b)\phi \end{pmatrix} \tag{2.4.121}$$

in terms of which the equations can be written as

$$(\dot{b}, \dot{\phi}) + \mathcal{X}(b,\phi) = 0. \tag{2.4.122}$$

Moreover, it turns out that $\mathcal{X}(b,\phi)$ can be considered as the gradient of a smooth function

$$\mathcal{L}\colon \mathcal{C}_0(Y) \to \mathbb{R}, \tag{2.4.123}$$

called the *Chern–Simons–Dirac functional* (CSD), with respect to the (real) L^2 inner product on $\mathcal{C}_0(Y)$. The CSD functional is defined as

$$\begin{aligned} \mathcal{L}(b,\phi) &= \tfrac{1}{2}(\phi, D_B\phi)_0 + \tfrac{1}{2}(b, *_Y d_Y b)_0 + \tfrac{1}{2}(b, *_Y F_{B_0^t})_0 \\ &= \tfrac{1}{2}(\phi, D\phi)_0 + \tfrac{1}{2}(b, *_Y d_Y b)_0 + \tfrac{1}{2}(\phi, \rho(b)\phi)_0 + \tfrac{1}{2}(b, *_Y F_{B_0^t})_0 \end{aligned} \tag{2.4.124}$$

Part II

Monopole Floer Homology and Seiberg–Witten–Floer homotopy types (WiSe 2023–2024)

Chapter 3

The Seiberg–Witten equations on cylinders revisited

3.1 Recollections from last semester

Lecture 1, 10.10.23

Notational conventions. Let's begin by reviewing with some notational ground rules from last semester:

- All manifolds are implicitly assumed to be smooth, oriented, and equipped with a Riemannian metric.
- All vector bundles are implicitly equipped with bundle metrics.
- M stands for any n–manifold as above (possibly non-compact and/or with non-empty boundary)
- X is reserved for 4–dimensional manifolds which are compact by default.
- Y is reserved for closed 3–manifolds.
- $\mathbb{T}$ is the unit circle group.
- Spin^c structures are represented by spinor bundles (S, ρ) (see Section 2.1)
- $\mathcal{A}(S)$ is the space of spin^c connections

Floer homology and Conley index theory in finite dimensions. We first studied how a Morse–Smale pair (f, ξ) on a closed manifold M gives rise to a chain complex, called *Floer complex*, which computes the homology $H_*(M)$ by studying the flow ϕ on M generated by the equation $\dot{x} + \xi(x) = 0$. A particularly important aspect was a compactness result for spaces of "broken ξ–trajectories":

Theorem (c.f. Theorem 1.7)**.** *Let (f, ξ) be a Morse–Smale pair and $p, q \in \mathrm{Crit}(f)$. The moduli spaces $\hat{M}(p, q)$ have compactifications given by*

$$\bar{M}(p,q) = \hat{M}(p,q) \cup \bigcup_{r=2}^{\mu(p)-\mu(q)} \bigcup_{p=p_0,p_1,\dots,p_r=q} \hat{M}(p_0,p_1) \times \cdots \times \hat{M}(p_{r-1},p_r) \tag{3.1.1}$$

with a suitable topology. The space $\bar{M}(p,q)$ has the structure of a smooth $(\mu(p) - \mu(q) - 1)$–manifold with corners.

We then introduced the concept of *isolated invariant sets* $S \subset X$ for ϕ and noticed that we can construct a Floer complex $CF(S, \phi)$ by simply restricting to the critical points in S. However, we realized that $CF(S, \phi)$ does not compute the homology of S, but rather of the *Conley index* of S with respect to ϕ. The latter was defined in terms of *index pairs* (N, E) for S as the based homotopy type $C(S, \phi) = [N/E]$.

We then added actions by a compact Lie group G to the mix and discussed equivariant generalizations. We realized that Conley index theory generalizes easily by "putting a G–everywhere", but noted that the story for Floer homology was less straight forward. On the one hand, there are technical problems related to the failure of transversality in the equivariant context. On the other hand, there is the philosophical question what "equivariant homology" should be. We opted for the notion of *Borel homology* which is defined for a G–space X as

$$H_*^G(X) = H_*(EG \times_G X), \tag{3.1.2}$$

where EG is a *universal G–space*. The space $X_{hG} = EG \times_G X$ is called the *Borel construction* and is the total space of a fiber bundle $p_G \colon X_{hG} \to BG$ over the *classifying space* $BG = EG/G$. We ended this discussion by indicating possible Morse theoretic descriptions $H_*^{\mathbb{T}}(M)$ for smooth G–manifolds M and emphasized the role of the circle group $\mathbb{T} \cong U_1$. We will come back to this soon.

The Seiberg–Witten equations on 4–manifolds. We then switched subjects and discussed the spin^c structures and the Seiberg–Witten equations

$$\frac{1}{2} F_{A^t}^+ = \rho^{-1}(\phi\phi^*)_0 \qquad D_A\phi = 0 \tag{3.1.3}$$

on a spin^c 4–manifold X with spinor bundle (S, ρ) representing a spin^c structure $\mathfrak{s}$. Here A is a *spin^c connection* and ϕ a *spinor*, that is, a section of S. Once we had learned how to read the equations properly, we mostly focused the case when X is closed.

We introduced the *configuration spaces*

$$\mathcal{C}(X, \mathfrak{s}) = \mathcal{A}(S) \times \Gamma(S) \quad \text{and} \quad \mathcal{C}_0(X, \mathfrak{s}) = i\Omega^1(X) \oplus \Gamma(S) \tag{3.1.4}$$

where $\mathcal{A}(X)$ is the space of spin^c connections which is an affine space over $i\Omega^1(X)$ and the affine structure gives a homeomorphism $\mathcal{C}_0(X, \mathfrak{s}) \cong \mathcal{C}(X, \mathfrak{s})$ sending (a, ϕ) to $(A_0 + a, \phi)$ where A_0 is any fixed spin^c connection. Moreover, there was an action by the *gauge group* $\mathcal{G}(X) = C^\infty(X, \mathbb{T})$ where $u \colon X \to \mathbb{T}$ acts on (A, ϕ) as $u(A, \phi) = (A - u^{-1}du, \phi)$ and on (a, ϕ) as $u(a, \phi) = (a - u^{-1}du, \phi)$.

Solutions to the Seiberg–Witten equations are the zero sets of the Seiberg–Witten map

$$\begin{aligned} \mathfrak{F} \colon \mathcal{A}(S) \times \Gamma(S^+) &\to i\Omega_+^2(X) \oplus \Gamma(S^-) \\ \mathfrak{F}(A, \phi) &= \left(\tfrac{1}{2} F_{A^t}^+ - q(\phi),\, D_A^+\phi\right). \end{aligned} \tag{3.1.5}$$

which is $\mathcal{G}(X)$–equivariant.

In the case of a closed 4–manifold, the main trick was to enlarge the Seiberg–Witten equations to the *Seiberg–Witten–Coulomb system*

$$d^+a - q(\phi) + \tfrac{1}{2} F_0^+ = 0 \qquad D_A\phi = 0 \qquad d^*a = 0 \tag{3.1.6}$$

whose solutions form the zero set of the map

$$\begin{aligned} \widetilde{\mathfrak{F}}_0 \colon \underbrace{i\Omega^1(X) \oplus \Gamma(S^+)}_{\mathcal{C}_0(X, \mathfrak{s})} &\to \underbrace{i\Omega_0^0(X) \oplus \Omega_+^2(X) \oplus \Gamma(S^-)}_{=:\widetilde{\mathcal{D}}(X, \mathfrak{s})} \\ \widetilde{\mathfrak{F}}_0(a, \phi) &= \left(d^*a, d^+a - q(\phi) + \tfrac{1}{2} F_0^t, D\phi + \rho(a)\phi\right). \end{aligned} \tag{3.1.7}$$

which has is equivariant with respect to the *harmonic gauge group*

$$\mathcal{G}^h(X) = \left\{ u \in \mathcal{G}(X) \,\middle|\, d^*(u^{-1}du) = 0 \right\} \cong \mathbb{T}^{b_0(X)} \times H^1(X;\mathbb{Z}). \tag{3.1.8}$$

In particular, if X is connected with $b_1(X) = 0$, then $\mathcal{G}^h(X) \cong \mathbb{T}$ which explains our interest in circle actions. The main insight was that the regular level sets of $\widetilde{\mathfrak{F}}_0$ are smooth, finite dimensional $\mathcal{G}^h(X)$–manifolds whose orbits spaces are compact and, at least for suitable regular values, represent homology classes in

$$H_*(\mathcal{B}^*(X,\mathfrak{s})), \quad \mathcal{B}^*(X,\mathfrak{s}) = \mathcal{C}^*(X,\mathfrak{s})/\mathcal{G}(X) \tag{3.1.9}$$

where $\mathcal{C}^*(X,\mathfrak{s}) = \{(A,\phi) \in \mathcal{C}(X,\mathfrak{s}) \,|\, \phi \neq 0\}$ is the space of *irreducible configurations*. Out of these, we obtained the *Seiberg–Witten invariants* of $X, \mathfrak{s}$ as the map

$$\mathfrak{m}(\cdot|X,\mathfrak{s})\colon H^*\big(\mathcal{B}^*(X,\mathfrak{s});\mathbb{Z}\big) \to \mathbb{Z}. \tag{3.1.10}$$

which evaluates a cohomology class on the homology classes obtained above.

We then went into a brief discussion about cutting the manifold X into two pieces $X = X_1 \cup_Y X_2$ along a hypersurface $Y \subset X$. The idea was to assume that the metric on X is cylindrical near Y, to stretch the length of the cylinder to infinite, and try to keep track of the Seiberg–Witten moduli spaces. The main problem is that the pieces X_1 and X_2 are not closed, which makes the analysis or the Seiberg–Witten equations considerably more complicated. Nevertheless, there was hope to be able to define *relative Seiberg–Witten invariants* of X_1 and X_2 which allow to recover those of X. The relative invariants take values in certain *monopole Floer homology* groups associated to the common boundary Y. The latter are constructed using the Seiberg–Witten equations on the infinite cylinder $\mathbb{R} \times Y$.

3.2 The Seiberg–Witten equations on cylinders revisited

We have already started discussing the Seiberg–Witten equations on cylinders in Section 2.4.7. Here's a review of what we've learned so far. Let Y be a connected Riemannian 3–manifold with spinor bundle (S_Y, ρ_Y) and $Z = \mathbb{R}\times Y$ the infinite cylinder with metric $g_Z = dt^2 + p^*gY$ where we write t for the $\mathbb{R}$–coordinate and $p\colon Z = \mathbb{R} \times Y \to Y$ for the projection onto Y.

(1) If $E \to Y$ is any vector bundle on Y, then sections of $\hat{E} = \mathbb{R} \times E \cong p^*E \to Z$ can be viewed as paths of sections on Y. Every connection ∇ on E determines a connection on $\widehat{E}$ which can be informally written as $\widehat{\nabla} = \frac{d}{dt} + \nabla$.

(2) Every $\omega \in \Omega^p(Z;\mathbb{C})$ can be uniquely written as

$$\omega = \hat{\eta} + dt \wedge \hat{\chi}. \tag{3.2.1}$$

where $\eta \in C^\infty(\mathbb{R}, \Omega^p(Y;\mathbb{C}))$ and $\chi \in C^\infty(\mathbb{R}, \Omega^{p-1}(Y;\mathbb{C}))$.

(3) Self-dual 2–forms on Z correspond to paths of 1–forms on Y via the bijection

$$C^\infty(\mathbb{R}, \Omega^1(Y)) \xrightarrow{\cong} \Omega^2_+(Z), \quad b \mapsto \widehat{*_Y b} + dt \wedge \hat{b}. \tag{3.2.2}$$

(4) If (S_Y, ρ_Y) is a spinor bundle for Y, then $S_Z = \widehat{S}_Y \oplus \widehat{S}_Y$ is a spinor bundle on Z with Clifford multiplication

$$\rho_Z(dt) = \begin{pmatrix} 0 & -\mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \quad \text{and} \quad \rho_Z(p^*a) = \begin{pmatrix} 0 & \rho_Y(a) \\ \rho_Y(a) & 0 \end{pmatrix} \quad \text{for } a \in T^*Y. \tag{3.2.3}$$

We fix a spin^c connection $B_0 \in \mathcal{A}(S_Y)$ and let $A_0 = \widehat{B}_0 \oplus \widehat{B}_0$. Then every spin^c connection $A \in \mathcal{A}(S_Z)$ can be uniquely written as

$$A = A_0 + a = A_0 + \hat{b} + \hat{c}dt. \tag{3.2.4}$$

where $a \in i\Omega^1(Z)$ corresponds to paths $b \in C^\infty(\mathbb{R}, i\Omega(Y))$ and $c \in C^\infty(\mathbb{R}, i\Omega^0(Y)) \cong iC^\infty(Z)$. We say that A is in *temporal gauge* if $c = 0$.

(5) The Seiberg–Witten equations for $(A, \phi) \in \mathcal{C}(Z)$ take the form

$$\begin{aligned} D_A^+\Phi &= 0 & \dot{\phi} + (D_B\phi + c\phi) &= 0 \\ \tfrac{1}{2}F_{A^t}^+ - \rho_Z^{-1}(\Phi\Phi^*)_0 &= 0 & \dot{b} + \big(*_Y\, d_Y b - dc + \rho_Y^{-1}(\phi\phi) + *_Y \tfrac{1}{2}F_{B_0^t}\big) &= 0 \end{aligned}$$

If A happens to be in temporal gauge, the equations simplify to

$$\begin{aligned} D_A^+\Phi &= 0 & \dot{\phi} + (D\phi + \rho(b)\phi) &= 0 \\ \tfrac{1}{2}F_{A^t}^+ - \rho_Z^{-1}(\Phi\Phi^*)_0 &= 0 & \dot{b} + \big(*_Y\, d_Y b + \rho_Y^{-1}(\phi\phi) + *_Y \tfrac{1}{2}F_{B_0^t}\big) &= 0 \end{aligned}$$

Lecture 2, 17.10.23

The Chern–Simons–Dirac functional. Note that the equations on the right hand side are formally a negative flow equation in the based configuration space $\mathcal{C}_0(Y)$. Th generator is the *Seiberg–Witten vector field*

$$\mathcal{X}\colon \mathcal{C}_0(Y) \to \mathcal{C}_0(Y), \quad \mathcal{X}(b,\phi) = \begin{pmatrix} *_Y d_Y b + \rho_Y^{-1}(\phi\phi)_0 + *_Y \tfrac{1}{2}F_{B_0^t} \\ D\phi + \rho(b)\phi \end{pmatrix} \tag{3.2.5}$$

in terms of which the equations can be written as

$$(\dot{b}, \dot{\phi}) + \mathcal{X}(b,\phi) = 0. \tag{3.2.6}$$

Moreover, it turns out that $\mathcal{X}(b,\phi)$ can be considered as the gradient of a smooth function

$$\mathcal{L}\colon \mathcal{C}_0(Y) \to \mathbb{R}, \tag{3.2.7}$$

called the *Chern–Simons–Dirac functional* (CSD), with respect to the (real) L^2 inner product on $\mathcal{C}_0(Y)$. The CSD functional is defined as

$$\begin{aligned} \mathcal{L}(b,\phi) &= \tfrac{1}{2}(\phi, D_B\phi)_0 + \tfrac{1}{2}(b, *_Y d_Y b)_0 + \tfrac{1}{2}(b, *_Y F_{B_0^t})_0 \\ &= \tfrac{1}{2}(\phi, D\phi)_0 + \tfrac{1}{2}(b, *_Y d_Y b)_0 + \tfrac{1}{2}(\phi, \rho(b)\phi)_0 + \tfrac{1}{2}(b, *_Y F_{B_0^t})_0 \end{aligned} \tag{3.2.8}$$

Lemma 3.1. *We have $\nabla\mathcal{L}(b,\phi) = \mathcal{X}(b,\phi)$.*

Proof. The derivative of $\frac{1}{2}(\phi, \rho(b)\phi)_0$ in the direction of (c,ψ) can be computed as

$$\begin{aligned} \tfrac{d}{dt}\big|_{t=0} \tfrac{1}{2}\big(\phi + t\psi, \rho(b+tc)(\phi+t\psi)\big)_0 &= \tfrac{1}{2}\big(\phi, \rho(c)\phi\big)_0 + \tfrac{1}{2}\big(\phi, \rho(b)\psi\big)_0 + \tfrac{1}{2}\big(\psi, \rho(b)\phi\big)_0 \\ &= \big(\rho^{-1}(\phi\phi^*)_0, c\big)_0 + \big(\rho(b)\phi, \psi\big)_0 \end{aligned}$$

Here we have used that $\rho(b)^* = \rho(b)$ and the identity $\frac{1}{2}(\phi, \rho(c)\phi)_0 = (\rho^{-1}(\phi\phi^*)_0, c)_0$ which was discussed in the exercises. The computation shows that $\frac{1}{2}(\phi, \rho(b)\phi)_0$ admits an L^2 gradient given by $\big(\rho^{-1}(\phi\phi^*)_0,\ \rho(b)\phi\big)$. The other summands of $\mathcal{L}$ can be treated similarly. The computations are straight forward and produce the remaining terms in $\mathcal{X}$. ☐

Temporal gauge fixing. We may or may not have already discussed the following lemma which allows to restrict our attention to configurations in temporal gauge.

Lemma 3.2 (Temporal gauge fixing).

(i) For every $A \in \mathcal{A}(S_Z)$ there is a gauge transformation of the form $u = e^{if} \in \mathcal{G}(Z)$ such that uA is in temporal gauge.

(ii) Let $A \in \mathcal{A}(S_Z)$ be in temporal gauge and $u \in \mathcal{G}(Z)$. Then $uA = A - u^{-1}du$ is also in temporal gauge if and only if $\partial_t u = 0$, that is, $u(t,y) = u_0(y)$ for some $u_0 \in \mathcal{G}(Y)$.

Proof. (i) Write $A = A_0 + \hat{b} + \hat{c}\,dt$ as in (3.2.4). For $u = e^{if}$ we have

$$u^{-1}d_Z u = i\,df = i(\partial_t f\,dt + d_Y f) \tag{3.2.9}$$

and thus

$$u^*A = A - u^{-1}du = A_0 + i(b - d_Y f) + i(c - \partial_t f)\,dt. \tag{3.2.10}$$

Define $u = e^{if}$ with $f \in C^\infty(Z)$ given by

$$f(t,y) = \int_0^t c(s,y)\,ds. \tag{3.2.11}$$

Then $\partial_t f = c$ so that u^*A is in temporal gauge.

(ii) For arbitrary $u \in \mathcal{G}(Z)$ and $A \in \mathcal{A}(S_Z)$ in temporal gauge, we find

$$u^*A = A - u^{-1}d_Y u - u^{-1}\partial_t u. \tag{3.2.12}$$

Since $u^{-1}d_Y u \in i\Gamma(p^*T^*Y)$, the connection u^*A is in temporal gauge iff $\partial_t u = 0$. □

Combining the maps $C^\infty(\mathbb{R}, \mathcal{A}(S_Y)) \to \mathcal{A}(S_Z)$ and $\Gamma(S_Z^+) \cong C^\infty(\mathbb{R}, \Gamma(S_Y))$ with Lemma 3.2, we arrive at the following conclusion:

Corollary 3.3. *The map $C^\infty(\mathbb{R}, \mathcal{C}(Y)) \to \mathcal{C}(Z)$ induces a homeomorphism*

$$C^\infty\big(\mathbb{R}, \mathcal{C}(Y)/\mathcal{G}(Y)\big) \xrightarrow{\cong} \mathcal{C}(Z)/\mathcal{G}(Z) = \mathcal{B}(Z). \tag{3.2.13}$$

Remark 3.4. While conceptually convenient, the temporal gauge condition is not perfect. Unlike the Coulomb condition on closed manifolds, it does not reduce the Seiberg–Witten equations to an ellitpic system. The temporal gauge condition is also generally incompatible with the Coulomb condition $d_Z^* a = 0$ on the cylinder. We will have to find tricks to work around this.

Gauge invariance of the Chern–Simons–Dirac functional. Let us see how the CSD functional behaves under gauge transformations.

Lemma 3.5. *For $(b,\phi) \in \mathcal{C}_0(Y)$ and $u \in \mathcal{G}(Y)$ we have*

$$\mathcal{L}(u(b,\phi)) - \mathcal{L}(b,\psi) = \tfrac{1}{2}(u^{-1}du, *F_{B_0^t})_0. \tag{3.2.14}$$

Proof. A straight forward computation using $D(u\phi) = uD\phi + \rho(du)\phi$ show that the sum $\frac{1}{2}(\phi, D\phi)_0 + \frac{1}{2}(b, *_Y d_Y b)_0 + \frac{1}{2}(\phi, \rho(b)\phi)_0$ is fully gauge invariant. The remaining summand $\frac{1}{2}(b, *_Y F_{B_0^t})_0$ changes as indicated. □

Using that $F_{B_0^t}$ and $u^{-1}du$ are de Rham representatives of $2\pi i\, c_1(S_Y)$ and the class $[u] \in H^1(Y;\mathbb{Z})$ obtained by pulling back the generator of $H^1(\mathbb{T};\mathbb{Z})$, we can also write the change of $\mathcal{L}$ as

$$\mathcal{L}(u(b,\phi)) - \mathcal{L}(b,\psi) = 2\pi^2 \left\langle [u] \cup c_1(S_Y), [Y] \right\rangle \in 2\pi^2\mathbb{Z}. \tag{3.2.15}$$

We can draw the following conclusions:

(a) $\mathcal{L}$ is invariant under the full gauge group if and only if $c_1(S_Y) = 0$.

(b) $\mathcal{L}$ is always invariant under the subgroup of constant gauge transformations $\mathcal{G}^c(Y) \cong \mathbb{T}$.

(c) $\mathcal{L}$ descends to a well-defined map $\mathcal{C}_0(Y)/\mathcal{G}(Y) \to \mathbb{R}/2\pi^2\mathbb{Z} \cong S^1$.

In summary, the Seiberg–Witten equations on Z are equivalent to the negative gradient flow equation

$$\dot{x} + \nabla\mathcal{L}(x) = \dot{x} + \mathcal{X}(x) = 0 \tag{3.2.16}$$

on the infinite dimensional space $\mathcal{C}_0(Y)$ for the $\mathbb{T}$–invariant CSD functional $\mathcal{L}\colon \mathcal{C}_0(Y) \to \mathbb{R}$.

Towards monopole Floer homology. Now if X is a compact 4–manifold with boundary Y, we can attach a cylindrical end and study the Seiberg–Witten equations on Y

$$X^\infty = X \cup_Y \mathbb{R}_+ \times Y. \tag{3.2.17}$$

As mentioned before, there is a notion of 'energy' for monopoles and – after choosing suitable perturbations – one can show that finite energy monopoles on X^∞ have asymptotic limits in $\mathcal{C}_0(Y)$ on the cylindrical end, well-defined up to gauge, which are critical points of $\mathcal{L}$. While all of this is admittedly rather sketchy, it hopefully gives a plausible explanation why it might be fruitful to try and define something like "$\mathbb{T}$–equivariant Floer homology" based on the equation $\dot{x} + \nabla\mathcal{L}(x) = 0$. This brings us back to the question how to define "$\mathbb{T}$–equivariant Floer homology" in finite dimensional setting.

Chapter 4

Morse theory for circle actions

We review a Morse theoretic description of the $\mathbb{T}$–equivariant Borel homology due to Kronheimer and Mrowka [KM07, Ch. 2] which will serve as a blueprint for the subsequent construction of monopole Floer homology. Following [KM07, Ch. 2.5, p. 31], we consider the following situation:

- ▶ P is a closed $\mathbb{T}$–manifold.
- ▶ $Q = P^{\mathbb{T}}$ is the fixed point set.
- ▶ $B = P/\mathbb{T}$ is the orbit space.
- ▶ We suppose that the $\mathbb{T}$–action is *semi-free* in the sense that $\mathbb{T}$ acts freely on $P \setminus P^{\mathbb{T}}$.

The goal is to obtain a Morse theoretic description of the Borel homology $H_*^{\mathbb{T}}(P)$ using a $\mathbb{T}$–invariant Morse–Smale pair (f, ξ). Let us think about the two extreme cases first:

(1) If $\mathbb{T}$ acts freely on P, that is, if $Q = \emptyset$, then the $P/\mathbb{T}$ is a smooth manifold and (f, ξ) descends to a Morse–Smale pair $(\bar{f}, \bar{\xi})$. We know from Lemma 1.38 that $H_*^{\mathbb{T}}(P) \cong H_*(P/\mathbb{T})$ and the right hand side can be computed from the Morse complex of $(\bar{f}, \bar{\xi})$ by Theorem 1.11.

(2) If $\mathbb{T}$ acts trivially on P, that is, if $P = Q$, then (f, ξ) is just a Morse–Smale pair in the ordinary sense. Since $\mathbb{T}$ acts trivially on $Q = P^{\mathbb{T}}$, we know from Lemma 1.37 that

$$H_*^{\mathbb{T}}(Q) \cong H_*^{\mathbb{T}}(Q \times \mathbb{C}\mathrm{P}^\infty) \cong H_*(Q) \otimes_{\mathbb{Z}} H_*(\mathbb{C}\mathrm{P}^\infty) \tag{4.0.1}$$

and we can at least compute $H_*(Q)$ directly using the Morse complex of (f, ξ).

The intuitive idea is to mix Morse theory on the fixed point set Q and on the orbit space $(P \setminus Q)/\mathbb{T}$, which is always a smooth manifold, but not compact unless $Q = \emptyset$. This is done by passing to an associated manifold with boundary P^σ on which $\mathbb{T}$ acts freely and to set up a notion of Morse homology for manifolds with boundary.

4.1 Morse complexes for manifolds with boundary

Let B be a Riemannian n–manifold with non-empty boundary and ν the outward unit normal field along ∂B. There is a standard approach to compute $H_*(B)$ and $H_*(B, \partial B)$ by Morse theoretic means, in when one considers Morse functions $f \colon B \to \mathbb{R}$ which are constant on ∂B and achieve their maximum or minimum on ∂B, respectively. This is *not* what we will do! Instead, we will work with certain Morse functions on B which also restrict to Morse functions on ∂B (c.f. [KM07, Ch. 2.4]).

We form the *double* $\widetilde{B}$ of B as

$$\widetilde{B} = (B \amalg B)/\sim \tag{4.1.1}$$

where every boundary point in the first copy of B is identified with its other copy in the second factor. A choice of collar for ∂B determines a (reasonably canonical) smooth structure on $\widetilde{B}$ such that the two embeddings of B are smooth. We consider B as a codimensions 0 submanifold using the first summand. The boundary ∂B then becomes the fixed point set of the smooth involution

$$i\colon \widetilde{B} \to \widetilde{B} \tag{4.1.2}$$

that interchanges the two factors. Note that $\widetilde{B}$ is always a closed smooth manifold.

Definition 4.1. Let B be a compact smooth manifold with boundary and $\widetilde{B}$ its double. We consider pairs $(\widetilde{f}, \widetilde{g})$ consisting of an i–invariant Morse function $\widetilde{f}\colon \widetilde{B} \to \mathbb{R}$ and an i–invariant Riemannian metric $\widetilde{g}$ on $\widetilde{B}$. Let (f, g) be the restriction to B and $\xi = \nabla^g f$ the gradient of f with respect to g. We call (f, ξ) a *vertical Morse pair.*

This definition allows f to have critical points on ∂B and we have to be careful The possible critical points of f on ∂B are then in one-to-one correspondence with the i–invariant critical points of $\widetilde{f}$.

Lemma 4.2. *The vector field $\xi = \nabla^g f$ is everywhere tangent to ∂B. Moreover, the restriction $f^\partial = f|_{\partial B}\colon \partial \to \mathbb{R}$ is a Morse function with* $\mathrm{Crit}(f^\partial) = \mathrm{Crit}(f) \cap \partial B$ *and $\xi^\partial = \xi|_{\partial B}$ is its gradient with respect to $g|_{\partial B}$.*

Proof. Let ν be the unit outward normal field along ∂B and $\widetilde{\nu}$ its canonical lift to $\widetilde{B}$. Then $i_*\widetilde{\nu} = -\nu$ and thus

$$\langle \xi, \nu \rangle = df(\nu) = d\widetilde{f}(\widetilde{\nu}) = d(\widetilde{f} \circ i)(\nu) = d\widetilde{f}(i_*\widetilde{\nu}) = -df(\nu) = 0. \tag{4.1.3}$$

Thus ξ is tangent to ∂B, which implies ξ^∂ considered as a vector field on ∂B is the gradient of f^∂ with respect to to $g|_{\partial B}$ so that

$$\mathrm{Crit}(f^\partial) = \mathrm{Crit}(f) \cap \partial B. \tag{4.1.4}$$

A similar computation shows that the Hessian $H_p f(\nu, w)$ at a critical point p of f on ∂B vanishes for $w \in T_p\partial B$. So in terms of the splitting $T_pB = \mathbb{R}\nu \oplus T_p\partial B$, we can write

$$H_p f = \begin{pmatrix} H_p f(\nu,\nu) & 0 \\ 0 & H_p(f^\partial) \end{pmatrix} \tag{4.1.5}$$

which shows that f^∂ is a Morse function. □

Lecture 3, 24.10.23

Based on the lemma, we can partition the set of critical points as follows:

Definition 4.3. We can decompose $\mathrm{Crit}(f)$ into three subsets:

$$\begin{aligned} \mathfrak{c}^o &= \{p \in \mathrm{Crit}(f) \,|\, p \in B \setminus \partial B\} \\ \mathfrak{c}^s &= \{p \in \mathrm{Crit}(f) \,|\, p \in \partial B,\, H_p(\nu,\nu) > 0\} \\ \mathfrak{c}^u &= \{p \in \mathrm{Crit}(f) \,|\, p \in \partial B,\, H_p(\nu,\nu) < 0\}\,. \end{aligned} \tag{4.1.6}$$

Points in $\mathfrak{c}^s$ and $\mathfrak{c}^u$ are called *boundary-stable* and *boundary-unstable*, respectively. For brevity, we henceforth write

$$\mathfrak{c} = \mathrm{Crit}(f) = \mathfrak{c}^o \cup \mathfrak{c}^s \cup \mathfrak{c}^u \quad \text{and} \quad \mathfrak{c}^\partial = \mathrm{Crit}(f^\partial) = \mathfrak{c}^s \cup \mathfrak{c}^u. \tag{4.1.7}$$

From here onward, we shift our focus to the gradient vector field $\xi = \nabla^g f$. After all, we learned last semester (see p. 8) that the classical Floer complexes of Morse–Smale pairs really only depend on the *downward gradient flow* generated by the equation $\dot{x} + \xi(x) = 0$, while the function f merely provides some control and guidance. We should expect the same in the new situation. Recall that we have

$$\mathfrak{c} = \mathrm{Crit}(f) = Z(\xi) = \{p \in M \mid \xi(p) = 0\} \tag{4.1.8}$$

and for a *stationary point* $p \in Z(\xi)$ we saw in an exercise that

$$H_p(v, w) = \langle v, D_p\xi(w)\rangle_g . \tag{4.1.9}$$

where $D_p\xi\colon T_pB \to T_pB$ is the linearization of ξ at p defined in (1.1.19). The latter is a self-adjoint isomorphism and the Morse index $\mu(p)$ is the number of negative eigenvalues of $D_p\xi$ counted with multiplicity.

Corollary 4.4. *Let (f, ξ) be a vertical Morse pair.*

(i) The equation $\dot{x} + \xi(x) = 0$ generates a flow on B, that is, all maximal integral curves are defined on all of $\mathbb{R}$.

(ii) The flow preserves ∂B and restricts to the flow generated by $\dot{x} + \xi^\partial(x) = 0$ on ∂B.

(iii) All flow trajectories $\gamma\colon \mathbb{R} \to B$ have asymptotic limits $\gamma(\infty) = \lim_{t\to\pm\infty} \gamma(t) \in \mathfrak{c}$.

This means that we can define stable and unstable manifolds and moduli spaces of trajectories as before, but we have to pay special attention to the interaction of flow trajectories with the boundary. The first observation is that we can partition flow trajectories as follows:

- Some trajectories γ stay entirely within ∂B and necessarily have limits $\gamma(\pm\infty) \in \mathfrak{c}^\partial$.
- Others stay entirely within the interior $B \setminus \partial B$ and necessarily have limits

$$\gamma(-\infty) \in \mathfrak{c}^o \cup \mathfrak{c}^u \quad \text{and} \quad \gamma(\infty) \in \mathfrak{c}^o \cup \mathfrak{c}^s. \tag{4.1.10}$$

Definition 4.5. Let (f, ξ) be a vertical Morse pair. For $p, q \in \mathfrak{c} = Z(\xi) = \mathrm{Crit}(f)$ let

$$\begin{aligned}
\mu(p) &= \text{index of } p \text{ with respect to } f \text{ (or equivalently } \xi)\\
U_p &= \text{unstable manifold of } p \text{ with respect to } \xi\\
S_q &= \quad \text{stable manifold of } q \text{ with respect to } \xi\\
M(p, q) &= U_p \cap S_q, \text{ moduli space of parameterized trajectories from } p \text{ to } q\\
\hat{M}(p, q) &= M(p, q)/\mathbb{R}, \text{ moduli space of unparameterized trajectories}
\end{aligned}$$

For $p, q \in \mathfrak{c}^\partial = \mathfrak{c} \cap \partial B$, we have analogues defined using $(f^\partial, xi^\partial)$ instead:

$$\mu^\partial(p), \quad U_p^\partial, \quad S_q^\partial, \quad M^\partial(p, q) = U_p^\partial \cap S_p^\partial, \quad \text{and} \quad \hat{M}^\partial(p, q) = M^\partial(p, q)/\mathbb{R}.$$

We make some observations about the relation of the two sets of data for $p \in \mathfrak{c}^\partial$, which follow from Lemma 4.2 and the description of the Hessians in its proof:

Theorem 4.6 (Stable manifold theorem, vertical case)**.** *Let (f, ξ) be a vertical Morse pair on an n–manifold with boundary B.*

(i) If $p \in \mathfrak{c}^o$, then U_p and S_p are smooth submanifolds of $B \setminus \partial B$ of dimensions $\mu(p)$ and $n - \mu(q)$, respectively.

(ii) If $p \in \mathfrak{c}^s$, then $\mu(p) = \mu^\partial(p)$ and

- ▶ $U_p = U_p^\partial$ *is a smooth submanifold of* ∂B *of dimension* $\mu(p)$.
- ▶ S_p *is a smooth submanifold of* B *of dimension* $n-\mu(p)$ *with (possibly empty) boundary* $\partial S_p = S_p^\partial$.

(iii) Similarly, if $p \in \mathfrak{c}^u$*, then* $\mu(p) = \mu^\partial(p) + 1$ *and*

- ▶ U_p *is a smooth submanifold of* B *of dimension* $\mu(p)$ *with (possibly empty) boundary* $\partial U_p = U_p^\partial$.
- ▶ $S_p = S_p^\partial$ *is a smooth submanifold of* ∂B *of dimension* $n - \mu(p)$.

$$U_p = U_p^\partial \tag{4.1.11}$$

This leaves us with a bit of a conundrum, since for $p \in \mathfrak{c}^s$ and $q \in \mathfrak{c}^u$ we have $U_p \subset \partial B$ and $S_q \subset \partial B$, and submanifolds of ∂B can never intersect transversely in B. However, in that case we have

$$M(p,q) = U_p \cap U_q = U_p^\partial \cap U_q^\partial = M^\partial(p,q) \subset \partial B \tag{4.1.12}$$

can never be transverse in B. Nevertheless, they can be transverse in ∂B, and we have

$$M(p,q) = M^\partial(p,q) \subset \partial B \tag{4.1.13}$$

This suggests the following vertical version of the Smale condition in Definition 1.4.

Definition 4.7 (c.f. [KM07, Def. 2.4.2])**.** A vertical Morse pair (f,ξ) is called *regular* or a *vertical Morse–Smale pair*, if for all $p,q \in \mathfrak{c}$ we have

$$\begin{aligned} &S_p \pitchfork U_q \quad \text{in } \partial B \quad \text{if } p \in \mathfrak{c}^s \text{ and } q \in \mathfrak{c}^u,\\ &S_p \pitchfork U_q \quad \text{in } B \quad\ \ \text{otherwise.}\end{aligned} \tag{4.1.14}$$

Pairs $p \in \mathfrak{c}^s$ and $q \in \mathfrak{c}^u$ as above are called *boundary-obstructed*.

The following is clear from the definition:

Lemma 4.8. *Let* (f,ξ) *be a vertical Morse–Smale pair. If* $p,q \in \mathfrak{c}$*, then* $M(p,q)$ *is a smooth manifold of dimension*

$$\dim M(p,q) = \begin{cases} \mu(p) - \mu(q) + 1, & p \in \mathfrak{c}^s \text{ and } q \in \mathfrak{c}^u \text{ (boundary-obstructed)}\\ \mu(p) - \mu(q), & \text{otherwise.}\end{cases} \tag{4.1.15}$$

If $p,q \in \mathfrak{c}^\partial$*, then* $M^\partial(p,q)$ *is a smooth manifold of dimension*

$$\dim M^\partial(p,q) = \begin{cases} \mu(p) - \mu(q) + 1, & p \in \mathfrak{c}^s \text{ and } q \in \mathfrak{c}^u \text{ (boundary-obstructed)}\\ \mu(p) - \mu(q) - 1, & p \in \mathfrak{c}^u \text{ and } q \in \mathfrak{c}^s\\ \mu(p) - \mu(q), & \text{else (i.e. if } p,q \in \mathfrak{c}^s \text{ or } p,q \in \mathfrak{c}^u\text{).}\end{cases} \tag{4.1.16}$$

Proof. The vertical Smale condition guarantees that all moduli spaces are manifolds. In the boundary obstructed case we have $M(p,q) = M^\partial(p,q)$. In all other cases, the formula for $\dim M(p,q)$ follows, since U_p has dimension $\mu(p)$ and S_q has codimension $\mu(q)$ in B. For $p,q \in \mathfrak{c}^\partial$ we have $\dim M^\partial(p,q) = \mu^\partial(p) - \mu^\partial(q)$. In the boundary obstructed case when $p \in \mathfrak{c}^s$ and $q \in \mathfrak{c}^u$, we have $\mu^\partial(p) = \mu(p)$ and $\mu^\partial(q) = \mu(q) - 1$, which implies the dimension formula in that case. The other cases are similar. □

We also have the following finiteness theorem for 0-dimensional moduli spaces:

Proposition 4.9. *Let* (f,ξ) *be a vertical Morse–Smale pair.*

(i) If $p, q \in \mathfrak{c}$ and $\dim M(p,q) = 1$, then $\hat{M}(p,q)$ is a finite set.

(ii) If $p, q \in \mathfrak{c}^\partial$ and $\dim M^\partial(p,q) = 1$, then $\hat{M}^\partial(p,q)$ is a finite set.

Proof. This follows from Proposition 1.6, the corresponding finiteness result in the horizontal case, applied to f^∂ and $\widetilde{f}$. □

We now have all ingredients to build Floer-style chain complexes. At this point, we ask two questions:

(Q1) What can we define by counting points in 0–dimensional moduli spaces?

(Q2) How does that help us?

Again, we work mod 2 to avoid the discussion of orientations.

Definition 4.10. Let (f, ξ) be a vertical Morse–Smale pair on B. We define

$$\begin{aligned} n(p,q) &= \#_2\ \hat{M}(p,q)\ \in \mathbb{Z}_2 \\ \bar{n}(p,q) &= \#_2 \hat{M}^\partial(p,q) \in \mathbb{Z}_2 \end{aligned} \tag{4.1.17}$$

whenever the moduli spaces are 0–dimensional and $n(p,q) = 0 = \bar{n}(p,q)$ otherwise.

For $\alpha \in \{o, s, u\}$ we let C^α be the $\mathbb{Z}_2$–vector space generated by $\mathfrak{c}^\alpha$ and write C^α_k for the subspace generated by the points $p \in \mathfrak{c}^\alpha$ with $\mu(p) = k$. The point counts $n(p,q)$ and $\bar{n}(p,q)$ give rise to linear maps

$$\partial^\alpha_\beta \colon C^\alpha \to C^\beta, \quad \partial\,\langle p\rangle = \sum_{q \in \mathfrak{c}^\beta} n(p,q)\,\langle q\rangle \tag{4.1.18}$$

$$\bar{\partial}^\alpha_\beta \colon C^\alpha \to C^\beta, \quad \bar{\partial}\,\langle p\rangle = \sum_{q \in \mathfrak{c}^\beta} \bar{n}(p,q)\,\langle q\rangle \tag{4.1.19}$$

for all combinations $\alpha, \beta \in \{o, s, u\}$ that make sense. Taking gradings into account, we have defined eight maps:

$$\begin{aligned} &\partial^o_o \colon C^o_k \to C^o_{k-1} && \partial^s_s = \bar{\partial}^s_s \colon C^s_k \to C^s_{k-1} \\ &\partial^o_s \colon C^o_k \to C^s_{k-1} && \partial^u_u = \bar{\partial}^u_u \colon C^u_k \to C^u_{k-1} \\ &\partial^u_o \colon C^u_k \to C^o_{k-1} && \partial^s_u = \bar{\partial}^s_u \colon C^s_k \to C^u_k \\ &\partial^u_s \colon C^u_k \to C^s_{k-1} && \bar{\partial}^u_s \colon C^u_k \to C^s_{k-2} \end{aligned}$$

Out of these linear maps we will eventually obtain Floer complexes computing the homology sequence of the pair (B, ∂). A few observations are in order:

- There are no maps $C^s \to C^o$ and $C^o \to C^u$, because nothing can flow from $\mathfrak{c}^s$ into the interior or from the interior into $\mathfrak{c}^u$.
- The coincidences $\partial^s_u = \bar{\partial}^s_u$, $\partial^s_s = \bar{\partial}^s_s$, and $\partial^u_u = \bar{\partial}^u_u$ hold, because $M(p,q) = M^\partial(p,q)$ in those cases.
- There are two maps $\partial^u_s, \bar{\partial}^u_s \colon C^u \to C^s$, because the moduli spaces $M(p,q)$ and $M^\partial(p,q)$ are not the same in that case.
- Most of the maps ∂^α_β and $\bar{\partial}^\alpha_\beta$ decrease the index by 1, as expected. However, the maps $\partial^s_u = \bar{\partial}^s_u$ and $\bar{\partial}^u_s$ behave unexpectedly.
- The peculiar behavior of ∂^s_u is not surprising, as the map counts precisely those trajectories that violate the ordinary Smale condition.

As a first step, we recognize the classical Floer complex of the pair $(f^\partial, \xi^\partial)$. Indeed, we find

$$C_k(f^\partial, \xi^\partial) = C_k^s \oplus C_{k+1}^u =: \bar{C}_k \tag{4.1.20}$$

and the usual Floer differential is given by

$$\bar{\partial}\colon \bar{C}_k \to \bar{C}_{k-1}, \quad \bar{\partial}\,\langle p\rangle = \sum_q \bar{n}(p,q)\,\langle q\rangle \tag{4.1.21}$$

which can be rewritten as

$$\bar{\partial} = \begin{pmatrix} \bar{\partial}_s^s & \bar{\partial}_s^u \\ \bar{\partial}_u^s & \bar{\partial}_u^u \end{pmatrix}. \tag{4.1.22}$$

As a consequence of Theorem 1.11, we get:

Lemma 4.11. *We have $\bar{\partial}\bar{\partial} = 0$ and $H_*(\bar{C}, \bar{\partial}) \cong H_*(\partial B)$.*

Now that we have managed to compute $H_*(\partial B)$ using the data (f, ξ) on B, it remains to find $H_*(B)$ and $H_*(B, \partial B)$. This turns out to be rather annoying, but possible. Before moving on, let us recall that how have proved $\bar{\partial}\bar{\partial} = 0$ in Proposition 1.9:

- ▶ We studied 2–dimensional moduli spaces $M(p,q)$ and noticed that sequences of trajectories therein may split into what we called *broken trajectories* in the limit.
- ▶ We noticed that the quotients $\hat{M}(p,q)$ have compactifications $\bar{M}(p,q)$ obtained by adding broken trajectories which are compact 1-dimensional manifolds with boundary (see Theorem 1.7).
- ▶ We noticed that the matrix entries of $\bar{\partial}\bar{\partial}$ count points in $\partial\bar{M}(p,q)$.

Alternatively, we could have proved $\bar{\partial}\bar{\partial} = 0$ by relating $(f^\partial, \xi^\partial)$ to a cell (or handle) decomposition of ∂B and arguing that the Floer differential agrees with the cellular differential. The key to this approach is to exhaust ∂B by sub-level sets of $\{f^\partial \le a\}$, $a \in \mathbb{R}$, and studying the effect of passing critical levels. It is instructive, to play this through for f and B. For simplicity, we assume that f is injective on $\mathfrak{c}$ so that each critical level contains exactly one critical point. By drawing 2–dimensional pictures, we can get an idea how the topology of the sub-level sets changes when crossing critical levels. With some effort the following table can be made precise:

type and index	effect on B	effect on ∂B
$\mathfrak{c}_0^o$	0–cell	—
$\mathfrak{c}_0^s$	0–cell	0–cell
$\mathfrak{c}_1^o$	1–cell	—
$\mathfrak{c}_1^s$	1–cell	1–cell
$\mathfrak{c}_1^u$	—	0–cell
$\vdots$		
$\mathfrak{c}_k^o$	k–cell	—
$\mathfrak{c}_k^s$	k–cell	k–cell
$\mathfrak{c}_k^u$	—	$(k-1)$–cell
$\vdots$		
$\mathfrak{c}_{n-1}^o$	$(n-1)$–cell	—
$\mathfrak{c}_{n-1}^s$	$(n-1)$–cell	$(n-1)$–cell
$\mathfrak{c}_{n-1}^u$	—	$(n-2)$–cell
$\mathfrak{c}_n^o$	n–cell	—
$\mathfrak{c}_n^u$	—	$(n-1)$–cell

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This suggests the following:

- $\bar{C}_k = C^s_k \oplus C^u_{k+1}$ should support a Floer-style differential $\hat{\partial}$ such that $(\bar{C}, \bar{\partial})$ is isomorphic to a cellular chain complex which computes $H_*(\partial B)$ This we already know.
- $\check{C}_k = C^o_k \oplus C^s_k$ should support a Floer-style differential $\check{\partial}$ such that $(\check{C}, \check{\partial})$ is isomorphic to a cellular chain complex which computes $H_*(\partial B)$
- There should also be a chain map $\bar{C} \to \check{C}$ inducing the map $H_*(\partial B) \to H_*(B)$.

We begin by writing down the chain complexes that will eventually do the job.

Definition 4.12 (c.f. [KM07, 2.4.4 & 22.2.1])**.** Let (f, ξ) be a vertical Morse–Smale pair on B. In addition to $(\bar{C}, \bar{\partial})$ (*"C–bar"*), we consider the graded $\mathbb{Z}_2$–vector spaces $\check{C}$ (*"C-to"*) and $\hat{C}$ (*"C–from"*) given by

$$\check{C}_k = C^o_k \oplus C^s_k \quad \text{and} \quad \hat{C}_k = C^o_k \oplus C^u_k \tag{4.1.23}$$

together with the following diagram of linear maps

$$\begin{array}{ccccccc} \hat{C}_{k+1} & \xrightarrow{p} & \bar{C}_k & \xrightarrow{i} & \check{C}_k & \xrightarrow{j} & \hat{C}_k \\ \downarrow{\scriptstyle \hat{\partial}} & & \downarrow{\scriptstyle \bar{\partial}} & & \downarrow{\scriptstyle \check{\partial}} & & \downarrow{\scriptstyle \hat{\partial}} \\ \hat{C}_k & \xrightarrow{p} & \bar{C}_{k-1} & \xrightarrow{i} & \check{C}_{k-1} & \xrightarrow{j} & \hat{C}_{k-1} \end{array} \tag{4.1.24}$$

defined by the matrices

$$\hat{\partial} = \begin{pmatrix} \partial^o_o & \partial^u_o \\ -\bar{\partial}^s_u & \partial^o_s - \bar{\partial}^s_u \partial^u_s \end{pmatrix}, \qquad \check{\partial} = \begin{pmatrix} \partial^o_o & -\partial^u_o \bar{\partial}^s_u \\ \partial^o_s & \bar{\partial}^s_s - \partial^u_s \bar{\partial}^s_u \end{pmatrix}, \qquad \bar{\partial} = \begin{pmatrix} \bar{\partial}^s_s & \bar{\partial}^u_s \\ \bar{\partial}^s_u & \bar{\partial}^u_u \end{pmatrix} \tag{4.1.25}$$

$$i = \begin{pmatrix} 0 & -\partial^u_o \\ 1 & -\partial^u_s \end{pmatrix}, \qquad j = \begin{pmatrix} 1 & 0 \\ 0 & -\partial^s_u \end{pmatrix}, \qquad p = \begin{pmatrix} \partial^o_s & \partial^u_s \\ 0 & 1 \end{pmatrix}. \tag{4.1.26}$$

Here's the punchline:

Theorem 4.13 (c.f. [KM07, 2.4.5 & 22.2.1])**.** *We have $\check{\partial}\check{\partial} = 0$ and $\hat{\partial}\hat{\partial} = 0$ and the diagram (4.1.24) commutes.*[1] *Furthermore, there are isomorphisms that make the following diagram commute:*

$$\begin{array}{ccccccc} H_{k+1}(\hat{C}, \hat{\partial}) & \xrightarrow{p_*} & H_k(\bar{C}, \bar{\partial}) & \xrightarrow{i_*} & H_k(\check{C}, \check{\partial}) & \xrightarrow{j_*} & H_k(\hat{C}, \hat{\partial}) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ H_{k+1}(B, \partial B) & \longrightarrow & H_k(\partial B) & \longrightarrow & H_k(B) & \longrightarrow & H_k(B, \partial B). \end{array}$$

Proof (sketch). The proof has three steps:

(1) Proving the identities $\check{\partial}^2 = 0$, $\hat{\partial}^2 = 0$, etc.

(2) Proving that of $H_*(\check{C})$ and $H_*(\hat{C})$ and the maps i_*, j_*, p_* are independent of (f, ξ).

(3) Identifying the homology groups and maps.

As in the standard case in Proposition 1.9, the identities in (1) can be proved by studying compactifications of 1–dimensional unparameterized moduli spaces $\breve{M}(p, q)$. The main difference is that trajectories can break more than once:

[1] The left square only commutes, since we are working mod 2. For integer coefficients, we have $\bar{\partial}p = -p\hat{\partial}$ while i and j are honest chain maps.

Lemma 4.14 (cf. [KM07, 2.4.3]). *Let $p, q \in \mathfrak{c}^o$ be interior stationary points of ξ with $\mu(p) = k$ and $\mu(q) = k - 2$. Then $\hat{M}(p,q)$ has a compactification $\bar{M}(p,q)$ obtained by adding broken trajectories from p to q. Every strictly broken trajectory in $\bar{M}(p,q)$ has either two or three components and takes form*

$$(\gamma_1, \gamma_2) \in \hat{M}(p,r) \times \hat{M}(r,q) \tag{4.1.27}$$

with $r \in \mathfrak{c}^o$ with $\mu(r) = k - 1$ or

$$(\gamma_1, \gamma_2, \gamma_3) \in \hat{M}(p,r_1) \times \hat{M}(r_1,r_2) \times \hat{M}(r_2,q) \tag{4.1.28}$$

with $r_1 \in \mathfrak{c}^s$ and $r_2 \in c^u$ is a boundary-obstructed pair. Furthermore, the number of strictly broken trajectories in $\bar{M}(p,q)$ is finite and even.

The independence of (f, ξ) in (2) not a trivial task, but it can be proved by adapting the arguments for the standard case (see [Jos17, Thm. 7.9.3]). Once (2) is established, one can make special choices for (f, ξ) such that the complexes $\check{C}$ and $\hat{C}$ behave like standard Morse complexes with ∂B is horizontal and the function takes a maximum or minimum, respectively. □

4.2 The blow-up construction for semi-free $\mathbb{T}$–actions

We now go back to semi-free circle actions. Recall the setup from the beginning of this chapter:

- P is a closed $\mathbb{T}$–manifold with semi-free action (i.e. $\mathbb{T}$ acts freely on $P \setminus Q$)
- $\tilde{g}$ is a $\mathbb{T}$–invariant Riemannian metric on P.
- $Q = P^{\mathbb{T}}$ is the fixed point set; we assume $Q \neq \emptyset$
- $B = P/\mathbb{T}$ is the orbit space.
- $q \colon P \to B$ is the orbit map
- $\bar{Q} = q(Q)$ is the image of the fixed points

The goal is to describe the Borel homology of P using $\mathbb{T}$–equivariant Morse pairs (f, ξ). The strategy is to pass to a manifold with boundary on which $\mathbb{T}$ acts freely so that $H_*^{\mathbb{T}}$ reduces to the ordinary homology of the quotient.

Disclaimer: This section was written hastily and is therefore a little terse.

We proceed a several step.

(1) Let $N \subset D(N) \subset S(N)$ be the normal bundle of $Q \subset P$. Since the $\mathbb{T}$ action is semi-free, it induces a complex structure on N. In particular, Q has even codimension in P, say $2k$.

(2) The exponential map for $\tilde{g}$ gives a $\mathbb{T}$–equivariant tube embedding

$$\tau \colon (N, Q) \hookrightarrow (P, Q) \tag{4.2.1}$$

The complement $P \setminus Q$ is non-compact with one (topologically) cylindrical end which we can parameterize by

$$\tau_0 \colon (0, \epsilon) \times S(N) \to P \setminus Q, \quad \tau_0(t, v) = \tau(tv). \tag{4.2.2}$$

(3) **The oriented blow-up:** We define the *oriented blow-up* of P along Q as

$$P^\sigma = \big([0,\varepsilon) \times S(N)\big) \cup_{\tau_0} (P \setminus Q). \tag{4.2.3}$$

This is a compact manifold with boundary $\partial P^\sigma \cong S(N)$ and the $\mathbb{T}$ action on $P \setminus Q$ extends canonically to a free $\mathbb{T}$–action on P^σ.

(4) **The blow-down maps:** The orbit space $B^\sigma = P^\sigma/\mathbb{T}$ is a smooth manifold with boundary $\partial B^\sigma \cong \mathbb{P}(N)$, the projectivization of N. We have a commutative diagram of $\mathbb{T}$-pairs

$$\begin{array}{ccc} (P^\sigma, \partial P^\sigma) & \xrightarrow{\pi} & (P, Q) \\ \big\downarrow{\scriptstyle q^\sigma} & & \big\downarrow{\scriptstyle q} \\ (B^\sigma, \partial B^\sigma) & \xrightarrow{\bar\pi} & (B, \bar Q) \end{array}$$

where the orbit map $q^\sigma\colon \partial P^\sigma \to \partial B^\sigma$ corresponds to $S(N) \to \mathbb{P}(N)$, and $\bar\pi\colon \partial B^\sigma \to \bar Q$ corresponds to the bundle projection $\mathbb{P}(N) \to Q$.

(5) **Blowing up gradients:** Let $\tilde f\colon P \to \mathbb{R}$ be a $\mathbb{T}$–invariant smooth function and $\tilde\xi = \nabla \tilde f$ its gradient with respect to $\tilde g$.

- According to [KM07, 2.5.2], the restriction of $\tilde\xi$ to $P \setminus Q$ extends to a smooth $\mathbb{T}$–invariant vector field $\tilde\xi^\sigma$ on P^σ which is everywhere tangent to ∂P^σ.
- $\tilde\xi^\sigma$ further descends to a smooth vector field ξ^σ on B^σ which is everywhere tangent to ∂B^σ

From here on, the idea is to do non-equivariant Floer theory for ξ^σ on B^σ, assuming the usual types of regularity conditions, and to related the results back to $H_*^{\mathbb{T}}(P)$.

(6) **The flow of $\boldsymbol{\xi^\sigma}$:** As noted in [KM07, p. 34], ξ^σ is not a gradient in any natural way. However, it behaves like one:

- The equation $\dot x + \xi^\sigma(x) = 0$ generates a complete flow on B^σ.
- All trajectories have asymptotic limits in the set $\mathfrak{c} = Z(\xi^\sigma) \subset B^\sigma$ of stationary points.
- For $p \in \mathfrak{c}$ the linearization $D_p\xi\colon T_pB^\sigma \to T_pB^\sigma$ has only real eigenvalues.
- The *index* $\mu(p)$ is the number of negative eigenvalues.
- We say that $p \in \mathfrak{c}$ is *non-degenerate* if $D_p\xi^\sigma$ is an isomorphism.
- The vertical stable manifold theorem Theorem 4.6 holds for all non-degenerate $p \in \mathfrak{c}$.

($6\frac{1}{2}$) **The flow of $\boldsymbol{\xi^\sigma}$:** It helps to take a closer look at the blown-up vector field ξ^σ along the boundary. Recall from (4) that $\partial B^\sigma \cong \mathbb{P}(N)$.

- A point $p \in \partial B^\sigma$ correspond to $(q, \mathbb{C}\phi) \in \mathbb{P}(N)$ with $q \in Q$ and $\phi \in S(N_q)$ and we have a splitting
$$T_pB^\sigma = T_p\partial B^\sigma \oplus \mathbb{R} \cong T_pQ \oplus \langle\phi\rangle^\perp \oplus \mathbb{R} \tag{4.2.4}$$
where $\langle\phi\rangle^\perp \subset N_q$ is the complex orthogonal complement with respect to the Hermitian metric on N_q given by $\tilde h(v,w) = \tilde g(v,w) - i\tilde g(v, iw)$
- The metric on P and the $\mathbb{T}$–invariance of $\tilde\xi$ give a $\mathbb{T}$–equivariant $(\nabla\tilde\xi)_q\colon T_qP \to T_qP$ which preserves N_q and thus gives a linear operator
$$L_q := (\nabla\tilde\xi)_q|_{N_q}\colon N_q \to N_q. \tag{4.2.5}$$
- Projecting further onto $\langle\phi\rangle^\perp$ gives an operator
$$\mathbb{L}_q\colon N_q \to \langle\phi\rangle^\perp, \quad \mathbb{L}_q(\psi) = L_q\psi - \langle\phi, L_q\psi\rangle_{\tilde h}\,\phi \tag{4.2.6}$$

- ▶ Using the splitting (4.2.4) the vector field ξ^σ on ∂B^σ can then be described as

$$\xi^\sigma(p) = (\tilde{\xi}(q), \mathbb{L}_q\phi, 0) \in T_pQ \oplus \langle\phi\rangle^\perp \oplus \mathbb{R}. \tag{4.2.7}$$

- ▶ If $\tilde{\xi}(p) = 0$, then $L_q = D_q\tilde{\xi}$ is self-adjoint and $\mathbb{L}_q\phi = L_q\phi - \langle\phi, L_q\phi\rangle_{\tilde{g}}\,\phi$. In particular, we find

$$\xi^\sigma(p) = 0 \quad\Leftrightarrow\quad \begin{cases} \tilde{\xi}(q) = 0 \text{ and} \\ \phi \in N_q \text{ is an eigenvector of } D_q\tilde{\xi} \end{cases} \tag{4.2.8}$$

- ▶ Lastly, the solutions of $\dot{x} + \xi^\sigma(x) = 0$ are the images of solutions of $\dot{\tilde{x}} + \tilde{\xi}^\sigma(\tilde{x}) = 0$ and writing $\tilde{x} = (q, \phi)$ using the identification $\partial P^\sigma \cong S(N)$ we get

$$\dot{\tilde{x}} + \tilde{\xi}^\sigma(\tilde{x}) = 0 \quad\Leftrightarrow\quad \begin{cases} \dot{q} + \tilde{\xi}(q) = 0 \\ (q^*\nabla)\phi + \mathbb{L}_q\phi = 0. \end{cases} \tag{4.2.9}$$

[Update (7.11.23): This part was added later.]

(7) **Equivariant Morse–Smale gradients:** To proceed, we make stronger assumptions on the function $\widetilde{f}$ on P.

- ▶ $\widetilde{f}$ is a $\mathbb{T}$–Morse function (i.e. its Hessian is non-degenerate normal to critical orbits)
- ▶ $\widetilde{f}|_Q$ is a Morse function in the ordinary sense
- ⇝ As a consequence, $\mathfrak{c} = Z(\xi^\sigma)$ is finite and all stationary point are non-degenerate.
- ▶ We require the vertical Smale condition from Definition 4.7 to hold for ξ^σ.
- ▶ For $q \in Q$ with $\tilde{x}(q) = 0$ we require that $L_q\colon N_q \to N_q$ has a complex basis of eigenvectors $\phi_1(q), \dots, \phi_k(q)$ with eigenvalues

$$\lambda_1(q) < \lambda_2(q) < \cdots < \lambda_k(q). \tag{4.2.10}$$

[Update (7.11.23): This conditions was previously stated incorrectly.]

In that case, we call ξ^σ **regular**.

[*Note:* All these assumptions can be arranged by careful choices of $\widetilde{f}$ and $\widetilde{g}$.]

(8) **Floer complexes for regular ξ^σ:** Assuming that ξ^σ is regular, we can apply the theory from Section 4.1 to obtain Floer complexes $\hat{C}$, $\bar{C}$, and $\check{C}$ which compute the homology sequence of the pair $(B^\sigma, \partial B^\sigma)$.

[*Note:* It no problem that ξ^σ does not arise as the gradient of a Morse function. The only thing that matters is that the structure of moduli spaces of trajectories is the same. And this is the case here.]

($7\frac{1}{2}$) **Comparing the indices:** Assuming that ξ^σ is regular with $\tilde{f}$ as in (7), it is a natural question how the indices of stationary points of ξ^σ are related to $\widetilde{f}$.

- ▶ Interior stationary points of ξ^σ correspond to critical $\mathbb{T}$–orbits of $\tilde{f}$ in $P \setminus Q$ and the index of the with respect to ξ^σ agrees with the index of the Hessian of $\widetilde{f}$ normal to the critical orbit.
- ▶ According to ($6\frac{1}{2}$) and (7), we see that the stationary points on the boundary have the form $p = (q, [\phi_i(q)])$ and the index is given by

$$\mu(p) = \begin{cases} \mu^Q(q) + 2i - 2, & \text{if } \lambda_i(q) > 0 \\ \mu^Q(q) + 2i - 1, & \text{if } \lambda_i(q) < 0 \end{cases} \tag{4.2.11}$$

where $\mu^Q(q)$ is the index of q as a critical point of $\tilde{f}|_Q$ (cf. [KM07, Lem.2.5.5]).

At interior stationary points of ξ^σ, the index with respect to to ξ^σ – that is, the number of negative eigenvalues of $D_p\xi^\sigma$ – agrees with the number of negative eigenvalues of $D_{\tilde{p}}\tilde{\xi} = H_{\tilde{p}}\tilde{f}$ restricted to the normal bundle of the corresponding crtivial orib

[Update (7.11.23): This part was added later.]

(9) **Relation to Borel homology:** The next task is to relate the complexes $\hat{C}$, $\bar{C}$, and $\check{C}$ to the Borel homology sequence of the pair (P, Q).

(10) **Identifying $H_*(\hat{C})$:** The homology of the complex $\hat{C}$ can be identified as follows:

$$H_*^{\mathbb{T}}(P, Q) \xleftarrow[\cong]{\pi_*} H_*^{\mathbb{T}}(P^\sigma, \partial P^\sigma) \xrightarrow[\cong]{} H_*(B^\sigma, \partial B^\sigma) \xrightarrow[(10)]{\cong} H_*(\hat{C}).$$

The first isomorphism follows from excision, the second from the freeness of the action, and the last one is part of Theorem 4.13.

(11) **Identifying $H_*(\bar{C})$:** We know from (10) that $H_*(\bar{C}) \cong H_*(\partial B^\sigma)$. Recall from (4) that $\partial B^\sigma \xrightarrow{\pi_*}$ is a $\mathbb{C}\mathrm{P}^{k-1}$–bundle and as such isomorphic to $\mathbb{P}(N) \to Q$. Using the Leray–Hirsch theorem (LH), one can construct a commutative diagram

$$\begin{array}{ccccc} H_*^{\mathbb{T}}(\partial P^\sigma) & \xrightarrow[\mathbb{T}\text{–free}]{\cong} & H_*(\partial B^\sigma) & \xrightarrow[(LH)]{\cong} & H_*(\mathbb{C}\mathrm{P}^{k-1}) \otimes H_*(Q) \\ \downarrow{\scriptstyle \pi_*} & & & & \downarrow{\scriptstyle \mathrm{incl}_* \times \mathrm{id}} \\ H_*^{\mathbb{T}}(Q) & \xrightarrow{\cong} & H_*(\mathbb{C}\mathrm{P}^\infty \times Q) & \xrightarrow{\cong} & H_*(\mathbb{C}\mathrm{P}^\infty) \otimes H_*(Q). \end{array}$$

The vertical map on the right hand side is an isomorphism in degrees $\leq 2k-2$, so that

$$H_{\leq 2k-2}(\bar{C}) \cong H_{\leq 2k-2}(\partial B^\sigma) \cong H_{\leq 2k-2}^{\mathbb{T}}(Q). \tag{4.2.12}$$

We can therefore consider $H_*(\bar{C})$ as an approximation to $H_*^{\mathbb{T}}(Q)$, the Borel homology of the fixed points.

(12) **Connectivity of $(P, P \setminus Q)$:** The identification of $H_*(\check{C})$ requires a detour. As noted in (1), Q has codimension $2k$ in P. It follows that the pair $(P, P \setminus Q)$ is non-equivariantly $(2k-1)$–connected, that is, the map

$$\pi_i(P \setminus Q) \to \pi_i(P) \quad \text{is} \quad \begin{cases} \text{an isomorphism} & \text{for } i < 2k-1 \\ \text{surjective} & \text{for } i = 2k-1 \end{cases} \tag{4.2.13}$$

This follows from transversality.

(13) **Connectivity of Borel constructions:** Recall that $P_{h\mathbb{T}} = E\mathbb{T} \times_{\mathbb{T}} P$ is a fiber bundle over $B\mathbb{T}$ with model fiber P, and similarly for $P \setminus Q$. Using (10) and the homotopy sequences of these fiber bundles, one can show that the pair $(P_{h\mathbb{T}}, (P \setminus Q)_{h\mathbb{T}})$ is also $(2k-1)$–connected. It follows that

$$H_i^{\mathbb{T}}(P, P \setminus Q) = H_i(P_{h\mathbb{T}}, (P \setminus Q)_{h\mathbb{T}}) = 0, \quad \text{for } i \leq 2k-1. \tag{4.2.14}$$

(14) **Identifying $H_*(\check{C})$:** Lastly, using (13) we can relate $H_*(\check{C})$ to $H_*^{\mathbb{T}}(P)$ again in a range:

$$H_{\leq 2k-2}^{\mathbb{T}}(P) \xleftarrow[(13)]{\cong} H_{\leq 2k-2}^{\mathbb{T}}(P \setminus Q) \xrightarrow{\cong} H_{\leq 2k-2}^{\mathbb{T}}(P^\sigma) \xrightarrow{\cong} H_{\leq 2k-2}(B^\sigma) \cong H_{\leq 2k-2}(\check{C}). \tag{4.2.15}$$

Again, we view $H_*(\check{C})$ as an approximation to $H_*^{\mathbb{T}}(P)$.

(15) **Stabilizing to raise the codimension:** There is a trick to increase the codimension of the fixed point set. Following [KM07, p. 42], we let Lecture 5, 7.11.23

$$P_{(r)} = P \times \mathbb{C}^r \quad \text{and} \quad \widetilde{f}_{(r)} = \widetilde{f} + \sum_{i=1}^{r} \mu_i |z_i|^2 \tag{4.2.16}$$

with real $0 < \mu_1 < \mu_2 < \dots$ and $\mu_1 > \max_{p \in \mathfrak{c}} \lambda_n(p)$.

- ▶ We can form $B^\sigma_{(r)}$ and $\xi^\sigma_{(r)}$ as before.
- ▶ Although $B^\sigma_{(r)}$ is non-compact for $r > 0$, the vector field $\xi^\sigma_{(k)}$ generates a complete flow which is sufficiently regular and has finitely many critical points.
- ▶ One can define complexes $\hat{C}_{(r)}$, $\bar{C}_{(r)}$, and $\check{C}_{(r)}$ which compute $H^{\mathbb{T}}_*(P, Q)$, $H^{\mathbb{T}}_*(Q)$, and $H^{\mathbb{T}}_*(P)$, the latter two in the increased range $* \leq 2(r + k - 1)$.
- ▶ Lastly, one can argue that there are chain inclusions $\check{C}_{(r)} \hookrightarrow \check{C}_{(r+1)}$ and similarly for the other flavors.
- ▶ One can thus form limit complexes $\hat{C}_{(\infty)}$, $\bar{C}_{(\infty)}$, and $\check{C}_{(\infty)}$ which compute the Borel homology sequence of the pair (P, Q).

(16) **The module structure:** Recall that $H^*(B\mathbb{T}) \cong H^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathbb{Z}_2[u]$ acts on all Borel homology groups via the ordinary cap product. The action of u can be realized by chain maps on the complexes $\hat{C}_{(r)}$, $\bar{C}_{(r)}$, and $\check{C}_{(r)}$, at least in suitable ranges of degrees. The details can be looked up in [LM18, Ch. 2.7].

Chapter 5

Monopole Floer homology

Throughout this chapter, we fix the following data:

- Y is a closed, connected, oriented, Riemannian 3–manifold
- (S, ρ) is a spinor bundle for Y representing a spin^c structure $\mathfrak{t} \in \mathrm{Spin}^c(Y)$
- $B_0 \in \mathcal{A}(S)$ is a fixed spin^c connection on S
- $y_0 \in Y$ is a base point

5.1 Outline of the construction

As mentioned, the Floer complexes for semi–free $\mathbb{T}$–actions serve as a blueprint for the construction of monopole Floer homology. The naive idea is that the Seiberg–Witten vector field

$$\mathcal{X}\colon \mathcal{C}_0(Y) \to \mathcal{C}_0(Y), \quad \mathcal{X}(b, \phi) = \begin{pmatrix} *db + \rho^{-1}(\phi\phi)_0 + *\frac{1}{2}F_{B_0^t} \\ D\phi + \rho(b)\phi \end{pmatrix} \tag{5.1.1}$$

shares sufficiently many properties with $\mathbb{T}$–equivariant Morse–Smale gradients after choosing sufficient perturbations and passing to a gauge subquotient. Let us try to make this a little more precise.

The CSD functional. Recall that the configuration spaces are given by

$$\mathcal{C}_0(Y) = \mathcal{A}(S) \times \Gamma(S) \quad \text{and} \quad \mathcal{C}(Y) = i\Omega^1(Y) \oplus \Gamma(S) \tag{5.1.2}$$

and are identified via $(b, \psi) \mapsto (B_0 + b, \psi)$. We have already seen that the CSD functional

$$\mathcal{L}\colon \mathcal{C}_0 \to \mathbb{R}, \quad \mathcal{L}(b, \psi) = \frac{1}{2}\langle d, *db\rangle + \frac{1}{2}\langle \psi, D_b\psi\rangle \tag{5.1.3}$$

descends to a (generally circle valued) function

$$\bar{\mathcal{L}}\colon \mathcal{B}(Y) = \mathcal{C}(Y)/\mathcal{G}(Y) \cong \mathcal{C}_0(Y)/\mathcal{G}(Y) \to \mathbb{R}/\mathfrak{d}(\mathfrak{t})\mathbb{Z}. \tag{5.1.4}$$

where $\mathfrak{d}(\mathfrak{t})\mathbb{Z} = 2\pi^2 \left\langle H^1(Y, \mathbb{Z}) \cup c_1(\mathfrak{t}), [Y]\right\rangle \subset 2\pi^2 Z$. Since the $\mathcal{G}(Y)$–action does not have constant stabilizers, the orbit space $\mathcal{B}(Y)$ cannot be a smooth manifold in any natural way. So we cannot work on $\mathcal{B}(Y)$ directly.

Adding a base point. Let $y_0 \in Y$ be a base point. We consider the following subgroups of the gauge group:

$$\mathcal{G}^h(Y) = \{u \in \mathcal{G}(Y) \,|\, d^*(u^{-1}du) = 0\} \qquad \textit{(“harmonic gauge group”)} \qquad (5.1.5)$$

$$\mathcal{G}_*(Y) = \{u \in \mathcal{G}(Y) \,|\, u(y_0) = 1\} \qquad \textit{(“based gauge group”)} \qquad (5.1.6)$$

$$\mathcal{G}^h_*(Y) = \mathcal{G}^h(Y) \cap \mathcal{G}_*(Y) \qquad \textit{(“based harmonic gauge group”)} \qquad (5.1.7)$$

$$\mathcal{G}^\perp(Y) = \exp(i\Omega^0_0(Y)) \qquad \textit{(“unnamed gauge group”)} \qquad (5.1.8)$$

Lemma 5.1. *The choice of a base point $y_0 \in Y$ gives rise to product splittings*

$$\mathcal{G}(Y) = \mathbb{T} \times \mathcal{G}_*(Y) = \mathbb{T} \times \mathcal{G}^h_*(Y) \times \mathcal{G}^\perp(Y) \qquad (5.1.9)$$

where $\mathbb{T}$ denotes the constant gauge transformation. Moreover, we have $\mathcal{G}^h_*(Y) \cong H^1(Y;\mathbb{Z})$.

We learned last semester that $\mathcal{G}_*(Y)$ acts freely on $\mathcal{C}_0(Y)$. Using suitable Sobolev completions, one can make sense of the orbit space

$$\widetilde{\mathcal{B}}(Y) = \mathcal{C}_0(Y)/\mathcal{G}_*(Y) \qquad (5.1.10)$$

as an infinite dimensional smooth manifold with a residual action of $\mathcal{G}(Y)/\mathcal{G}_*(Y) \cong \mathbb{T}$ with orbit space $\mathcal{B}(Y) = \widetilde{\mathcal{B}}/\mathbb{T}$. Furthermore, the CSD functional descends to a $\mathbb{T}$–invariant map

$$\widetilde{\mathcal{L}} \colon \widetilde{\mathcal{B}} \to \mathbb{R}/\mathfrak{d}(\mathfrak{t})\mathbb{Z}. \qquad (5.1.11)$$

Moreover, the L^2 gradient

$$\mathcal{X} = \nabla\mathcal{L} \colon \mathcal{C}_0(Y) \to \mathcal{C}_0(Y), \quad \mathcal{X}(b,\phi) = \begin{pmatrix} *db + \rho^{-1}(\phi\phi)_0 + *\frac{1}{2}F_{B_0^t} \\ D\phi + \rho(b)\phi \end{pmatrix} \qquad (5.1.12)$$

descends to a $\mathbb{T}$–invariant vector field $\widetilde{\mathcal{X}}$ on $\widetilde{\mathcal{B}}(Y)$.

The basic strategy. The construction then proceeds as follows:

(1) Perturb the CSD functional to $\mathcal{L}_\mathfrak{q} = \mathcal{L} + \mathfrak{q}$ using suitable functions $\mathfrak{q} \colon \mathcal{C}_0(Y) \to \mathbb{R}$ which, among other things, admit L^2 gradients that are $\mathcal{G}(Y)$–invariant.

(2) As in the finite dimensional situation, form a *blown-up configuration space* $\widetilde{\mathcal{B}}^\sigma(Y)$ on which $\mathbb{T}$ acts freely and consider $\mathcal{B}^\sigma(Y) = \widetilde{\mathcal{B}}^\sigma(Y)/\mathbb{T}$. This will be an infinite dimensional smooth manifold.

(3) Note that the gradient $\mathcal{X}_\mathfrak{q} = \nabla\mathcal{L}_\mathfrak{q}$ on $\mathcal{C}_0(Y)$ descends to a $\mathbb{T}$–invariant vector field $\widetilde{\mathcal{X}}_\mathfrak{q}$ on $\widetilde{\mathcal{B}}(Y)$ and gives rise to a vector field $\mathcal{X}^\sigma_\mathfrak{q}$ on $\mathcal{B}^\sigma(Y)$ as in the finite dimensional case.

(4) Consider the negative flow equation $\dot{x} + \mathcal{X}^\sigma_\mathfrak{q}(x) = 0$ for paths $x \colon \mathbb{R} \to \mathcal{B}^\sigma(Y)$.

(5) Argue that for suitable choices of $\mathfrak{q}$ the moduli spaces of trajectories share sufficiently many properties with those of equivariant Morse–Smale gradients for semi-free $\mathbb{T}$–actions in finite dimensions.

(6) Use this to build chain complexes $\widehat{\mathrm{CM}}_*(Y)$, $\overline{\mathrm{CM}}_*(Y)$, $\widecheck{\mathrm{CM}}_*(Y)$.

(7) Prove that the homology groups $\widehat{\mathrm{HM}}_*(Y)$, $\overline{\mathrm{HM}}_*(Y)$, and $\widecheck{\mathrm{HM}}_*(Y)$ depend only on $(Y, \mathfrak{t})$.

(8) Throughout all of this, keep track of the relation to the Seiberg–Witten equations on the infinite cylinder $\mathbb{R} \times Y$.

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The obstacles. There are several problems with the strategy outlined above.

(1) First of all, after Sobolev completions the "vector field" $\mathcal{X}^\sigma_\mathfrak{q}$ is not actually a vector field in any strict sense.

(2) The flow equation $\dot{x}+\mathcal{X}^\sigma_\mathfrak{q} = 0$ nevertheless makes sense, but it does not actually generate a flow in any strict sense.

(3) Ideally, we would want trajectories $x\colon \mathbb{R} \to \mathcal{B}^\sigma(Y)$ to have stationary points of $\mathcal{X}^\sigma_\mathfrak{q}$ as **asymptotic limits**. This is not at all clear!

(4) Next on the wish list are stable and unstable manifolds. Their existence can be guaranteed by careful choice of $\mathfrak{q}$. But it will turn out that they are necessarily infinite dimensional. So there is no reasonable notion of Morse index! This makes **gradings** a somewhat complicated story.

(5) The way out will be a *relative index* $\mu(x, y)$ which corresponds to the difference $\mu(x)-\mu(y)$ in the finite dimensional theory. Eventually, it will be possible to obtain smooth, finite dimensional moduli spaces of trajectories. This issue is commonly referred to as **achieving transversality**.

(6) Next come **compactness**. With transversality in place, compactness of the moduli space of broken trajectories most be proved.

(7) And then there is **gluing** which refers to the question which broken trajectories can actually be realized as limits of unbroken trajectories.

5.2 Blown-up configuration spaces

Recall that for a closed spin^c 3–manifold Y with a base point $y_0 \in Y$ we have found that

$$\widetilde{\mathcal{B}}(Y) = \mathcal{C}(Y)/\mathcal{G}_*(Y) \tag{5.2.1}$$

carries a semi-free $\mathbb{T}$–action whose fixed points are the $\mathcal{G}_*(Y)$–orbits of reducible configurations $(A, 0) \in \mathcal{C}(Y)$. The idea is to mimic the Morse theoretic constructions from Chapter 4 for the CSD functional and its L^2 gradient. We could try to construct the blow-up $\widetilde{\mathcal{B}}^\sigma(Y)$ directly, but this his two drawbacks:

- First, this would require an a priori discussion of a smooth structure on $\widetilde{\mathcal{B}}(Y)$.
- Second, the orbit space $\widetilde{\mathcal{B}}(Y)$ typically does not have a linear structure.

From a technical perspective, it is more convenient to work with $\mathcal{C}(Y)$ and the full gauge group $\mathcal{G}(Y)$, although this blurs the analogy with Section 4.2 a little. The discussion below closely follows [KM07, Chs. 6 & 9].

5.2.1 The σ–model for 3–manifolds

While the $\mathcal{G}(Y)$–action on $\mathcal{C}(Y)$ is not semi-free, it only fails to be free on reducible configurations $(A, 0)$ which are stabilized by the constant gauge transformation (see Lemma 2.33). We define the *blown-up configuration space* as

$$\begin{aligned} \mathcal{C}^\sigma(Y) &= \mathcal{A}(S_Y) \times \mathbb{R}_+ \times \mathbb{S}(\Gamma(S_Y)) \\ &= \{(A, s, \phi) \in \mathcal{A}(S_Y) \times \mathbb{R} \times \Gamma(S_Y) \mid s \geq 0,\ \|\phi\|_{L^2} = 1\} \end{aligned} \tag{5.2.2}$$

where $\mathbb{S}(\Gamma(S))$ the unit sphere with respect to the L^2–norm. The corresponding *blow-down map* is given by

$$\pi\colon \mathcal{C}^\sigma(Y) \to \mathcal{C}(Y), \quad \pi(A, s, \phi) = (A, s\phi). \tag{5.2.3}$$

This is a bijection over the irreducible locus $\mathcal{C}^*(Y)$ and

$$\pi^{-1}(A,0) = \{(A,0)\} \times \mathbb{S}(\Gamma(S)) \cong S(\Gamma(S)). \tag{5.2.4}$$

The gauge group $\mathcal{G}(Y)$ acts on $\mathcal{C}^\sigma(Y)$ by

$$u(A,s,\phi) = (A - u^{-1}du,\, s,\, u\phi) \tag{5.2.5}$$

The action is free, because $u \neq 0$, and the blow-down map is $\mathcal{G}(Y)$–equivariant.

5.2.2 The σ–model for 4–manifolds

Now let X be a compact spin^c 4–manifold, possibly with boundary. The same discussion as above applies and gives a blown-up configuration space and blow down map

$$\mathcal{C}^\sigma(X) = \mathcal{A}(S_X) \times \mathbb{R}_+ \times \mathbb{S}(\Gamma(S_X^+)), \quad \pi\colon \mathcal{C}^\sigma(X) \to \mathcal{C}(X) \tag{5.2.6}$$

where $\mathbb{S}(\Gamma(S_X^+))$ is the L^2–unit sphere. We want to have a blown-up version of the monopole map

$$\mathfrak{F}\colon \mathcal{C}(X) \to i\Omega^2_+(X) \oplus \Gamma(S_X^-), \quad \mathfrak{F}(A,\phi) = \big(\tfrac{1}{2}F^+_{A^t} - \rho_X^{-1}(\phi\phi^*)_0,\, D_A^+\phi\big). \tag{5.2.7}$$

For that purpose, we think of $\mathfrak{F}$ as a section of the trivial bundle

$$\mathcal{V}(X) = \mathcal{C}(X) \times \big(i\Omega^2_+(X) \oplus \Gamma(S_X^-)\big) \tag{5.2.8}$$

and consider the pull-back $\mathcal{V}^\sigma(X) = \pi^*\mathcal{V}(X)$ over $\mathcal{C}^\sigma(X)$ (which is again just a trivial bundle). We define the *blown-up monopole map* as

$$\mathfrak{F}^\sigma\colon \mathcal{C}^\sigma(X) \to \mathcal{V}^\sigma(X), \quad \mathfrak{F}^\sigma(A,s,\phi) = \big(\tfrac{1}{2}F^+_{A^t} - s^2\rho_X^{-1}(\phi\phi^*)_0,\, D_A^+\phi\big) \tag{5.2.9}$$

and note that it is $\mathcal{G}(Y)$–equivariant with respect to the obvious $\mathcal{G}(Y)$–action on $\mathcal{V}^\sigma(X)$. Moreover, we have

$$\mathfrak{F}^\sigma(A,s,\phi) = 0 \quad \Leftrightarrow \quad \begin{cases} \mathfrak{F}(A,s\phi) = 0, & \text{if } s \neq 0 \\ \mathfrak{F}(A,0) = 0 \quad \text{and} \quad D_A^+\phi = 0, & \text{if } s = 0. \end{cases} \tag{5.2.10}$$

The blow-down map sends the locus $r = 0$ to the reducible locus of $\mathcal{C}(X)$. The equation $D_A^+\phi = 0$ in $\mathfrak{F}^\sigma(A,0,\phi) = 0$ should be thought of as including normal information to the reducible locus that is invisible to the equation $\mathfrak{F}(A,0) = 0$.

Restriction maps and unique continuation. The use of the L^2–norms in the construction of the blown-up configuration causes some trouble with restrictions:

- ► Similarly, if $Z = I \times Y$ and $\phi \in \mathbb{S}(\Gamma(S_Z))$, then for $t \in I$ it might happen that $\phi_t = \phi|_{\{t\}\times Y} \equiv 0$.
- ► If $X' \subset X$ is an interior domain and $\phi \in \mathbb{S}(\Gamma(S_X))$, then it is possible that $\phi|_{X'} \equiv 0$.

As a consequence, there are only partially defined restriction maps

$$\{(A,s,\phi) \,|\, \phi_t \neq 0\} \to \mathcal{C}^\sigma(Y), \qquad (A,s,\phi) \mapsto (A, s\,\|\phi_t\|_{L^2(Y)}, \phi_t/\|\phi_t\|_{L^2(Y)}) \tag{5.2.11}$$

$$\{(A,s,\phi) \,|\, \phi|_{X'} \neq 0\} \to \mathcal{C}^\sigma(X'), \qquad (A,s,\phi) \mapsto (A, s\,\|\phi\|_{L^2(X')}, \phi/\|\phi\|_{L^2(X')}) \tag{5.2.12}$$

The first point is particularly awkward, since general elements of $\mathcal{C}^\sigma(I \times Y)$ will not give rise to paths in $\mathcal{C}^\sigma(Y)$. However, for solutions $\gamma^\sigma = (A,s,\phi)$ of $\mathfrak{F}^\sigma(\gamma^\sigma) = 0$ this trick still works. This is guaranteed by the following 'unique continuation theorem' for spin^c Dirac operators.

Theorem 5.2 (Unique continuation, cf. [KM07, Prop. 7.1.2 & 7.1.4])**.**

(i) Suppose that $(A,\phi) \in \mathcal{C}(I\times Y)$ satisfies $D_A^+\phi = 0$. If $\phi_t = 0$ for some $t \in I$, then $\phi = 0$.

(ii) Suppose that $(A,\phi) \in \mathcal{C}(X)$ satisfies $D_A^+\phi = 0$. If ϕ vanishes on an open subset of X, then $\phi = 0$.

In particular, for $\gamma^\sigma = (A,s,\phi) \in \mathcal{C}^\sigma(I\times Y)$ with $\mathfrak{F}^\sigma(\gamma^\sigma) = 0$ we obtain a smooth path

$$\check{\gamma}^\sigma : I \to \mathcal{C}^\sigma(Y). \tag{5.2.13}$$

As before, we can recover γ^σ from $\check{\gamma}^\sigma$ if and only if A is in temporal gauge.

The blown-up Seiberg–Witten equations as a flow. We now focus exclusively on the case of a compact cylinder $Z = I \times Y$. Suppose that $\gamma^\sigma = (A,s,\phi) \in \mathcal{C}^\sigma(Z)$ satisfies $\phi_t \neq 0$ for all $t \in I$. Then the corresponding path is given by

$$\check{\gamma}^\sigma(t) = \big(\check{A}(t), s\left\|\phi_t\right\|_{L^2(Y)}, \phi_t/\left\|\phi_t\right\|_{L^2(Y)}\big) =: (B(t), r(t), \psi(t)). \tag{5.2.14}$$

We would like to view the equation $\mathfrak{F}^\sigma(\gamma^\sigma) = 0$ as a flow equation for the path.

Lemma 5.3. *Suppose that $\gamma^\sigma \in \mathcal{C}^\sigma(Z)$ is in temporal gauge. Then $\mathfrak{F}^\sigma(\gamma^\sigma) = 0$ if and only if γ^σ corresponds to a path $\check{\gamma}^\sigma = (B = B_0 + b, r, \psi)$ in $\mathcal{C}^\sigma(Y)$ satisfies*

$$\begin{aligned} \dot{b} &= - * db - r^2\rho^{-1}(\psi\psi)_0 - *\tfrac{1}{2}F_{B_0^t} \\ \dot{r} &= -\Lambda(B,r,\psi)\, r \\ \dot{\psi} &= -D_B\psi + \Lambda(B,r,\psi)\psi \end{aligned} \tag{5.2.15}$$

where $\Lambda(B,r,\psi) = \langle \psi, D_B\psi\rangle$ is defined using the real L^2 inner product on $\Gamma(S_Y)$.

Note that (5.2.15) can be written of the form $\dot{x} + \mathcal{X}^\sigma(x) = 0$ with

$$\begin{aligned} &\widetilde{\mathcal{X}}^\sigma : \mathcal{C}^\sigma(Y) \to i\Omega^1(Y) \oplus \mathbb{R} \oplus \Gamma(S), \\ &\widetilde{\mathcal{X}}^\sigma(B,r,\psi) = \begin{pmatrix} *db + r^2\rho^{-1}(\psi\psi)_0 + *\tfrac{1}{2}F_{B_0^t} \\ \Lambda(B,r,\psi)\, r \\ D_B\psi - \Lambda(B,r,\psi)\psi \end{pmatrix}. \end{aligned} \tag{5.2.16}$$

Informally, we can consider this as a vector field on $\mathcal{C}^\sigma(Y)$. Indeed, for $(B,r,\psi) \in \mathcal{C}^\sigma(Y)$ we have canonical isomorphisms

$$T_B\mathcal{A}(S_Y) \cong i\Omega^1(Y), \quad T_r\mathbb{R}_+ \cong \mathbb{R}, \tag{5.2.17}$$

and the finite dimensional intuition $T_pS^n = \langle p\rangle^\perp \subset \mathbb{R}^{n+1}$ suggest that

$$T_\psi\mathbb{S}(\Gamma(S_Y)) \cong \langle\psi\rangle^\perp = \{\kappa \in \Gamma(S_Y) \,|\, \langle\psi,\kappa\rangle = 0\}. \tag{5.2.18}$$

We thus define 'tangent spaces'

$$T_{(B,r,\psi)}\mathcal{C}^\sigma(Y) = i\Omega^1(Y) \oplus \mathbb{R} \oplus \langle\psi\rangle^\perp \subset i\Omega^1(Y) \oplus \mathbb{R} \oplus \Gamma(S) \tag{5.2.19}$$

and assemble them into a 'tangent bundle'

$$T\mathcal{C}^\sigma(Y) \subset \mathcal{C}^\sigma(Y) \times \big(i\Omega^1(Y) \oplus \mathbb{R} \oplus \Gamma(S)\big). \tag{5.2.20}$$

Now, the last component of $\mathcal{X}^\sigma(B,r,\psi)$ satisfies

$$\begin{aligned} \langle\psi, D_B\psi - \Lambda(B,r,\psi)\psi\rangle &= \langle\psi, D_B\psi - \langle\psi, D_B\psi\rangle\,\psi\rangle \\ &= \langle\psi, D_B\psi\rangle - \langle\psi, D_B\psi\rangle \underbrace{\left\|\psi\right\|^2}_{=1} = 0 \end{aligned} \tag{5.2.21}$$

and we can therefore view it as a 'vector field'

$$\widetilde{\mathcal{X}}^\sigma : \mathcal{C}^\sigma(Y) \to T\mathcal{C}^\sigma(Y). \tag{5.2.22}$$

Moreover, if we think of $\mathcal{C}^\sigma(Y)$ as an infinite dimensional manifolds with boundary

$$\partial\mathcal{C}^\sigma(Y) = \{(B, r, \psi) \in \mathcal{C}^\sigma(Y) \,|\, r = 0\}\,, \tag{5.2.23}$$

then $\widetilde{\mathcal{X}}^\sigma$ is tangent to the boundary. Keeping on with the spirit of pretending, we might just as well compute the 'derivative' of the blow-down map Lecture 7, 21.11.23

$$\pi_*\colon T\mathcal{C}^\sigma(Y) \to T\mathcal{C}(Y) = \mathcal{C}(Y) \times \big(i\Omega^1(Y) \times \Gamma(S_Y)\big). \tag{5.2.24}$$

Since π is given by restricting the product of the identity on $\mathcal{A}(X)$ and the scalar product map $\mathbb{R} \times \Gamma(S_Y) \to \Gamma(S_Y)$ to $\mathbb{R}_+ \times \mathbb{S}(\Gamma(S_Y))$, we get

$$T_{(B,r,\psi)}\mathcal{C}^\sigma(Y) \ni (b, s, \kappa) \quad \mapsto \quad \pi_*(b, s, \kappa) = \big(b, s\psi + r\kappa\big) \in T_{(B,r\psi)}\mathcal{C}(Y) \tag{5.2.25}$$

We can also view the L^2 gradient $\widetilde{\mathcal{X}} = \nabla\mathcal{L}$ of the functional $\mathcal{L}\colon \mathcal{C}(Y) \to \mathbb{R}$ as a vector field on $\mathcal{C}(Y)$. From this we see that

$$\pi_*\widetilde{\mathcal{X}}^\sigma(B, r, \psi) = \begin{pmatrix} *db + r^2\rho^{-1}(\psi\psi^*)_0 + *\frac{1}{2}F_{B_0^t} \\ \langle\psi, D_B\psi\rangle\, r\psi + r(D_B\psi - \langle\psi, D_B\psi\rangle\,\psi) \end{pmatrix} = \widetilde{\mathcal{X}}(B, r\psi) \tag{5.2.26}$$

In particular, $\widetilde{\mathcal{X}}^\sigma$ corresponds to $\widetilde{\mathcal{X}}$ over the irreducible locus $\mathcal{C}^*(Y)$ where π is a 'diffeomorphism'. As for the stationary points of $\widetilde{\mathcal{X}}^\sigma$ and $\widetilde{\mathcal{X}}$, we find:

Corollary 5.4. *For $(B, r, \psi) \in \mathcal{C}^\sigma(Y)$ we have*

$$\mathcal{X}^\sigma(B, r, \psi) = 0 \quad \Leftrightarrow \quad \begin{cases} \mathcal{X}(B, r\psi) = 0, & r \neq 0 \\ \mathcal{X}(B, 0) \text{ and } \psi \text{ is an eigenvector of } D_B, & s = 0. \end{cases} \tag{5.2.27}$$

In the case $r = 0$, the eigenvalue is $\Lambda(B, 0, \phi)$

This should be compared with (4.2.8) in the finite dimensional toy case. The main difference is that D_B is a self-

Remark 5.5. Lastly, we can pretend that $\mathcal{G}(Y)$ is an infinite dimensional Lie group and that its action on the various spaces if sufficiently well behaved so that the quotient

$$\mathcal{B}^\sigma(Y) = \mathcal{C}^\sigma(Y)/\mathcal{G}(Y) \tag{5.2.28}$$

is an infinite dimensional smooth manifold. The vector field $\widetilde{\mathcal{X}}^\sigma$ would then descend to a vector field

$$\mathcal{X}^\sigma : \mathcal{B}^\sigma(Y) \to T\mathcal{B}^\sigma(Y). \tag{5.2.29}$$

Moreover, $\widetilde{\mathcal{B}}(Y) = \mathcal{C}(Y)/\mathcal{G}_*(Y)$ would be a semi-free $\mathbb{T}$–manifold and the construction would factor through $\widetilde{\mathcal{B}}^\sigma(Y) = \mathcal{C}^\sigma(Y)/\mathcal{G}_*(Y)$, realizing the $\mathcal{X}^\sigma$ as the blow-up of the gradient of the $\mathbb{T}$–invariant CSD functional on $\widetilde{\mathcal{B}}(Y)$ in full analogy with Section 4.2.

5.2.3 The τ–model for cylinders

For a cylinder $Z = I \times Y$ there is another way to write the flow equations

$$\dot{x} + \mathcal{X}^\sigma(x) = 0, \quad x\colon I \to \mathcal{C}^\sigma(Y) \tag{5.2.30}$$

in terms of a 4–dimensional configuration space. This is the so-called *τ–model* for the blown-up configuration space on Z:

$$\mathcal{C}^\tau(Z) = \Big\{(A, s, \phi) \,\Big|\, \forall t : s(t) \geq 0,\, \|\phi(t)\|_{L^2(Y)} = 1\Big\} \subset \mathcal{A}(Z) \times C^\infty(I) \times \Gamma(S_Z^+) \tag{5.2.31}$$

This comes with a similar blow-down map also denoted by

$$\pi\colon \mathcal{C}^\tau(Z) \to \mathcal{C}(Z), \quad (A,s,\phi) \mapsto (A, s\phi) \tag{5.2.32}$$

which is equivariant with respect to the $\mathcal{G}(Z)$ action on $\mathcal{C}^\tau(Z)$ defined by

$$u(A,s,\phi) = (A - u^{-1}du, s, u\phi). \tag{5.2.33}$$

We begin with a few observations on the relation of $\mathcal{C}^\tau(Z)$ and $\mathcal{C}^\sigma(Z)$.

- A path $I \to \mathcal{C}^\sigma(Y)$ uniquely determined elements of $\mathcal{C}^\sigma(Z)$ and $\mathcal{C}^\tau(Z)$ in temporal gauge.
- Unlike as for $\mathcal{C}^\sigma(Z)$, *every* element $\gamma = (A,s,\phi) \in \mathcal{C}^\tau(X)$ determines a path $\check{\gamma}\colon I \to \mathcal{C}^\sigma(Y)$ in the obvious way, and γ is determined by $\check{\gamma}$ if and only if it is in temporal gauge.
- The definition of $\mathcal{C}^\tau(Z)$ does not require Z to be compact.

There is also a τ–version of a blown-up Seiberg–Witten map:

$$\mathfrak{F}^\tau \colon \mathcal{C}^\tau(Z) \to i\Omega^2_+(Z) \oplus C^\infty(I,\mathbb{R}) \oplus \Gamma(S^-_Z)$$
$$\mathfrak{F}^\tau(A,s,\phi) = \begin{pmatrix} \frac{1}{2}F^+_{A^t} - s^2\rho_Z^{-1}(\phi\phi^*)_0 \\ \dot{s} + \left\langle D^+_A\phi, \rho_Z(dt)^{-1}\phi\right\rangle_{L^2(Y)} s \\ D^+_A\phi - \left\langle D^+_A\phi, \rho_Z(dt)^{-1}\phi\right\rangle_{L^2(Y)} \phi \end{pmatrix} \tag{5.2.34}$$

Lemma 5.6 (cf. [KM07, p. 119 f.])**.** *(i) Let $\gamma\colon I \to \mathcal{C}^\sigma(Y)$ be a smooth path and $\gamma^\tau \in \mathcal{C}^\tau(Z)$ (and $\gamma^\sigma \in \mathcal{C}^\sigma(X)$ if I is compact) the corresponding element in temporal gauge. Then*

$$\dot{\gamma} + \mathcal{X}^\sigma(\gamma) = 0 \quad \Leftrightarrow \quad \mathcal{F}^\tau(\gamma^\tau) = 0 \quad \Leftrightarrow \quad \mathcal{F}^\tau(\gamma^\tau) = 0 \quad (\Leftrightarrow \quad \mathcal{F}^\sigma(\gamma^\sigma) = 0). \tag{5.2.35}$$

(ii) If I is compact, then there is a one-to-one correspondence between the solutions of $\mathcal{F}^\tau = 0$ and $\mathcal{F}^\sigma = 0$.

Proof. (i) follows from a direct computation based on the considerations in Section 2.4.7. As for (ii), note that $\mathfrak{F}^\tau(A,s,\phi) = 0$ implies that s is either identically identically zero or everywhere positive. In the latter case, one can show that

$$\mathfrak{F}^\tau(A,s,\phi) = 0 \quad \Leftrightarrow \quad \mathfrak{F}(A,s\phi) = 0 \quad \Leftrightarrow \quad \mathfrak{F}^\sigma(A, \|s\phi\|_{L^2(Z)}, s\phi/\|s\phi\|_{L^2(Z)}) = 0. \tag{5.2.36}$$

In the case $s = 0$, suppose that $\mathcal{F}^\tau(A,0,\phi) = 0$ and fix some $t_0 \in I$. Let $s_0\colon I \to \mathbb{R}$ be the unique solution of the initial value problem

$$\dot{s}_0 + \left\langle D^+_A\phi, \rho_Z(dt)^{-1}\phi\right\rangle_{L^2(Y)} s_0 = 0, \quad s_0(t_0) = 1. \tag{5.2.37}$$

Then s_0 is everywhere positive and a quick computation shows that $D^+_A(s_0\phi) = 0$. In addition, $\mathcal{F}^\tau(A,0,\phi) = 0$ implies $\mathfrak{F}(A,0) = 0$ and thus $\mathfrak{F}^\sigma(A,0,s_0\phi/\|s_0\phi\|_{L^2(Z)}) = 0$ by (5.2.10). ☐

Remark 5.7. (i) The map $\mathfrak{F}^\tau$ can be viewed as a section of a vector bundle $\mathcal{V}^\tau \to \mathcal{C}^\tau(Z)$ with fibers

$$\mathcal{V}^\tau_{(A,s,\phi)} = \left\{(\eta,r,\psi) \in i\Omega^2_+(Z) \oplus C^\infty(I,\mathbb{R}) \oplus \Gamma(S^-_Z) \,\middle|\, \forall t : \langle\phi(t),\psi(t)\rangle = 0\right\}. \tag{5.2.38}$$

(ii) For technical reasons, it is convenient to introduce the larger space

$$\tilde{\mathcal{C}}^\tau(Z) = \left\{(A,s,\phi) \in \mathcal{A}(Z) \times C^\infty(I) \times \Gamma(S^+_Z) \,\middle|\, \forall t : \|\phi(t)\|_{L^2(Y)} = 1\right\} \tag{5.2.39}$$

where the function is allowed to take arbitrary values. Reversing the sign on functions gives an involution

$$\iota\colon \tilde{\mathcal{C}}^\tau(Z) \to \tilde{\mathcal{C}}^\tau(Z), \quad (A,s,\phi) \mapsto (A,-s,\phi) \tag{5.2.40}$$

The space $\mathcal{C}^\tau(Z)$ can be viewed either as a subspace of $\tilde{\mathcal{C}}^\tau(Z)$ or as the orbit space of the involution ι. The blow-down map and the map $\mathfrak{F}^\tau$ can be extended to $\tilde{\mathcal{C}}^\tau(Z)$ by the same formulas.

5.3 Sobolev completions

It's time to get a little more serious about the infinite dimensional analytic setup. We have already touched upon Sobolev spaces in Section 2.4.2 in the context of vector bundles over closed manifolds.

(1) Let $E \to M$ be a real or complex vector bundle over a smooth n–manifold M. We are mostly interested in the following cases:

- Spinor bundles or bundles of forms over a closed 3–manifold Y.
- Spinor bundles or bundles of forms over a compact 4–manifold X, possibly with boundary
- Bundles over a 4–dimensional cylinder $Z = I \times Y$ pulled back from Y

Let $\Gamma_0(E)$ be the set of smooth sections of E with compact support in the interior of M.

(2) A choice of metrics and connections gives rise to *Sobolev norms* $\|\cdot\|_{L^p_k}$ on $\Gamma_0(E)$

$$\|\phi\|_{L^p_k} = \left(\int_M \big(|\phi|^p + |\nabla\phi|^p + \cdots + |\nabla^k\phi|^p\big)d\mu_g.\right)^{\frac{1}{p}} \quad (p \geq 1,\, k \geq 0) \tag{5.3.1}$$

which obviously depend on the choices.

(3) Let $L^p_k(E)$ be the completion of $\Gamma_0(E)$ with respect to $\|\cdot\|_{L^p_k}$. By construction, these *Sobolev spaces* of sections are Banach spaces and for $p = 2$ they are Hilbert spaces.

- If M is compact, then $L^p_k(E)$ is independent of the chosen metrics and connections.
- If M is not compact, then $L^p_k(E)$ generally depends on these choices!

(4) In the non-compact setting, there are canonically defined *local Sobelev spaces* $L^p_{k,\mathrm{loc}}(E)$ which can be described as the completion of $\Gamma(E)$ with respect to the semi-norms given by $\phi \mapsto \|\kappa_n\phi\|_{L^p_k}$ where $\kappa_n\colon M \to [0,1]$ is a sequence of smooth functions with compact supports such that $K_n = \kappa_n^{-1}(1)$ is a compact exhaustion of M.

- If M is compact, then $L^p_{k,\mathrm{loc}}(E) = L^p_k(E)$.
- If M is not compact, then the inclusion $L^p_k(E) \subset L^p_{k,\mathrm{loc}}(E)$ is strict and the right hand side is not a Banach space.

(5) If $Z = I \times Y$ is a cylinder over a closed manifold and E is the pull-back of a bundle $F \to Y$, we make the convention that Sobolev spaces $L^p_k(E)$ are defined using a cylindrical metric on E and a connection on E pulled back from one on F. Then the compactness of Y ensures that $L^p_k(E)$ is canonically defined, even if I is not compact.

5.3.1 Sobolev completions of configuration spaces

In what follows, let M be a Riemannian spin^c n–manifold with spinor bundle S_M and $k \geq 0$ be an integer.

(1) We define Sobolev spaces of connections as

$$\mathcal{A}_k(S_M) = A_0 + L^2_k(iT^*M) \quad \text{and} \quad \mathcal{A}_{k,\mathrm{loc}}(S_M) = A_0 + L^2_{k,\mathrm{loc}}(iT^*M) \tag{5.3.2}$$

where A_0 is a fixed smooth spin^c connection. The space $\mathcal{A}_{k,\mathrm{loc}}(S_M)$ is always independent of A_0 (and all other choices), where is $\mathcal{A}_k(S_M)$ might depend on the choice of A_0 if M is non-compact.

(2) From this we get Sobolev configuration spaces

$$\mathcal{C}_k(M) = \begin{cases} \mathcal{A}_k(M) \times L^2_k(S_M), & n \text{ odd} \\ \mathcal{A}_k(M) \times L^2_k(S^+_M), & n \text{ even} \end{cases} \tag{5.3.3}$$

and $\mathcal{C}_{k,\mathrm{loc}}(M)$ defined analogously.

(3) Similarly, if M is compact, we have blown-up versions

$$\mathcal{C}^\sigma_k(M) = \mathcal{A}_k(M) \times \mathbb{R}_+ \times \mathbb{S}(L^2_k(S^{(+)}_M)) \tag{5.3.4}$$

where $\mathbb{S}$ still refers to the L^2–unit sphere.

(4) For a cylinder $Z = I \times Y$ of the usual type, we define

$$\begin{aligned} \tilde{\mathcal{C}}^\tau_k(Z) &= \left\{ (A, s, \phi) \in \mathcal{A}_k(S_Z) \times L^2_k(I; \mathbb{R}) \times L^2_k(S^+_Z) \,\middle|\, \|\phi(t)\|_{L^2(Y)} = 1 \; \forall t \in I \right\} \\ \mathcal{C}^\tau_k(Z) &= \left\{ (A, s, \phi) \in \tilde{\mathcal{C}}^\tau_k(Z) \,\middle|\, s \geq 0 \text{ almost everywhere} \right\} \end{aligned} \tag{5.3.5}$$

There are also $L^2_{k,\mathrm{loc}}$ versions which become relevant if I is not compact.

(5) Now let $2(k+1) > n$. We have a continuous embedding and multiplication maps

$$L^2_{k+1} \hookrightarrow C^0 \quad \text{and} \quad L^2_{k+1} \times L^2_j \to L^2_j, \quad 0 \leq j \leq k+1 \tag{5.3.6}$$

and similarly for the $L^2_{k,\mathrm{loc}}$ versions. Using this, we define Sobolev gauge groups

$$\begin{aligned} \mathcal{G}_{k+1}(M) &= \left\{ u \in L^2_{k+1}(M; \mathbb{C}) \,\middle|\, |u| = 1 \text{ pointwise} \right\} \quad \text{and} \\ \mathcal{G}_{k+1,\mathrm{loc}}(M) &= \left\{ u \in L^2_{k+1,\mathrm{loc}}(M; \mathbb{C}) \,\middle|\, |u| = 1 \text{ pointwise} \right\}. \end{aligned} \tag{5.3.7}$$

where the group operation is point-wise multiplication.

For the moment, we focus on the compact case. The actions of $\mathcal{G}(M)$ and $\mathcal{G}(Z)$ on $\mathcal{C}(M)$, $\mathcal{C}^\sigma(M)$ and $\mathcal{C}^\tau(Z)$ extend to continuous actions of the $(k+1)$–completed gauge groups on the k–completed configuration spaces. We define the orbit spaces

$$\begin{aligned} \mathcal{B}_k(M) &= \mathcal{C}_k(M)/\mathcal{G}_{k+1}(M) \\ \mathcal{B}^\sigma_k(M) &= \mathcal{C}^\sigma_k(M)/\mathcal{G}_{k+1}(M) \end{aligned} \tag{5.3.8}$$

and

$$\begin{aligned} \mathcal{B}^\tau_k(Z) &= \mathcal{C}^\tau_k(Z)/\mathcal{G}_{k+1}(Z) \\ \tilde{\mathcal{B}}^\tau_k(Z) &= \tilde{\mathcal{C}}^\tau_k(Z)/\mathcal{G}_{k+1}(Z). \end{aligned} \tag{5.3.9}$$

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Proposition 5.8 (cf. [KM07, Ch. 9]). *Let M be a compact spin^c n–manifold and $2(k+1) > n$.*

(i) $\mathcal{G}_{k+1}(M)$ is a Hilbert Lie Group.

(ii) $\mathcal{C}_k(M)$ is a smooth Hilbert manifold on which $\mathcal{G}_{k+1}(M)$ acts smoothly. The orbit space $\mathcal{B}_k(M)$ is Hausdorff.

(iii) $\mathcal{C}^\sigma_k(M)$ is a smooth Hilbert manifold with boundary on which $\mathcal{G}_{k+1}(M)$ acts smoothly and freely. The orbit space $\mathcal{B}^\sigma_k(M)$ is a smooth Hilbert manifold.

(iv) If $Z = I \times Y$ is a compact cylinder and $2(k+1) > 4$, then $\tilde{\mathcal{C}}^\tau_k(Z)$ is a smooth Hilbert manifold on which $\mathcal{G}_{k+1}(Z)$ acts smoothly and freely. The orbit space $\tilde{\mathcal{B}}^\tau_k(M)$

With this analytic setup in place, we can now make sense of the bundles $T\mathcal{C}_k(M)$ and $T\mathcal{C}_k^\sigma(M)$ and give a precise meaning to our ad hoc considerations before passing to the completions.

(1) For $\mathcal{C}_k(M)$ we consider the trivial bundles

$$\mathcal{T}_j = \mathcal{C}_k(M) \times L^2_j(iT^*M \oplus S_M^{(+)}) \quad (j \geq 0) \tag{5.3.10}$$

and for $\mathcal{C}_k^\sigma(M)$ the sub-bundles

$$\mathcal{T}_j^\sigma \subset \mathcal{C}_k^\sigma(M) \times \big(L^2_j(iT^*M) \oplus \mathbb{R} \oplus L^2_k(S_M^{(+)})\big) \quad (j \geq 0) \tag{5.3.11}$$

whose fibers over $\gamma = (A, s, \phi)$ are

$$\mathcal{T}_{j,\gamma}^\sigma = \Big\{(b, r, \psi) \in L^2_j(iT^*M) \oplus \mathbb{R} \oplus L^2_k(S_M^{(+)}) \,\Big|\, \langle \phi, \psi \rangle_{L^2} = 0\Big\} \tag{5.3.12}$$

We then have canonical identifications

$$T\mathcal{C}_k(M) = \mathcal{T}_k \quad \text{and} \quad T\mathcal{C}_k^\sigma(M) = \mathcal{T}_k^\sigma. \tag{5.3.13}$$

(2) In the case of a 3–manifold, the 'vector fields' $\widetilde{\mathcal{X}} = \nabla\mathcal{L}$ and $\widetilde{\mathcal{X}}^\sigma$ extend to smooth sections

$$\widetilde{\mathcal{X}} \colon \mathcal{C}_k(Y) \to \mathcal{T}_{k-1} \quad \text{and} \quad \widetilde{\mathcal{X}}^\sigma \colon \mathcal{C}_k^\sigma(Y) \to \mathcal{T}_{k-1}^\sigma \tag{5.3.14}$$

and cannot be factored through th actual tangent bundles $\mathcal{T}_k^{(\sigma)} \subset \mathcal{T}_{k-1}^{(\sigma)}$. The reason is that the formulas for $\widetilde{\mathcal{X}}^{(\sigma)}$ involve differential operators of order 1 which map L^2_k continuously into L^2_{k-1} but not into L^2_k. In that sense, $\widetilde{\mathcal{X}}$ is not a vector field and the same discussion applies to the blown-up version $\widetilde{\mathcal{X}}^\sigma$.

(3) Lastly, we note that $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{X}}^\sigma$ are equivariant with respect to the obvious $\mathcal{G}_{k+1}(Y)$ actions and the latter descends to a smooth section

$$\mathcal{X}^\sigma \colon \mathcal{B}_k^\sigma(Y) \to \bar{\mathcal{T}}_{k-1}^\sigma = \mathcal{T}_{k-1}^\sigma / \mathcal{G}_{k+1}(Y). \tag{5.3.15}$$

5.4 Invariants of closed 4–manifolds revisited

Let X be a closed, connected spin^c 4–manifold with $b_2^+(X) \geq 2$. In Section 2.4.5 we defined the classical Seiberg–Witten invariants by studying the moduli spaces

$$N(X) = \mathfrak{F}^{-1}(2\eta, 0)/\mathcal{G}_{k+1}(X) \subset \mathcal{B}_k(X) = \mathcal{C}_k(X)/\mathcal{G}_{k+1} \tag{5.4.1}$$

of solutions to the Seiberg–Witten equations with perturbation $\eta \in i\Omega^2_+(X)$. We proved that for suitable η the space $N(X)$ is a closed smooth manifold consisting entirely of irreducible solutions, thus representing a homology class

$$[N(X)]_2 \in H_*(\mathcal{B}_k^*(X); \mathbb{Z}_2) \cong H_*(\mathbb{CP}^\infty \times Pic(X); \mathbb{Z}_2), \tag{5.4.2}$$

which is independent of η, the Sobolev order k, and the chosen metric on X. Recall that the homotopy type of $\mathcal{B}$ was identified in Proposition 2.52 as

$$\mathcal{B}_k^*(X) \simeq \mathbb{CP}^\infty \times Pic(X) \tag{5.4.3}$$

where $Pic(X) = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$.

We can recast this story using the blown-up monopole map

$$\mathfrak{F}^\sigma(A, s, \phi) = \big(\tfrac{1}{2}F^+_{A^t} - s^2\rho_X^{-1}(\phi\phi^*)_0,\, D_A^+\phi\big). \tag{5.4.4}$$

and similarly defined moduli spaces

$$M(X) = (\mathfrak{F}^\sigma)^{-1}(2\eta, 0)/\mathcal{G}_{k+1} \subset \mathcal{B}_k^\sigma(X) \tag{5.4.5}$$

The same ideas that were used to study $N(X)$ gives the following result:

Theorem 5.9 (cf. [KM07, Ch. 27])**.** *Let X be a closed, connected spin^c 4–manifold.*

(i) There is a dense set of perturbations η such that $M(X)$ is a compact manifold with (possibly empty) boundary of dimension

$$\dim M(X) = \frac{1}{4}\big(c_1^2(S_X^+)[X] - 2\chi(X) + 3\sigma(X)\big). \tag{5.4.6}$$

(ii) If $b_2^+(X) \geq 1$, then there is a dense set of perturbations η as in (i) such that $\partial M(X) = \emptyset$.

(iii) If $b_2^+(X) \geq 2$ and $M_0(X)$ and $M_1(X)$ are defined using different Riemannian metrics on X and perturbations as in (ii), then $M_0(X)$ and $M_1(X)$ are cobordant in $\mathcal{B}_k^\sigma(X)$.

The relation to the classical approach is given as follows:

(1) The blow-down map $\mathcal{B}_k^\sigma(X) \to \mathcal{B}_k(X)$ is a diffeomorphism over $\mathcal{B}_k^*(X)$ and $\partial B_k^\sigma(X)$ is the preimage of the reducible locus. In particular, if $b_2^+(X) \geq 1$ the for η as in (ii), π maps $M(X)$ diffeomorphically onto $N(X)$

(2) The homotopy type of $\mathcal{B}_k^\sigma(X)$ can be identified as

$$\mathcal{B}_k^\sigma(X) \xleftarrow[\simeq]{incl} \mathcal{B}_k^\sigma(X) \setminus \partial B_k^\sigma(X) \xrightarrow[\simeq]{\pi} \mathcal{B}_k^*(X) \simeq \mathbb{C}\mathrm{P}^\infty \times Pic(X) \tag{5.4.7}$$

In particular, we have a an isomorphism

$$H_*(\mathcal{B}_k^\sigma(X); \mathbb{Z}_2) \to H_*(\mathcal{B}_k^*(X)) \tag{5.4.8}$$

which sends $[M(X)]$ to $[N(X)]$.

5.5 Perturbations of the CSD functional

Let Y be a closed, connected spin^c 3–manifold. Just as we had to perturb the Seiberg–Witten equations on closed 4–manifolds to obtain meaningful invariants, we should expect the same necessity on the infinite cylinder $\mathbb{R} \times Y$. Since the Seiberg–Witten equations on $\mathbb{R} \times Y$ are formally the negative gradient flow equations of the CSD functional

$$\mathcal{L}\colon \mathcal{C}(Y) \to \mathbb{R}, \quad \mathcal{L}(\underbrace{B_0 + b}_{=B}, \psi) = \tfrac{1}{2}\langle \psi, D_B\psi\rangle + \tfrac{1}{2}\langle b, *db\rangle + \tfrac{1}{2}\left\langle b, *F_{B_0^t}\right\rangle \tag{5.5.1}$$

one might hope to be able to realize the necessary perturbations on $\mathbb{R} \times Y$ as perturbations of $\mathcal{L}$ of the form

$$\mathcal{L}_f = \mathcal{L} + f\colon \mathcal{C}(Y) \to \mathbb{R} \tag{5.5.2}$$

where $f\colon \mathcal{C}(Y) \to \mathbb{R}$ is some function. This turns out to be possible, but finding a suitable class of such functions is a longer story (told in [KM07, Chs. 10&11]). We limit the discussion to a brief outline decorated with some motivation.

(1) First of all, f should be $\mathcal{G}(Y)$–invariant so that it $\mathcal{L}_f$ has the same invariance properties as $\mathcal{L}$.

(2) Just as $\mathcal{L}$, the function f should have a formal L^2 gradient which can be viewed as a smooth section

$$\mathfrak{q} = \nabla f\colon \mathcal{C}_k(Y) \to \mathcal{T}_{k-1}, \quad k \geq 3. \tag{5.5.3}$$

(3) Given f an in (1) and (2), we get a perturbation of the Seiberg–Witten vector field

$$\widetilde{\mathcal{X}}_{\mathfrak{q}} := \widetilde{\mathcal{X}} + \mathfrak{q} = \nabla\mathcal{L}_f\colon \mathcal{C}_k(Y) \to \mathcal{T}_{k-1} \tag{5.5.4}$$

which is really the main character of the story. This is usually reflected in terminology:

- ▶ $\mathfrak{q} = \nabla f$ is called a *perturbation*.
- ▶ f is called a *perturbation potential*.

(4) The blow-up procedure gives a smooth section $\mathfrak{q}^\sigma\colon \mathcal{C}_k^\sigma(Y) \to \mathcal{T}_{k-1}^\sigma$ and thus a perturbation

$$\widetilde{\mathcal{X}}_{\mathfrak{q}}^\sigma := \widetilde{\mathcal{X}}^\sigma + \mathfrak{q}^\sigma\colon \mathcal{C}_k^\sigma(Y) \to \mathcal{T}_{k-1}^\sigma \tag{5.5.5}$$

(5) The goal is that for sufficiently many $\mathfrak{q}$ the equation $\dot{\gamma} + \mathcal{X}_{\mathfrak{q}}^\sigma = 0$ in $\mathcal{B}^\sigma(Y)$ is sufficiently well-behaved in the sense that one can mimic the construction of the Floer complexes in vertical Morse theory.

(6) Further regularity conditions on $\mathfrak{q}$ are necessary to carry to guarantee desirable properties of the flow equation $\dot{x} + \mathcal{X}_{\mathfrak{q}}^\sigma(x) = 0$ on $\mathcal{B}^\sigma(Y)$. Narrowing down precise conditions eventually leads to the definition of *tame perturbations* in [KM07, Def. 10.5.1]. For example, $f(B,\psi) = \|\psi\|^2$ is such a tame perturbation.

(7) The existence of sufficiently many $\mathfrak{q}$ should follow from the Sard–Smale theorem. This would require a sufficiently large Banach space of tame perturbations. The construction of such spaces is carried out in [KM07, Ch. 11].

5.6 Non-degeneracy of critical points

We now consider a perturbation $\mathfrak{q}$ as above and the corresponding 'vector fields'

$$\widetilde{\mathcal{X}}_{\mathfrak{q}}^\sigma\colon \mathcal{C}_k^\sigma(Y) \to \mathcal{T}_{k-1}^\sigma \quad \text{and} \quad \mathcal{X}_{\mathfrak{q}}^\sigma\colon \mathcal{B}_k^\sigma(Y) \to \bar{\mathcal{T}}_{k-1}^\sigma. \tag{5.6.1}$$

Recall that $\mathcal{X}_{\mathfrak{q}}^\sigma$ is supposed to behave like the gradient of a vertical Morse function. In particular, its stationary points should be non-degenerate in a suitable sense.

5.6.1 Finite dimensional intuition

The non-equivariant case. We begin by recasting the classical notion of non-degeneracy of critical points.

Lemma 5.10. *Let $f\colon P \to \mathbb{R}$ be a smooth function on a closed Riemannian manifold P and $\xi = \nabla f$. The following conditions are equivalent:*

(i) f is a Morse function.

(ii) $H_p f\colon T_pP \times T_pP \to \mathbb{R}$ is non-degenerate whenever $df(p) = 0$.

(iii) $D_p\xi\colon T_pP \to T_pP$ is an isomorphism whenever $\xi(p) = 0$.

(iv) $\xi\colon P \to TP$ is transverse to the zero section.

Proof. The equivalence of the first three conditions follows from the definition of Morse functions and the formula $H_pf(v,w) = \langle v, D_p\xi(w)\rangle$ that we proved in an exercise last semester. The equivalence of (iii) and (iv) follows from unraveling the definition of transversality. □

The equivariant case. Suppose that a Lie group G acts smoothly on a manifold P.

(1) If G acts properly, then the orbits $Gx \subset P$ for $x \in P$ are smoothly embedded submanifolds diffeomorphic to G/G_x where G_x is the stabilizer of x.

(2) If G acts properly and freely, then the orbit space $B = P/G$ is a smooth manifold with a unique smooth structure such that the orbit map $q\colon P \to B$ is a submersion.

From now on we assume that G acts properly and freely on P.

(3) Combining (1) and (2) shows that all orbits Gx are diffeomorphic to G and the fibers of the sub-bundle $J = \ker(dq)$ can be canonically identified as

$$J_x = \ker(dq_x) = T_x G_x \underset{\cong}{\overset{dL_x}{\longleftarrow}} T_e G \tag{5.6.2}$$

where $L_x\colon G \to P$ is given by $g \mapsto gx$.

(4) The tangent space at $y = q(x) \in B$ can be identified as

$$T_y B \cong T_x P / J_x = T_x P / T_x Gx. \tag{5.6.3}$$

More globally, there is a short exact sequence of vector bundles over P

$$0 \to J \to TP \xrightarrow{dq} q^* TB \to 0. \tag{5.6.4}$$

(5) The G action on P lifts to a free and proper action on TP which leaves J invariant (more precisely, we have $g_* J_x = J_{gx}$) and we can identify the tangent bundle of B as

$$TB \cong (TP/J)/G \tag{5.6.5}$$

where the inner quotient is one of vector spaces while the outer means the passage to G–orbits.

(6) If P carries a G–invariant Riemannian metric, we get a G–invariant orthogonal splitting

$$TP = J \oplus K, \quad K = J^\perp \tag{5.6.6}$$

and an identification $TB \cong K/G$.

(7) Alternatively, suppose that there is a smooth submanifold $S \subset P$ such that for each $x \in S$ there is a splitting

$$T_x P = J_x \oplus T_x S. \tag{5.6.7}$$

In other words, S is transverse to all G–orbits that pass through it. Such a submanifold is called a *(local) slice* for the action. In that case we have $T_{q(x)} B \cong T_x S$ for all $x \in S$.

(8) If we drop the assumption that G acts freely, the tangent spaces to the orbits still from a set

$$J = \bigcup_{x \in P} J_x \subset TP, \quad J_x = T_x Gx. \tag{5.6.8}$$

However, this is generally not a sub-bundle, since the dimension of J_x depends on the stabilizer. The same applies to the orthogonal complements K_x taken with respect to a G–invariant metric on P.

With this understood, we obtain the following equivariant analogue of Lemma 5.10

Lemma 5.11. *Let $f\colon P \to \mathbb{R}$ be a G–invariant smooth function and $\xi = \nabla f$ its gradient with respect to a G–invariant metric. Then the following are equivalent:*

(i) f is a G–Morse function.

(ii) $H_x f\colon K_x \times K_x \to \mathbb{R}$ is non-degenerate whenever $df(x) = 0$

(iii) $D_x \xi\colon K_x \to K_x$ is an isomorphism whenever $\xi(x) = 0$.

(iv) $\xi\colon P \to TP$ is transverse to the subset $J = \bigcup_{x \in P} J_x$ along the zero section $z\colon P \to TP$ in the sense that whenever $\xi(x) = 0$ we have

$$T_{(x,0)} TP \, \big(= z_* T_x P \oplus T_0 T_x P \big) = \xi_* T_x P + z_* T_x P + J_x. \tag{5.6.9}$$

If G acts freely, then either condition is equivalent the induced map $\bar{f}\colon B \to \mathbb{R}$ being a Morse function.

5.6.2 The gauge theoretic setting

Lecture 9, 12.12.23

Now let us come back to the perturbed Seiberg–Witten 'vector fields'

$$\widetilde{\mathcal{X}}_{\mathfrak{q}} = \nabla \mathcal{L}_f \colon \mathcal{C}_k(Y) \to \mathcal{T}_{k-1} \quad \text{and} \quad \widetilde{\mathcal{X}}^{\sigma}_{\mathfrak{q}} \colon \mathcal{C}^{\sigma}_k(Y) \to \mathcal{T}^{\sigma}_{k-1} \tag{5.6.10}$$

with perturbation $\mathfrak{q} = \nabla f$ with potential $f \colon \mathcal{C}_k(Y) \to \mathbb{R}$. Both vector fields will play a role. We use the following notational convention (cf. [KM07]):

- ▶ Stationary points of $\widetilde{\mathcal{X}}^{\sigma}_{\mathfrak{q}}$ are denoted by $\mathfrak{a}, \mathfrak{b}$, etc.
- ▶ Stationary points of $\widetilde{\mathcal{X}}_{\mathfrak{q}}$ are denoted by α, β, etc.

The goal of this section is to define a reasonable notion of non-degeneracy for stationary points which can be achieved by carefully choosing $\mathfrak{q}$.

Defining non-degeneracy. Recall that $\mathcal{G}_{k+1}(Y)$ is a Hilbert Lie group. In the light of Lemma 5.11, we should be interested in the tangent spaces to $G_{k+1}(Y)$ orbits. To begin with, we have an identification

$$T_1\mathcal{G}_{k+1}(Y) \stackrel{\cong}{\longleftarrow} iL^2_{k+1}(Y;\mathbb{R}), \quad \xi \mapsto \frac{d}{dt}|_{t=0} e^{t\xi}. \tag{5.6.11}$$

Consequently, we can compute the differentials of the maps

$$L^{(\sigma)}_{\gamma} \colon \mathcal{G}_{k+1}(Y) \to \mathcal{C}^{(\sigma)}_k(Y), \quad u \mapsto u\gamma, \tag{5.6.12}$$

for example in the case of $\gamma = (B,\psi) \in \mathcal{C}_k(Y)$ and $\xi \in iL^2_{k+1}(Y,\mathbb{R})$ as

$$dL_{\gamma}\big|_1(\xi) = \tfrac{d}{dt}\big|_{t=0} e^{t\xi}(B,\psi) = \tfrac{d}{dt}\big|_{t=0}(B - td\xi, e^{t\xi}\psi) = (-d\xi, \xi\psi). \tag{5.6.13}$$

and similarly for $\gamma = (B, r, \psi) \in \mathcal{C}^{\sigma}_k(Y)$

$$dL^{\sigma}_{\gamma}\big|_1(\xi) = \cdots = (-d\xi, 0, \xi\psi). \tag{5.6.14}$$

The tangent spaces to the orbits are thus

$$\mathcal{J}_{k,\gamma} := \left\{(-d\xi, \xi\psi) \,\middle|\, \xi \in iL^2_{k+1}(Y;\mathbb{R})\right\} = T_{\gamma}\mathcal{G}_{k+1}\gamma \subset T_{\gamma}\mathcal{C}_k(Y) \qquad \gamma = (B,\psi) \in \mathcal{C}_k(Y)$$
$$\mathcal{J}^{\sigma}_{k,\gamma} := \left\{(-d\xi, 0, \xi\psi) \,\middle|\, \xi \in iL^2_{k+1}(Y;\mathbb{R})\right\} = T_{\gamma}\mathcal{G}_{k+1}\gamma \subset T_{\gamma}\mathcal{C}^{\sigma}_k(Y) \quad \gamma = (B, r, \psi) \in \mathcal{C}^{\sigma}_k(Y)$$

Similarly, we can define $\mathcal{J}^{(\sigma)}_{k,\gamma}$ for lower Sobolev orders $0 \le j \le k$. With this understood, Lemma 5.11 suggests the following definition.

Definition 5.12. A stationary point $\mathfrak{a}$ of $\widetilde{\mathcal{X}}^{\sigma}_{\mathfrak{q}} \colon \mathcal{C}^{\sigma}_k(Y) \to \mathcal{T}^{\sigma}_{k-1}$ is called *non-degenerate* if $\widetilde{\mathcal{X}}^{\sigma}_{\mathfrak{q}}$ is transverse to $\mathcal{J}^{\sigma}_{k-1}$ at $\mathfrak{a}$. An analogous definition applies to stationary points α of $\widetilde{\mathcal{X}}$.

Characterizing non-degeneracy. We can also mimic the formulation in Lemma 5.11(iii) of non-degeneracy in terms of linearizations. Thinking of the 'vector field' $\widetilde{\mathcal{X}}$ as a map

$$\widetilde{\mathcal{X}}_{\mathfrak{q}} \colon \mathcal{C}_k(Y) \to L^2_{k-1}(iT^*Y \oplus S_Y) \tag{5.6.15}$$

we have a canonical notion of derivative

$$D_{\gamma}\widetilde{\mathcal{X}}_{\mathfrak{q}} \colon \underbrace{L^2_k(iT^*Y \oplus S_Y)}_{=\mathcal{T}_{k,\gamma}} \to \underbrace{L^2_{k-1}(iT^*Y \oplus S_Y)}_{=\mathcal{T}_{k-1,\gamma}}, \quad \gamma \in \mathcal{C}_k(Y). \tag{5.6.16}$$

Since questions of non-degeneracy are concerned with directions complementary to the tangent spaces of orbits, we consider the fiberwise L^2–orthogonal splittings

$$\mathcal{T}_j = \mathcal{J}_j \oplus \mathcal{K}_j \qquad (0 \leq j \leq k) \tag{5.6.17}$$

where $\mathcal{K}_j$ is defined as the fiberwise L^2–orthogonal complement. By restricting and L^2 projecting $D_\gamma \widetilde{\mathcal{X}}_{\mathfrak{q}}$ we obtain a linear operator

$$\mathrm{Hess}_{\mathfrak{q},\gamma} \colon \mathcal{K}_{k,\gamma} \hookrightarrow \mathcal{T}_{k,\gamma} \xrightarrow{D_\gamma \widetilde{\mathcal{X}}_{\mathfrak{q}}} \mathcal{T}_{k-1,\gamma} \to \mathcal{K}_{k-1,\gamma} \tag{5.6.18}$$

which is an ad hoc version of the Hessian of $\mathcal{L}_f$.

This construction has a blown-up analogue, but this comes with an extra twist that lies in the definition of complementary sub-bundles to $\mathcal{J}_j^\sigma$. Instead of taking orthogonal complements, one proceeds as follows:

- Let $\mathcal{K}_j^*$ be the restriction of $\mathcal{K}_j$ to $\mathcal{C}_k^*(Y)$. This is a sub-bundle of the restriction $\mathcal{T}_j^*$.
- The blow-down map $\pi \colon \mathcal{C}_k^\sigma(Y) \to \mathcal{C}_k^*(Y)$ is a diffeomorphism over $\mathcal{C}_k^*(Y)$ and induces maps $\pi_* \colon \mathcal{T}_j^\sigma \to \mathcal{T}_j$ for $0 \leq j \leq k$.
- Define $\mathcal{K}_j^\sigma$ over $\mathcal{C}_k^*(Y)$ be requiring $\pi_* \mathcal{K}_j^\sigma = \mathcal{K}_j^*$.
- According to [KM07, 9.3.5] $\mathcal{K}_j^\sigma$ extends to a bundle over $\mathcal{C}_k^\sigma(X)$ such that there is a splitting $\mathcal{T}_j^\sigma = \mathcal{J}_j^\sigma \oplus \mathcal{K}_j^\sigma$. This splitting, however, is *not* orthogonal with respect to to any natural scalar product!

With this in place, a similar construction as above gives operators

$$\mathrm{Hess}^\sigma_{\mathfrak{q},\gamma} \colon \mathcal{K}^\sigma_{k,\gamma} \hookrightarrow \mathcal{T}^\sigma_{k,\gamma} \xrightarrow{D_\gamma \widetilde{\mathcal{X}}^\sigma} \mathcal{T}^\sigma_{k-1,\gamma} \to \mathcal{K}^\sigma_{k-1,\gamma} \tag{5.6.19}$$

where $D_\gamma \widetilde{\mathcal{X}}^\sigma$ is defined by viewing $\mathcal{C}_k^\sigma(Y)$ as a Hilbert submanifold of the affine Hilbert manifold $B_0 + L^2_k(i^*TY \oplus \mathbb{R} \oplus S_Y)$. Unraveling the transversality condition in Theorem 5.15 gives the following:

Lemma 5.13 (cf. [KM07, 12.4.1])**.** *A stationary point $\mathfrak{a}$ of $\widetilde{\mathcal{X}}^\sigma$ is non-degenerate if and only if the operator*

$$\mathrm{Hess}^\sigma_{\mathfrak{q},\mathfrak{a}} \colon \mathcal{K}^\sigma_{k,\gamma} \to \mathcal{K}^\sigma_{k-1,\gamma} \tag{5.6.20}$$

is surjective. An analogous statement holds for $\widetilde{\mathcal{X}}$ and $\mathrm{Hess}_{\mathfrak{q},\alpha}$.

We record an important property of the operators $\mathrm{Hess}_{\mathfrak{q},\gamma}$ without proof.

Proposition 5.14 (cf. [KM07, 12.3.1])**.** $\mathrm{Hess}_{\mathfrak{q},\gamma}$ *is a self–adjoint Fredholm operator.*

Achieving non-degeneracy. The next step is to show that $\widetilde{\mathcal{X}}^\sigma_{\mathfrak{q}}$ is non-degenerate for sufficiently many $\mathfrak{q}$. Recall from p. 40 that a countable intersection of dense, open subsets of a given space is called a *Baire set*[1] and that Baire sets in separable Banach spaces are dense.

Theorem 5.15 (cf. [KM07, 12.1.2])**.** *Let $\mathcal{P}$ be a large Banach space of tame perturbations. Then the perturbations $\mathfrak{q} \in \mathcal{P}$ for which all stationary points of $\widetilde{\mathcal{X}}^\sigma$ are non-degenerate form a Baire set.*

The proof is based on the following lemma

[1] Baire sets are called *residual* in [KM07].

Lemma 5.16 (cf. [KM07, 12.5.1]). *Let $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{P}$ be separable Banach spaces, $\mathcal{S} \subset F$ a closed submanifold, and*

$$F\colon \mathcal{E} \times \mathcal{P} \to \mathcal{F} \tag{5.6.21}$$

a smooth map. For fixed $p \in \mathcal{P}$ write $F_p = F(\cdot, p)\colon \mathcal{E} \to \mathfrak{F}$. Suppose that the following conditions are satisfied:

(a) F is transverse to $\mathcal{S}$.

(b) For all $(e,p) \in F^{-1}(\mathcal{S})$ the following composite is a Fredholm operator:

$$T_e\mathcal{E} \xrightarrow{dF_p|_e} T_f\mathcal{F} \xrightarrow{\text{quot}} T_f\mathcal{F}/T_f\mathcal{S}, \quad f = F(p,e). \tag{5.6.22}$$

Then the set of $p \in \mathcal{P}$ for which F_p is transverse to $\mathcal{S}$ is a Baire set.

Proof. The proof follows a common strategy (cf. [Nic11, Ch. 1.2]). The main steps are:

- Condition (a) ensures that $F^{-1}(\mathcal{S})$ is a Banach submanifold.
- Condition (b) ensures that the composition

$$Q\colon F^{-1}(\mathcal{S}) \hookrightarrow \mathcal{E} \times \mathcal{P} \xrightarrow{\text{pr}_2} \mathcal{P} \tag{5.6.23}$$

  is a Fredholm map.
- The Sard–Smale theorem (Theorem 2.32) gives a Baire set of regular values of Q.
- If $p \in \mathcal{P}$ is a regular value of Q, then F_p is transverse to $\mathcal{S}$. □

Proof of Theorem 5.15 (sketch). The proof has two parts:

(1) *Irreducible case:* points of the form (B, r, ψ) with $r \neq 0$

(2) *Reducible case:* points of the form $(B, 0, \psi)$.

The argument in the **irreducible case** goes as follows:

(1.1) The goal is to show that there is a Baire set of $\mathfrak{q} \in \mathcal{P}$ such that all irreducible zeros of $\widetilde{\mathcal{X}}^\sigma_\mathfrak{q}$ are non-degenerate.

(1.2) An irreducible configuration (B, r, ψ) is a non-degenerate zero of $\widetilde{\mathcal{X}}^\sigma_\mathfrak{q}$ if and only if $(B, r\psi)$ is a non-degenerate zero of $\widetilde{\mathcal{X}}_\mathfrak{q}$. So we can work with $\widetilde{\mathcal{X}}_\mathfrak{q}$ on the irreducible locus $\mathcal{C}^*_k(Y)$.

(1.3) We consider the 'parameterized zero sets'

$$\mathcal{Z}^* = \left\{(\alpha, \mathfrak{q}) \,\middle|\, \widetilde{\mathcal{X}}_\mathfrak{q}(\alpha) = 0\right\} \subset \mathcal{C}^*_k(Y) \times \mathcal{P} \tag{5.6.24}$$

We want to show that these are Banach manifolds.

(1.4) Recall that we have an L^2–orthogonal splitting $\mathcal{T}_j = \mathcal{J}_j \oplus \mathcal{K}_j$ where $\mathcal{J}_j$ consists of the tangent spaces of $\mathcal{G}_{k+1}(Y)$–orbits. Write $\mathfrak{q} \in \mathcal{P}$ as an L^2 gradient $\mathfrak{q} = \nabla f$. Then $\widetilde{\mathcal{X}}_\mathfrak{q} = \nabla(\mathcal{L} + f)$ and, since $\mathcal{L}$ and f are invariant under the identity component of $\mathcal{G}_{k+1}(Y)$, it follows that $\widetilde{\mathcal{X}}_\mathfrak{q}$ is orthogonal to $\mathcal{J}^*_j$ for all $\mathfrak{q} \in \mathcal{P}$.

(1.5) The restriction $\mathcal{K}^*_j$ of $\mathcal{K}_j$ to $\mathcal{C}^*_k(Y)$ is a Hilbert vector bundle and, in particular, a Hilbert manifold. By the above we can assemble all $\widetilde{\mathcal{X}}_\mathfrak{q}$ into a smooth map

$$\mathfrak{g}\colon \mathcal{C}^*_k(Y) \times \mathcal{P} \to \mathcal{K}^*_{k-1}, \quad \mathfrak{g}(\alpha, \mathfrak{q}) = \widetilde{\mathcal{X}}_\mathfrak{q}(\alpha) \tag{5.6.25}$$

and we have $\widetilde{\mathcal{Z}}^* = \mathfrak{g}^{-1}(0)$. Moreover, $\mathfrak{g}$ is transverse to the zero section:

- ▶ The transversality condition for $(\alpha, \mathfrak{q}) \in \widetilde{\mathcal{Z}}^*$ is equivalent to the surjectivity of

$$\mathcal{K}^*_{k,\alpha} \times \mathcal{P} \mapsto \mathcal{K}^*_{k-1,\alpha}, \quad \big((b,\psi),\mathfrak{h}\big) \mapsto \mathrm{Hess}_{\mathfrak{q},\alpha}(b,\psi) + \mathfrak{h}(\alpha). \tag{5.6.26}$$

- ▶ Since $\mathrm{Hess}_{\mathfrak{q},\alpha}$ is a self-adjoint Fredholm operator by Proposition 5.14, its cokernel and kernel agree and are both finite dimensional.
- ▶ Is thus suffices to produce for every $0 \neq v \in \ker \mathrm{Hess}_{\mathfrak{q},\alpha}$ and element $\mathfrak{h} \in \mathcal{P}$ with $\langle v, \mathfrak{h}(\alpha)\rangle_{L^2} \neq 0$.
- ▶ Writing $\mathfrak{h} = \nabla h$ for some $h\colon \mathcal{C}(Y) \to \mathbb{R}$ this is equivalent to $dh|_\alpha(v) \neq 0$.
- ▶ Lastly, the definition of 'large Banach spaces of perturbations' is made to ensure this property.

(1.6) It follows that $\widetilde{\mathcal{Z}}^*$ is a Banach manifold and, and so is $\mathcal{Z}^* = \widetilde{\mathcal{Z}}^*/\mathcal{G}_{k+1}(Y)$.

(1.7) We want to apply Lemma 5.16 to this situation:

- ▶ We have just verified the transversality condition (a) for $\mathfrak{g}$ and the zero section of $\mathcal{K}^*_{k-1}$.
- ▶ The condition (b) turns out be equivalent to the Fredholm property of $\mathrm{Hess}_{\mathfrak{q},\alpha}$.
- ▶ The conclusion is that the set of $\mathfrak{q} \in \mathcal{P}$ for which $\widetilde{\mathcal{X}}_\mathfrak{q}$, considered as a section of $\mathcal{K}^*_j$, is transverse to the zero section. Let us write $\mathcal{P}^*$ for this set.
- ▶ But this is equivalent to all irreducible zeros of $\widetilde{\mathcal{X}}_\mathfrak{q}$ being non-degenerate.

It remains to treat the **reducible case** which is considerably more involved. Lecture 10, 19.12.23

(2.1) Non-degeneracy for reducible zeros $(B, 0, \psi)$ of $\widetilde{\mathcal{X}}^\sigma_\mathfrak{q}$ can be characterized as follows:

- ▶ Let $\mathcal{T}^{\mathrm{red}}_j = \mathcal{A}_k \times L^2_j(iT^*Y)$ be the 'L^2_j tangent bundle' of $\mathcal{A}_k(S_Y)$.
- ▶ The action of $\mathcal{G}_{k+1}(Y)$ on $\mathcal{A}_k(S_Y)$ gives a fiberwise L^2-orthogonal splitting $\mathcal{T}^{\mathrm{red}}_j = \mathcal{J}^{\mathrm{red}}_j \oplus \mathcal{K}^{\mathrm{red}}_j$ where $\mathcal{J}^{\mathrm{red}}_j$ is tangent to the orbits.
- ▶ The 1–form component of the $\widetilde{\mathcal{X}}_\mathfrak{q} = \nabla(\mathcal{L}+f)$ gives defines a section $\widetilde{\mathcal{X}}^{\mathrm{red}}_\mathfrak{q}\colon \mathcal{A}_k(S_Y) \to \mathcal{T}^{\mathrm{red}}_{k-1}$.
- ▶ For fixed $B \in \mathcal{A}_k(S_Y)$ the linearization of the spinor component of $\widetilde{\mathcal{X}}_\mathfrak{q}$ at $(B, 0)$ gives rise to a linear operator $D_{B,\mathfrak{q}}\colon L^2_k(S_Y) \to L^2_{k-1}(S_Y)$ which is a compact perturbation of the Dirac operator D_B.

According to [KM07, 12.2.5], a reducible zero $\mathfrak{a} = (B, 0, \psi)$ of $\widetilde{\mathcal{X}}^\sigma_\mathfrak{q}$ is non-degenerate if and only if the following hold:

(a) B is a non-degenerate zero of $\widetilde{\mathcal{X}}^{\mathrm{red}}_\mathfrak{q}$

(b) ψ is a eigenvector of $D_{B,\mathfrak{q}}$ for a simple eigenvalue $\lambda \neq 0$ (i.e. the λ–eigenspace is 1–dimensional).

(2.2) A similar argument as in the irreducible case gives a Baire set $\mathcal{P}^{red}$ for which all zeros of $\widetilde{\mathcal{X}}^{\mathrm{red}}_\mathfrak{q}$ are non-degenerate.

(2.3) A more elaborate argument shows that achieving condition (b) at all zeros of $\widetilde{\mathcal{X}}^{\mathrm{red}}_\mathfrak{q}$ requires countably further conditions, indexed by $n \in \mathbb{N}$, say, each of which is satisfied for $\mathfrak{q}$ in a certain Baire set $\mathcal{P}_n$.

Altogether, we see that for $\mathfrak{q}$ in the intersection of the Baire sets $\mathcal{P}^*$, $\mathcal{P}^{\mathrm{red}}$ and $\mathcal{P}_n$ for all $n \in \mathbb{N}$ all zeros of $\widetilde{\mathcal{X}}^\sigma_\mathfrak{q}$ are non-degenerate. Since the intersection of countably many Baire sets is again a Baire sets, we are done. □

5.7 Energy and compactness

Energy on compact 4–manifolds. Let X be a compact spin^c 4–manifold with boundary $Y = \partial X$ and spinor bundle S_X.

(1) Recall that the induced spin^c structure on Y is represented by the spinor bundle

$$S_Y = S_X^+|_Y, \quad \rho_Y(b) = \rho_X(\nu^\#)^{-1}\rho_X(b), \quad b \in T^*Y \tag{5.7.1}$$

where $\nu^\#$ is the metric dual of the outward unit normal vector field. In what follows, let A be a spin^c connection on S_X and B the induced connection on S_Y.

(2) The Dirac operators on S_X and S_Y are related by the formula

$$D_B(\phi|_Y) = \big(\rho_X(\nu^\#)^{-1}D_A^+\phi - \nabla_\nu^A\phi\big)\big|_Y + \frac{H}{2}\phi|_Y, \quad \phi \in \Gamma(S_X) \tag{5.7.2}$$

where H is the mean curvature of Y in X. A general discussion of the concept can be found in [Jos17, Ch. 5.2] and a proof of the formula is given in [KM07, Lemma 4.5.1]. For the present purposes, it suffices to know that the mean curvature vanishes in the case that X is cylindrical near Y.

(3) The key to compactness results in Seiberg–Witten theory is the Weitzenböck formula

$$D_A^2\phi = (\nabla^A)^*\nabla^A\phi + \frac{1}{2}\rho_X(F_{A^t}^+) + \frac{s}{4}\phi \tag{5.7.3}$$

where s is the scalar curvature of X. The proof is a direct computation (cf. [Jos17, Thm. 4.4.2]). Note that (5.7.3) involves two terms in the Seiberg–Witten equations on X, namely $D_A\phi$ and $\frac{1}{2}F_{A^t}^+$.

(4) Following [KM07, Def. 4.5.4] we define the notions of *analytic* and *topological energy* of a configuration $(A, \phi) \in \mathcal{C}(X)$ as

$$\mathcal{E}^{\text{an}}(A,\phi) = \frac{1}{4}\int_X |F_{A^t}|^2 + \int_X |\nabla^A\phi|^2 + \frac{1}{4}\int_X (|\phi|^2 + s/2)^2 - \frac{1}{16}\int_X s^2 \tag{5.7.4}$$

$$\mathcal{E}^{\text{top}}(A,\phi) = \frac{1}{4}\int_X F_{A^t}\wedge F_{A^t} - \int_Y \langle \phi|_Y, D_B(\phi|_Y)\rangle + \int_Y \frac{H}{2}|\phi|^2. \tag{5.7.5}$$

It is straight forward to check that both these quantities are invariant under the action of $\mathcal{G}(X)$. Note that if X is closed, then the boundary terms in (5.7.5) vanish and $\mathcal{E}^{\text{top}}(A,\phi)$ is constant with value $-\pi^2 c_1^2(S_X)[X]$ which is a topological invariant of the spin^c structure.

(5) Using the formulas in (2) and (3) above, one can establish the **main energy identity**

$$\mathcal{E}^{\text{an}}(A,\phi) = \mathcal{E}^{\text{top}}(A,\phi) + \|\mathfrak{F}(A,\phi)\|^2_{L^2(X)} \tag{5.7.6}$$

where $\mathfrak{F}(A,\phi) = (\frac{1}{2}F_{A^t}^+ - \rho_X^{-1}(\phi\phi^*)_0, D_A\phi)$ is the usual Seiberg–Witten map.

Energy and compactness on compact cylinders. Now let Y be a closed spin^c 3–manifold and $Z = [t_1, t_2] \times Y$ a compact spin^c cylinder. As usual, given a configuration $\gamma = (A, \phi) \in \mathcal{C}(Z)$ we write $\check{\gamma}\colon [t_1, t_2] \to \mathcal{C}(Y)$ for the corresponding path in $\mathcal{C}(Y)$.

(6) In the cylinder case, the topological energy takes the more intuitive form

$$\mathcal{E}^{\text{top}}(\gamma) = 2\big(\mathcal{L}(\check{\gamma}(t_1)) - \mathcal{L}(\check{\gamma}(t_2))\big). \tag{5.7.7}$$

In words, the topological energy measures twice the change of $\mathcal{L}$ along the cylinder.

(7) If $\gamma \in \mathcal{C}(Z)$ is in temporal gauge, then the analytic energy can be expressed as

$$\mathcal{E}^{\mathrm{an}}(\gamma) = \int_{t_1}^{t_2} \left\|\dot{\check{\gamma}}(t)\right\|^2_{L^2(Y)} + \|\nabla\mathcal{L}(\check{\gamma}(t))\|^2_{L^2(Y)}\, dt. \tag{5.7.8}$$

In the light of the equivalence of the equations $\mathfrak{F}(\gamma) = 0$ and $\dot{\check{\gamma}} + \nabla\mathcal{L}(\check{\gamma}) = 0$, the main energy identity boils down to the observation that solutions of the latter (formal) downward gradient flow equations are characterized by the equality

$$2\big(\mathcal{L}(\check{\gamma}(t_1)) - \mathcal{L}(\check{\gamma}(t_2))\big) = \int_{t_1}^{t_2} \left\|\dot{\check{\gamma}}(t)\right\|^2_{L^2(Y)} + \|\nabla\mathcal{L}(\check{\gamma}(t))\|^2_{L^2(Y)}\, dt. \tag{5.7.9}$$

(8) The point of the discussion in (6) and (7) is an a posteriori justification for the admittedly out-of-the-blue definitions of $\mathcal{E}^{\mathrm{an}}$ and $\mathcal{E}^{\mathrm{top}}$ in (4). In hindsight, we could have started with the more intuitive identity (5.7.9) for $\dot{\gamma}$ and noted that the two sides of the equations can be expressed in terms of γ as in (4).

(9) The expression of $\mathcal{E}^{\mathrm{an}}$ in (4) has three main advantages over that in (7):

- It is defined for all configurations in $\mathcal{C}(Z)$ and not only those in temporal gauge.
- The resulting function $\mathcal{E}^{\mathrm{an}} : \mathcal{C}(Z) \to \mathbb{R}$ is invariant under the full gauge group $\mathcal{G}(Z)$ whereas the right hand side in (7) is only invariant under $\mathcal{G}(Y)$.
- The definition in (4) works for arbitrary compact 4–manifolds.

For the record, we note that the $\mathcal{G}(Z)$–invariant formula in (4) for an arbitrary configuration $\gamma = (A, \phi) \in \mathcal{C}(Z)$ with $A = \check{A} + c\,dt$ can be rewritten as

$$\mathcal{E}^{\mathrm{an}}(\gamma) = \int_{t_1}^{t_2} \left\|\dot{\check{A}}(t) - d_Y c\right\|^2 + \left\|\dot{\check{\phi}}(t) - c\check{\phi}\right\|^2 + \|\nabla\mathcal{L}(\check{\gamma}(t))\|^2\, dt \tag{5.7.10}$$

with L^2 norms understood everywhere.

The following two theorems indicate the usefulness of the notions of energy.

Theorem 5.17 (Finiteness theorem, compact cylinder case, cf. [KM07, 5.1.1(i)]). *Let Y be a closed, oriented, connected Riemannian 3–manifold and $Z = [t_1, t_2] \times Y$ a compact cylinder with base Y. For every $C \in \mathbb{R}$ there are only finitely many spin^c structures on Y (and hence on Z) such that the equation $\mathfrak{F}(\gamma) = 0$ has solutions $\gamma \in \mathcal{C}(Z)$ with $\mathcal{E}^{\mathrm{top}}(\gamma) \leq C$.*

Proof. Assuming that $\mathcal{F}(\gamma) = 0$ and $\mathcal{E}^{\mathrm{top}}(\gamma) \leq C$, the main energy identity gives $\mathcal{E}^{\mathrm{an}}(\gamma) \leq C$ which, among other things gives an upper bound

$$\int_Z |F_{A^t}|^2 \leq C + \frac{1}{16}\int_Z s^2 = C'. \tag{5.7.11}$$

The right hand side is constant as long as the metric on Y is fixed. This, in turn, gives upper bounds

$$\int_Z F_{A^t} \wedge \omega \leq C' \,\|\omega\|_{L^2(Y)}\,, \quad \omega \in \Omega^2(Z). \tag{5.7.12}$$

Since $F_{A^t}/2\pi i$ represents $c_1(S_Z)$ and de Rham cohomology with compact supports in the interior of Z computes $H^*(Z, \partial Z; \mathbb{R})$, the above bounds leave only finitely many possibilities for $c_1(S_Z)$ and thus for spin^c structures on Y. □

Theorem 5.18 (Compactness theorem for compact cylinders, cf. [KM07, 5.1.8]). *Let $Z = [t_1, t_2] \times Y$ be a compact spin^c cylinder with Y closed and connected. Suppose that the following is given:*

- *$\gamma_n \in \mathcal{C}(Z)$ is a sequence of smooth solutions of $\mathcal{F}(\gamma_n) = 0$.*
- *$\mathcal{E}^{\mathrm{top}}(\gamma_n) = 2\big(\mathcal{L}(\check{\gamma}_n(t_1)) - \mathcal{L}(\check{\gamma}_n(t_2)\big) \leq C$ for some $C \in \mathbb{R}$ uniformly in n.*

Then there exists a sequence of smooth gauge transformations $u_n \in \mathcal{G}(Z)$ such that $u_n\gamma_n$ has a subsequence that converges uniformly in the C^∞ topology in $\mathcal{C}(Z')$ for every compact sub-cylinder $Z' = [t_1', t_2'] \times Y$ with $t_1 < t_1' < t_2' < t_2$.

Energy and perturbations. Let Y and $Z = [t_1, t_2] \times Y$ be as above. The previous compactness theorem only deals with the unperturbed flow equation $\dot{x} + \nabla\mathcal{L}(x) = 0$ in $\mathcal{C}(Y)$. Unfortunately, the unperturbed equations generally suffer from non-degeneracies that prohibit a direct adaptation of the Floer homology construction, making perturbations strictly necessary.

(10) Let $\mathfrak{q}\colon \mathcal{C}(Y) \to L^2(iT^*Y \oplus S_Y)$ be a continuous map. From this we get a map

$$\hat{\mathfrak{q}}\colon \mathcal{C}(Z) \to L^2(i\Lambda^2_+ Z \oplus S_Z^-) \tag{5.7.13}$$

defined as follows:

- For $\gamma = (A, \phi) \in \mathcal{C}(Z)$ consider the continuous path $\check{\gamma}\colon [t_1, t_2] \to \mathcal{C}(Y)$.
- Compose with $\mathfrak{q}$ to get a continuous path $\mathfrak{q} \circ \check{\gamma}\colon [t_1, t_2] \to L^2(iT^*Y \oplus S_Y)$.
- The path interpretations of $\Omega^2_+(Z)$ and $\Gamma(S_Z^-)$ discussed in Section 2.4.7 gives rise to a continuous map

$$C^0\big([t_1, t_2], L^2(iT^*Y \oplus S_Y)\big) \to L^2(i\Lambda^2_+ Z \oplus S_Z^-). \tag{5.7.14}$$

Define $\mathfrak{q}(\gamma)$ as the image of $\mathfrak{q} \circ \check{\gamma}$.

(11) If $\mathfrak{q}$ is a tame perturbation in the sense of [KM07, 10.5.1], then the construction in (9) determines a smooth maps

$$\hat{\mathfrak{q}}\colon \mathcal{C}_k(Z) \to L^2_k(i\Lambda^2_+ Z \oplus S_Z^-) \quad (\forall k \geq 2) \tag{5.7.15}$$

Combined with the inclusion $L^2_k \hookrightarrow L^2_{k-1}$ we obtain a perturbed monopole map

$$\mathfrak{F}_{\mathfrak{q}} = \mathfrak{F} + \hat{\mathfrak{q}}\colon \mathcal{C}_k(Z) \to L^2_{k-1}(i\Lambda^2_+ Z \oplus S_Z^-) \tag{5.7.16}$$

and for $\gamma \in \mathcal{C}(Z)$ in temporal gauge we have

$$\mathfrak{F}_{\mathfrak{q}}(\gamma) = 0 \quad \Leftrightarrow \quad \dot{\check{\gamma}} + \widetilde{\mathcal{X}}_{\mathfrak{q}}(\check{\gamma}) = 0 \tag{5.7.17}$$

where $\widetilde{\mathcal{X}}_{\mathfrak{q}} = \nabla\mathcal{L} + \mathfrak{q}$. Lecture 11, 9.1.24

(12) Now let $\mathfrak{q} = \nabla f$ be a tame perturbation with potential $f\colon \mathcal{C}(Y) \to \mathbb{R}$. If we write $\mathcal{L}_f = \mathcal{L} + f$, we get $\widetilde{\mathcal{X}}_{\mathfrak{q}} = \nabla\mathcal{L}_f$. Given $\gamma = (A, \phi) \in \mathcal{C}(Z)$, we can simply take the formulas in (6) and (9), replace $\mathcal{L}$ with $\mathcal{L}_f$, and define

$$\begin{aligned}
\mathcal{E}^{\mathrm{an}}_{\mathfrak{q}}(\gamma) &= \int_{t_1}^{t_2} \left\|\dot{\check{A}}(t) - d_Y c\right\|^2 + \left\|\dot{\check{\phi}}(t) - c\check{\phi}\right\|^2 + \|\nabla\mathcal{L}_f(\check{\gamma}(t))\|^2 \, dt \\
\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(\gamma) &= 2\big(\mathcal{L}_f(\check{\gamma}(t_1)) - \mathcal{L}_f(\check{\gamma}(t_2))\big).
\end{aligned}$$

(13) The relation of $\mathcal{E}^{\mathrm{an}}_{\mathfrak{q}}$, $\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}$, and $\mathfrak{F}_{\mathfrak{q}}$ is not as straight forward as in the unperturbed case. However, it is true that $\mathcal{F}_{\mathfrak{q}}(\gamma) = 0$ implies $\mathcal{E}^{\mathrm{an}}_{\mathfrak{q}}(\gamma) = \mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(\gamma)$ and that $\mathcal{E}^{\mathrm{an}}_{\mathfrak{q}}(\gamma)$ controls the L^2 norms of F_{A^t} and $\nabla^A\phi$ (cf. [KM07, 10.6.1]). This is enough to prove the following refined compactness theorem:

Theorem 5.19 (Compactness for compact cylinder with perturbations, cf. [KM07, 5.1.8])**.** *Let $Z = [t_1, t_2] \times Y$ be a compact spin^c cylinder with Y closed and connected. Suppose that the following is given:*

- *$\mathfrak{q}$ is a tame perturbation with potential f (i.e. $\mathfrak{q} = \nabla f$).*
- *$\gamma_n \in \mathcal{C}_k(Z)$ is a sequence of solutions of $\mathfrak{F}_{\mathfrak{q}}(\gamma_n) = 0$ for some $k \geq 3$.*

▶ $\mathcal{E}^{\text{top}}_{\mathfrak{q}}(\gamma_n) = 2\big(\mathcal{L}_f(\check{\gamma}_n(t_1)) - \mathcal{L}_f(\check{\gamma}_n(t_2)\big) \leq C$ *for some* $C \in \mathbb{R}$ *uniformly in* n.

Then there exist gauge transformations $u_n \in \mathcal{G}_{k+1}(Z)$ *such that* $u_n\gamma_n$ *has a subsequence that converges uniformly in* $\mathcal{C}_{k+1}(Z')$ *for every compact sub-cylinder* $Z' \subset Z$.

The proof also gives a regularity result.

Proposition 5.20 (cf. [KM07, 10.7.2&3]). *Let* $\mathfrak{q}$ *be a tame perturbation and* $\gamma \in \mathcal{C}_k(Z)$ *a solution of* $\mathfrak{F}_{\mathfrak{q}}(\gamma) = 0$. *Then there exist a gauge transformation* $u \in \mathcal{G}_{k+1}(Z)$ *such that:*

(i) The restriction of $u\gamma$ *to any compact sub-cylinder* Z' *in the interior of* Z *is contained in* $\mathcal{C}_{k+1}(Z') \to \mathcal{C}_k(Z)$.

(ii) $u\gamma$ *determines an* $L^2_{1,\text{loc}}$ *path* $(t_1, t_2) \to \mathcal{C}_k(Y)$.

As an application, we obtain a compactness result for the zero sets of the vector field $\widetilde{\mathcal{X}}_{\mathfrak{q}} = \nabla\mathcal{L} + \mathfrak{q}$.

Compactness and blowing up. We are not quite done yet, since we are still lacking a compactness theorem that applies to the blown-up configurations spaces.

(1) We fix a tame perturbation $\mathfrak{q}$ and write it as $\mathfrak{q} = \nabla f$. Recall that applying the blow-up construction $\widetilde{\mathcal{X}}_{\mathfrak{q}} = \nabla\mathcal{L} + \mathfrak{q}$ produces a 'vector field'

$$\widetilde{\mathcal{X}}^{\sigma}_{\mathfrak{q}} \colon \mathcal{C}^{\sigma}_k(Y) \to \mathcal{T}^{\sigma}_{k-1}. \tag{5.7.18}$$

A smooth path $\check{\gamma}^{\sigma} \colon [t_1, t_2] \to \mathcal{C}^{\sigma}_k(Y)$ gives rise to an element in the τ-blow-up of $\mathcal{C}(Z)$

$$\gamma^{\tau} \in \mathcal{C}^{\tau}(Z) = \left\{(A, s, \phi) \in \mathcal{A}_k(S_Z) \times L^2_k(R) \times L^2_k(S^+_Z) \,\middle|\, s \geq 0\right\}. \tag{5.7.19}$$

The perturbation $\mathfrak{q}$ gives rise to a perturbed τ-version of the Seiberg–Witten map

$$\mathfrak{F}^{\tau}_{\mathfrak{q}} = \mathfrak{F}^{\tau} + \hat{\mathfrak{q}}^{\tau} \colon \mathcal{C}_k(Z) \to L^2_{k-1}(i\Lambda^2_+ Z \oplus \mathbb{R} \oplus S^-_Z) \tag{5.7.20}$$

where $\hat{\mathfrak{q}}^{\tau}$ is defined using $\mathfrak{q}^{\sigma}$ similarly as $\hat{\mathfrak{q}}$ was defined using $\mathfrak{q}$ (cf. [KM07, p. 158]). With these definitions, we get

$$\dot{\check{\gamma}}^{\sigma} + \widetilde{\mathcal{X}}^{\sigma}_{\mathfrak{q}}(\check{\gamma}^{\sigma}) = 0 \quad \Leftrightarrow \quad \mathfrak{F}^{\tau}_{\mathfrak{q}}(\gamma^{\tau}) = 0. \tag{5.7.21}$$

Again, the point is that the equation $\mathfrak{F}^{\tau}_{\mathfrak{q}}(\gamma^{\tau}) = 0$ makes sense for arbitrary $\gamma^{\tau} \in \mathcal{C}^{\tau}_k(Z)$ and is invariant under $\mathcal{G}_{k+1}(Z)$, whereas the flow equation only applies to configurations in temporal gauge.

(2) Given a smooth configuration $\gamma^{\tau} \in \mathcal{C}^{\tau}(Z)$ write

▶ $\check{\gamma}^{\sigma}$ for the corresponding path in $\mathcal{C}^{\sigma}(Y)$,

▶ $\gamma \in \mathcal{C}_k(Z)$ for the blow-down,

▶ $\check{\gamma}$ for the corresponding path in $\mathcal{C}^{\sigma}(Y)$.

In order to prove a compactness result for solutions of $\mathfrak{F}^{\tau}_{\mathfrak{q}}(\gamma^{\tau}) = 0$, we have to control the function $\mathcal{L}$ along $\check{\gamma}$ (aka the topological energy of γ) and another function

$$\Lambda_{\mathfrak{q}} \colon \mathcal{C}^{\sigma}_k(Y) \to \mathbb{R}, \quad \Lambda_{\mathfrak{q}}(B, r, \psi) = \big\langle \psi, D_B\psi + \tilde{\mathfrak{q}}^1(B, r\psi)\big\rangle_{L^2} \tag{5.7.22}$$

where $\mathfrak{q}^1$ is the spinor component of $\mathfrak{q}$ and

$$\tilde{\mathfrak{q}}^1(B, r\psi) = \int_0^1 D\mathfrak{q}^1(B, sr\psi)(0, \psi)ds. \tag{5.7.23}$$

Theorem 5.21 (Compactness for blow-ups on compact cylinders with perturbation, cf. [KM07, 10.9.2])**.** *Let $Z = [t_1, t_2] \times Y$ be a compact spin^c cylinder with Y. Suppose that the following is given:*

- ▶ *$\mathfrak{q}$ is a tame perturbation with potential f (i.e. $\mathfrak{q} = \nabla f$).*
- ▶ *$\gamma_n^\tau \in \mathcal{C}_k^\tau(Z)$ is a sequence of solutions of $\mathfrak{F}_\mathfrak{q}^\tau(\gamma_n^\tau) = 0$ for some $k \geq 3$.*
- ▶ *$\mathcal{E}_\mathfrak{q}^{\mathrm{top}}(\gamma_n) = 2\big(\mathcal{L}_f(\check{\gamma}_n(t_1)) - \mathcal{L}_f(\check{\gamma}_n(t_2)\big) \leq C_1$ for some $C_1 \in \mathbb{R}$ uniformly in n.*
- ▶ *$\Lambda_\mathfrak{q}(\check{\gamma}^\tau(t_1 + \epsilon)) \leq C_2$ and $\Lambda_\mathfrak{q}(\check{\gamma}^\tau(t_2 - \epsilon)) \geq -C_2$ for some $0 < \epsilon < (t_2 - t_1)/2$ and $C_2 \in \mathbb{R}$.*

Then there exist gauge transformations $u_n \in \mathcal{G}_{k+1}(Z)$ such that $u_n \gamma_n^\tau$ has a subsequence that converges uniformly in $\mathcal{C}_{k+1}(Z')$ for every compact sub-cylinder $Z' = [t_1', t_2'] \times Y$ with $t_1 + \epsilon < t_1' < t_2' < t_2 - \epsilon$.

An application of the compactness theorem. Lecture 12, 23.1.24

Corollary 5.22 (cf. [KM07, 10.7.4])**.** *Let $\mathfrak{q}$ be a tame perturbation. Then image in $\mathcal{B}_k(Y)$ of the zero set of $\widetilde{\mathcal{X}}_\mathfrak{q}$ is compact. In particular, it is finite if all zeroes of $\widetilde{\mathcal{X}}_\mathfrak{q}$ are non-degenerate.*

Proof. (1) Let $\alpha_n \in \mathcal{C}_k(Y)$ with $\widetilde{\mathcal{X}}_\mathfrak{q}(\alpha_n) = 0$.

(2) Let $\gamma_n \in \mathcal{C}_k([-3,3] \times Y)$ be the corresponding sequence of translation invariant solutions of $\mathfrak{F}_\mathfrak{q}(\gamma_n) = 0$ on Y on $\mathbb{R} \times Y$ restricted to $[-3,3] \times Y$.

(3) Since $\check{\gamma}_n(t) = \alpha$ for all t, we have $\mathcal{E}_\mathfrak{q}^{\mathrm{top}}(\gamma_n) = 0$.

(4) Theorem 5.19 gives $u_n \in \mathcal{G}_{k+1}([-1,1] \times Y)$ such that $u_n \gamma_n$ has a subsequence that converges in $\mathcal{C}_{k+1}([-1/2, 1/2] \times Y)$.

(5) Restricting to $\{0\} \times Y$ and passing to $\mathcal{B}_k(Y)$ gives a convergent subsequence of $[\alpha_n]$. □

5.8 Towards monopole Floer homology

Before we narrow in on the missing pieces for the definition of monopole Floer homology, we take a look back to remind us what we already have. We begin with the diagram

$$\begin{array}{ccc} \mathcal{C}_k^\sigma(Y) & \xrightarrow{\pi} & \mathcal{C}_k(Y) \\ \big\downarrow{\scriptstyle q^\sigma} & & \big\downarrow{\scriptstyle q} \\ \mathcal{B}_k^\sigma(Y) & \xrightarrow{\bar{\pi}} & \mathcal{B}_k(Y) \end{array}$$

involving the various completed configuration spaces for Y.

(1) Let I be any interval. The ordinary Seiberg–Witten equations (before blow-ups, perturbations, and completions) for smooth configurations $\gamma \in \mathcal{C}(I \times Y)$ in temporal gauge can be expressed in the two equivalent ways

$$\mathfrak{F}(\gamma) = 0 \quad \Leftrightarrow \quad \dot{\check{\gamma}} + \widetilde{\mathcal{X}}(\check{\gamma}) = 0 \tag{5.8.1}$$

where $\check{\gamma} \colon I \to \mathcal{C}(Y)$ is the path interpretation of γ. We refer to the left and right hand sides as the *4d and 3d versions* of the Seiberg–Witten equations on $I \times Y$. The general philosophy is this:

- ▶ The 3d equations give intuition for the constructions.
- ▶ The 4d equations are the objects of interest and the main tools for proofs.

(2) Before moving on, let us recap the roles of perturbations, blow-ups, and completions:

- ▶ Perturbations give the necessarily regularity of (spaces of) solutions of the equations.
- ▶ The blow-up process is a means to deal with the gauge equivariance.
- ▶ The Sobolev completions provide Hilbert manifold structures that facilitate the infinite dimensional analysis.

(3) The considerations about $\mathbb{T}$–equivariant Morse theory and Floer homology suggest that we should study the perturbed and blown-up versions of 3d equations

$$\dot{x} + \mathcal{X}_{\mathfrak{q}}^{\sigma}(x) = 0, \quad x\colon I \to \mathcal{B}^{\sigma}(Y) \tag{5.8.2}$$

for C^1 curves in the quotient space $\mathcal{B}^{\sigma}(Y)$ defined on intervals $I \subset \mathbb{R}$. Monopole Floer homology should arise from chain complexes with. . .

- ▶ . . . chain groups generated by the zeros of $\mathcal{X}_{\mathfrak{q}}^{\sigma}$ in $\mathcal{B}^{\sigma}(Y)$, and
- ▶ . . . differential counting solutions of $\dot{x} + \mathcal{X}_{\mathfrak{q}}^{\sigma}(x) = 0$ asymptotic to zeros of $\mathcal{X}_{\mathfrak{q}}^{\sigma}$.

Moreover, the finite dimensional theory suggests that we should study solutions of (5.8.2) with domain $I = \mathbb{R}$ along which the functions $\mathcal{L} \circ \pi$ and $\Lambda_{\mathfrak{q}}$ defined in (5.7.22) are bounded.

(4) Let us take a closer look at the the zero sets of the various vector fields:

- ▶ $\mathcal{Z}_{\mathfrak{q}}^{\sigma}$ is the zero set of $\mathcal{X}_{\mathfrak{q}}^{\sigma}$ in $\mathcal{B}_k^{\sigma}(Y)$.
- ▶ $\widetilde{\mathcal{Z}}_{\mathfrak{q}}^{\sigma}$ is the zero set of $\widetilde{\mathcal{X}}_{\mathfrak{q}}^{\sigma}$ in $\mathcal{C}_k^{\sigma}(Y)$.
- ▶ $\widetilde{\mathcal{Z}}_{\mathfrak{q}}$ is the zero set of $\widetilde{\mathcal{X}}_{\mathfrak{q}}$ in $\mathcal{C}_k(Y)$.

From Theorem 5.15 and Corollary 5.22 we know:

- ▶ For generic $\mathfrak{q}$ all zeros of $\mathcal{X}_{\mathfrak{q}}^{\sigma}$ will be non-degenerate.
- ▶ Assuming this, $\pi(\mathcal{Z}_{\mathfrak{q}}^{\sigma}) = q(\widetilde{\mathcal{Z}}_{\mathfrak{q}})$ is finite.

Remembering how zeros of $\widetilde{\mathcal{X}}_{\mathfrak{q}}^{\sigma}$ relate to those of $\widetilde{\mathcal{X}}_{\mathfrak{q}}$, we can conclude:

- ▶ $\widetilde{\mathcal{Z}}_{\mathfrak{q}}$ consists of finitely many gauge orbits.
- ▶ Each irreducible gauge orbit in $\widetilde{\mathcal{Z}}_{\mathfrak{q}}$ contributes an irreducible zero of $\mathcal{X}_{\mathfrak{q}}^{\sigma}$ and vice versa. In particular, $\mathcal{Z}_{\mathfrak{q}}^{\sigma}$ contains only finitely many irreducible zeros.
- ▶ Each reducible gauge orbit in $\widetilde{\mathcal{Z}}_{\mathfrak{q}}$, say $[B, 0]$, contributes countably many reducible zeros in $\mathcal{Z}_{\mathfrak{q}}^{\sigma}$ correspond to the eigenvalues of the operator $D_{B,\mathfrak{q}}$ that appeared in the proof of Theorem 5.15.
- ▶ To sum up, $\mathcal{Z}_{\mathfrak{q}}$ has a finite irreducible part and a countably infinite reducible part

(5) One could try to make all of this precise using only smooth configurations in the language of Fréchet manifolds. However, it is technically more convenient to use Sobolev completions and to work 'upstairs' in $\mathcal{C}^{\sigma}(Y)$ in the affine space. From the 3d perspective, this puts the equations the following equations on the map:

$$\dot{x} + \widetilde{\mathcal{X}}_{\mathfrak{q}}^{\sigma}(x) = 0, \quad x\colon I \to \mathcal{C}_k^{\sigma}(Y) \tag{5.8.3}$$

Note that the natural habitat for these would be something like $L^2_{1,\mathrm{loc}}(I, \mathcal{C}_k^{\sigma}(Y))$, the space of $L^2_{1,\mathrm{loc}}$ paths in $\mathcal{C}_k^{\sigma}(Y)$, which requires some sense making that we have not and will not do.

(6) Time for a word of warning about the shortcomings of the path interpretation with respect to blow-ups and Sobolev completions:

- ▶ As we have seen, temporal gauge configurations in $\mathcal{C}^\sigma(I \times Y)$ do not have path interpretations, in general. One way out was to use the τ–blow-up of $\mathcal{C}^\tau(I \times Y)$ which has 'underlying path' and 'associated temporal gauge configuration' maps

$$\mathcal{C}^\tau(I \times Y) \to C^\infty(I, \mathcal{C}^\sigma(Y)) \to \mathcal{C}^\tau(I \times Y) \tag{5.8.4}$$

  whose composition restricts to the identity on temporal gauge configurations.

- ▶ Another shortcoming of the path interpretation is that it does not interact well with Sobolev completions. While an elaboration on Fubini's theorem provides continuous extensions to L^2_{loc} completions

$$\mathcal{C}^\tau_{0,\mathrm{loc}}(I \times Y) \to L^2_{\mathrm{loc}}(I, \mathcal{C}^\sigma_0(Y)) \to \mathcal{C}^\tau_{0,\mathrm{loc}}(I \times Y) \tag{5.8.5}$$

  Unfortunately, the images of the subspaces $\mathcal{C}^\tau_{0,\mathrm{loc}}(I \times Y)$ and $L^2_{r,\mathrm{loc}}(I, \mathcal{C}^\sigma_s(Y))$ under these maps are not easily characterized.

The bottom line is to take the philosophy in (1) seriously. It would be ill advised to base the entire analysis on the 3d equations, since the natural habitat for the 4d equations is much simpler and we are ultimately interested in the solutions to the 4d equations anyway.

(7) Speaking of 4d equations, the remarks in (6) highlight the importance of the equivalence

$$\mathfrak{F}^\tau_{\mathfrak{q}}(\gamma^\tau) = 0 \quad \Leftrightarrow \quad \dot{\check{\gamma}}^\sigma + \widetilde{\mathcal{X}}^\sigma(\check{\gamma}^\sigma) = 0 \tag{5.8.6}$$

for smooth $\gamma^\tau \in \mathcal{C}^\tau(I \times Y)$ in temporal gauge and its underlying path $\check{\gamma}^\sigma$. The main selling points of the 4d equations $\mathfrak{F}^\tau_{\mathfrak{q}}(\gamma^\tau) = 0$ are:

- ▶ They can be studied in the Sobolev completions $\mathcal{C}^\tau_{k,\mathrm{loc}}(I \times Y)$ in which all derivatives (in I and Y directions) are treated equally.
- ▶ They are $\mathcal{G}_{k+1,\mathrm{loc}}(I \times Y)$ invariant.
- ▶ They are elliptic modulo the action of $\mathcal{G}_{k+1,\mathrm{loc}}(I \times Y)$.

Among other things, ellipticity implies that all solutions of $\mathfrak{F}^\tau_{\mathfrak{q}}(\gamma^\tau)$ are smooth (see Theorem 5.23 below) so that the Sobolev completions are merely technical baggage.

5.9 Moduli spaces of trajectories

Theorem 5.23. *Let $\gamma^\tau \in \mathcal{C}^\tau_{k,\mathrm{loc}}(I \times Y)$ be a solution of $\mathfrak{F}^\tau_{\mathfrak{q}}(\gamma^\tau) = 0$ for some $k \geq 3$ and some tame perturbation $\mathfrak{q}$. Then there is a gauge transformation $u \in \mathcal{G}_{k+1,\mathrm{loc}}(Z)$ such that $u\gamma^\tau$ is smooth in the interior of $I \times Y$.*

Proof. If I is compact, we can argue similarly as for closed manifolds. Let $\gamma^\tau = (A, s, \phi)$ and write $A = A_0 + a$ for some smooth reference connection $\mathcal{A}_0$ with $a \in L^2_k(iT^*Z)$. We may assume that A is in *Coulomb–Neumann gauge* with respect to A_0, meaning that

$$d^*a = 0 \text{ on } Z \quad \text{and} \quad a(\nu) = 0 \text{ on } \partial Z \tag{5.9.1}$$

where ν is the unit outward normal field. For if not, then we can find a gauge transformation of the form $u = e^\xi$ where $\xi \in L^2_{k+1}(Z; i\mathbb{R})$ is the unique a solution of the Neumann boundary value problem with normalization

$$\Delta\xi = d^*a, \quad d\xi(\nu) = a(\nu), \quad \int_{\{0\}\times Y} \xi = 0. \tag{5.9.2}$$

Since the equation d^*a together with $\mathfrak{F}^\tau(A, s, \phi) = 0$ is an elliptic system, a bootstrapping argument similar to the closed case shows that every Coulomb–Neumann solution in the interior of $I \times Z$.

If I is non-compact, we can cover it with countably many intervals I_n, $n \in \mathbb{Z}$ such that non-consecutive intervals are disjoint (i.e. $I_n \cap I_{n+2}$) consecutive intervals intersect such that the right end of I_n lies in the interior of I_{n+1} and the left end of I_{n+1} lies in the iterior of I_n. For each n write $\gamma_n \in L^2_k(I_n \times Y)$ for the restriction of γ and choose $u_n = e^{\xi_n}$ as above such that $u_n\gamma_n$ is smooth on the interior of $I_n \times Y$. The difference $\xi_{n+1} - \xi_n$ is necessarily smooth on the interior of $(I_{n+1} \cap I_n) \times Y$. From here on, one can patch together to functions ξ_n using a smooth partition of unity for I subordinate the open cover given by the interiors of the I_n to obtain $\xi \in iL^2_{k,\mathrm{loc}}(I \times Z)$ such that $e^f\gamma$ is smooth on the interior of I. □

Part III

Appendix

Appendix A

Background Material

A.1 Riemannian geometry

Let M be an oriented Riemannian n–manifold.

The Levi–Civita connection. We write ∇ for the Levi–Civita connection on TM, its dual T^*M, and tensor products involving the two. Recall that the dual connection on T^*M and $T^{r,s}M = (TM)^{\otimes r} \otimes (T^*M)^{\otimes s}$ is determined by the Leibniz rules

$$\nabla_X(\alpha(Y)) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y) \quad \text{and} \tag{A.1.1}$$

$$\nabla(S \otimes T) = (\nabla S) \otimes T + S \otimes (\nabla T). \tag{A.1.2}$$

where $\alpha \in \Omega^1(X)$ and S, T are sections of tensor bundles. If $e_1, \dots, e_n$ is an oriented local orthonormal frame for TM, we write $e^1, \dots, e^n$ for the dual coframe for T^*M determined by $e^i(e_j) = \delta^i_j$ and abbreviate the Levi–Civita connection as $\nabla_i = \nabla_{e_i}$.

Exterior calculus. We think of $\Lambda^* T^*M$ as the bundle of alternating multilinear maps on TM. The *wedge product* or *exterior multiplication*

$$\wedge \colon \Lambda^p T^*M \otimes \Lambda^q T^*M \to \Lambda^{p+q} T^*M \tag{A.1.3}$$

is defined using the convention

$$\omega \wedge \eta(Y_1, \dots Y_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} (-1)^\sigma \omega(Y_{\sigma(1)}, \dots, Y_{\sigma(p)}) \eta(Y_{\sigma(p+1)}, \dots, Y_{\sigma(p+q)}). \tag{A.1.4}$$

The wedge product is associative and graded commutative in the sense that

$$(\omega \wedge \eta) \wedge \xi = \omega \wedge (\eta \wedge \xi) \quad \text{and} \quad \omega \wedge \eta = (-1)^{|\omega||\eta|} \eta \wedge \omega. \tag{A.1.5}$$

There is another operation on $\Lambda^* T^*M$ known as *interior multiplication* or *contraction* with a vector $v \in T_x M$ defined by

$$v \lrcorner \colon \Lambda^p T^*_M \to \Lambda^{p-1} T^*_M, \quad (v \lrcorner \omega)(w_1, \dots, w_{p-1}) = \omega(v, w_1, \dots, w_{p-1}). \tag{A.1.6}$$

Interior and exterior multiplication are adjoint in the sense that

$$\langle v \lrcorner \omega, \eta \rangle = \left\langle \omega, v^\flat \wedge \eta \right\rangle \tag{A.1.7}$$

where $v^\flat = \langle v, \cdot \rangle$ is the metric dual of v. The contraction of a wedge product can be computed using the graded Leibniz rule

$$v \lrcorner (\omega \wedge \eta) = (v \lrcorner \omega) \wedge \eta + (-1)^{|\omega|} \omega \wedge (v \lrcorner \eta). \tag{A.1.8}$$

Differential forms. Let $\Omega^p(M) = \Gamma(\Lambda^p T^*M)$. The *de Rham differential* or *exterior derivative*

$$d\colon \Omega^p(M) \to \Omega^{p+1}(M) \tag{A.1.9}$$

is defined by requiring df to be the usual derivative for $f \in C^\infty(M) = \Omega^0(M)$ and the graded Leibniz rule

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge (d\eta). \tag{A.1.10}$$

The de Rham differential and codifferential can be expressed locally in terms of the Levi–Civita connection by the formulas

$$d = \sum_{i=1}^n e^i \wedge \nabla_i \quad \text{and} \quad d^* = -\sum_{i=1}^n e_i \llcorner \nabla_i. \tag{A.1.11}$$

A.2 Spin geometry

A.2.1 Complex Clifford algebras and their representations

Throughout, let V be a finite dimensional real inner product space. The *complex Clifford algebra* $\mathbb{C}\mathrm{l}(V)$ is defined as the associative unital $\mathbb{C}$–algebra generated by all $v \in V$ subject to the *Clifford relations* $v^2 = -|v|^2$. We have a canonical embedding $i\colon V \hookrightarrow \mathbb{C}\mathrm{l}(V)$ which can be used to identify V with its image in $\mathbb{C}\mathrm{l}(V)$.

Lemma A.1 (Universal property, cf. [LM89, Prop. I.1.1])**.** *Let E be a complex vector space and $\rho\colon V \to \mathrm{End}_{\mathbb{C}}(E)$ an $\mathbb{R}$–linear map such that $\rho(v)^2 = -|v|^2 \,\mathrm{id}_E$. Then there exists a unique $\mathbb{C}$–algebra homomorphism $\tilde{\rho}\colon \mathbb{C}\mathrm{l}(V) \to \mathrm{End}_{\mathbb{C}}(E)$ such that $\rho = \tilde{\rho} \circ i$.*

Lemma A.2 (cf. [LM89, Prop. I.1.3])**.** *There is a vector space isomorphism*

$$\mathbb{C}\mathrm{l}(V) \cong \Lambda^* V \otimes \mathbb{C}. \tag{A.2.1}$$

The Clifford algebra has a canonical $\mathbb{Z}_2$–grading by $\mathbb{C}\mathrm{l}(V) = \mathbb{C}\mathrm{l}^0(V) \oplus \mathbb{C}\mathrm{l}^1(V)$ where the *even part* $\mathbb{C}\mathrm{l}^0(V)$ is the sub-algebra generated by products $vw \in \mathbb{C}\mathrm{l}(V)$ with $v, w \in V$.

Lemma A.3 (cf. [LM89, Thm. I.3.7])**.** *There is an isomorphism of $\mathbb{C}$–algebras*

$$\mathbb{C}\mathrm{l}(V) \cong \mathbb{C}\mathrm{l}^0(\mathbb{R} \oplus V), \quad v \mapsto e_0 v. \tag{A.2.2}$$

For brevity, we write $\mathbb{C}\mathrm{l}_n = \mathbb{C}\mathrm{l}(\mathbb{R}^n)$ and $\mathbb{C}(n)$ for the algebra of complex $n \times n$–matrices. These algebras can be identified as follows.

Theorem A.4 (cf. [LM89, Ch. I.4])**.** *There are isomorphisms of $\mathbb{C}$–algebras*

$$\mathbb{C}\mathrm{l}_1 \cong \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{C}\mathrm{l}_2 \cong \mathbb{C}(2), \quad \mathbb{C}\mathrm{l}_{m+n} \cong \mathbb{C}\mathrm{l}_m \otimes \mathbb{C}\mathrm{l}_n, \quad \mathbb{C}(r) \otimes \mathbb{C}(s) \cong \mathbb{C}(rs).$$

In particular, this gives the periodicity property $\mathbb{C}\mathrm{l}_{n+2} \cong \mathbb{C}\mathrm{l}_n \otimes \mathbb{C}(2)$ and the classification

$$\mathbb{C}\mathrm{l}_n \cong \begin{cases} \mathbb{C}(2^k) & n = 2k \\ \mathbb{C}(2^k) \oplus \mathbb{C}(2^k), & n = 2k+1. \end{cases}$$

A $\mathbb{C}\mathrm{l}_n$–module E is called *irreducible* if it cannot be written as the direct sum of non-trivial $\mathbb{C}\mathrm{l}_n$–modules of smaller rank. Equivalently, E does not have any non-trivial, proper $\mathbb{C}\mathrm{l}_n$–invariant sub-modules. We also refer to (left) $\mathbb{C}\mathrm{l}_n$–modules as *$\mathbb{C}\mathrm{l}_n$–representations.*

Theorem A.5 (cf. [LM89, Thm. I.5.6])**.** *The canonical representation of $\mathbb{C}(r)$ on $\mathbb{C}^r$ is, up to isomorphism, the only irreducible representation of $\mathbb{C}(r)$. The algebra $\mathbb{C}(r) \oplus \mathbb{C}(r)$ has two inequivalent irreducible representations given by the canonical representations of the two summands.*

Theorems A.4 and A.5 can be used to give a classification of irreducible $\mathbb{C}l_n$–modules. The canonical orientatin of $\mathbb{R}^n$ determines the real and complex *volume elements*

$$\mathrm{vol}_n = e_1 \cdots e_n \in \mathbb{C}l_n, \quad \omega_n^{\mathbb{C}} = \begin{cases} i^k \,\mathrm{vol}_n, & n = 2k \\ i^{k+1} \,\mathrm{vol}_n, & n = 2k+1 \end{cases} \tag{A.2.3}$$

The normalization guarantees that $(\omega_n^{\mathbb{C}})^2 = 1$.

Theorem A.6 (cf. [LM89, Props. I.5.10 & 15])**.**

(i) If $n = 2k$ is even, then $\mathbb{C}l_n$ has, up to isomorphism, a unique irreducible complex representation. Any such representaion Δ_n has dimension 2^k. The subspaces $\Delta_n^{\pm} = (1 \pm \omega_n^{\mathbb{C}})\Delta_n$ are $\mathbb{C}l_n^0$–invariant and constitute irreducible $\mathbb{C}l_n^0$–modules of dimension 2^{k-1}. The element $\omega_n^{\mathbb{C}}$ acts on $\Delta_n^{\pm}$ as $\pm\,\mathrm{id}$.

(ii) If $n = 2k+1$ is odd, then $\mathbb{C}l_n$ has, up to isomorphism, two irreducible complex representations both of which have dimension 2^k. The two isomorphism classes are distinguished by the action of $\omega_n^{\mathbb{C}}$, which either acts as id *or* $-\mathrm{id}$. *If $\Delta_n^{\pm}$ are irreducible $\mathbb{C}l_n$–representations on which $\omega_n^{\mathbb{C}}$ acts as $\pm\,\mathrm{id}$. If $\Delta_n^{\pm}$ denotes one such representation in each isomorphism class are isomorphic as $\mathbb{C}l_n^0$–representations.*

More abstractly, if V is an oriented real inner product space of dimension n, then we have volume elements $\mathrm{vol}_V, \omega_V^{\mathbb{C}} \in \mathbb{C}l(V)$. In odd dimensions, we can use the orientation to single out one of the two irreducible $\mathbb{C}l(V)$–modules. Unfortunately, this is a matter of convention.

Definition A.7. Suppose that V has odd dimension $n = 2k+1$. We say that an irreducible $\mathbb{C}l(V)$–module Δ is *positively (resp. negatively) oriented* if $\omega_V^{\mathbb{C}}$ acts by $+\,\mathrm{id}$ (resp. $-\,\mathrm{id}$).

Concrete models for Δ_n can be obtained as follows. For even $n = 2k$, we identify $\mathbb{R}^{2k} \cong \mathbb{C}^k$ and let

$$\Delta_{2k} = \Lambda^* \mathbb{C}^k \tag{A.2.4}$$

with Clifford multiplication given $\rho_{2k} \colon \mathbb{C}^k \to \mathrm{End}_{\mathbb{C}}(\Lambda^*\mathbb{C}^k)$ given by

$$\rho_{2k}(\xi)\omega = v \wedge \xi - v \llcorner \xi. \tag{A.2.5}$$

For odd $n = 2k-1$ we can take

$$\Delta_{2k-1}^{\pm} = \Delta_{2k}^{\pm} \tag{A.2.6}$$

with Clifford action induced by the isomorphism $\mathbb{C}l_{2k-1} \cong \mathbb{C}l_{2k}^0$.

A.2.2 Spin^c structures on vector bundles

We now generalize the notions for vector spaces to Euclidean[1] vector bundles $V \to B$ over a sufficiently well-behaved space B (e.g. a manifold). We then have a *Clifford bundle* $\mathbb{C}l(V)$ whose fiber over $x \in M$ is $\mathbb{C}l(V_x)$. A *Clifford module* or $\mathbb{C}l(V)$–module is a complex vector bundle $E \to B$ together with a bundle map $\rho \colon \mathbb{C}l(V) \to \mathrm{End}_{\mathbb{C}}(E)$ which equips each fiber E_x with a $\mathbb{C}l(V_x)$–module structure (i.e. ρ is a homomorphism of $\mathbb{C}$–algebra bundles). By the universal property of Cliffod algebras, the so-called *Clifford multiplication* ρ is uniquely determined by its restriction to V which is a map of real vector bundles

$$\rho \colon V \to \mathrm{End}_{\mathbb{C}}(E) \tag{A.2.7}$$

and satisfies $\rho(v)^2 = -|v|^2\,\mathrm{id}_E$ for all $v \in V$. The argument in [LM89, Prop. I.5.16] shows that we can always find a Hermitian bundle metric $\langle\ ,\ \rangle$ on E such that $\rho(v)^* = -\rho(v)$

[1] A *Euclidean vector bundle* is a real vector bundle of finite rank equipped with a bundle metric.

for all $v \in V$. The triple $(E, \rho, \langle\ ,\ \rangle)$ is then called a *Hermitian* $\mathbb{C}\mathrm{l}(V)$*–module*. A Clifford module (E, ρ) is called *irreducible* if E_x is irreducible as a $\mathbb{C}\mathrm{l}(V_x)$–module for each $x \in B$. If V is oriented, then the orientation gives fiberwise well-defined volume elements as in (A.2.3) which assemble into sections

$$\mathrm{vol}_n, \omega_n^{\mathbb{C}} \in \Gamma(B; \mathbb{C}\mathrm{l}(V)). \tag{A.2.8}$$

If V has odd rank and B is connected, then for irreducible (E, ρ) we have $\rho(\omega_V^{\mathbb{C}}) = \pm 1$. Depending on the sign, we call (E, ρ) *positively* or *negatively oriented*.

Definition A.8 (Spinor bundles and spin^c structures)**.**
Let $V \to B$ be an oriented Euclidean vector bundle over a locally compact space B.

(a) A *spinor bundle* for V is a Hermitian $\mathbb{C}\mathrm{l}(V)$–module $\boldsymbol{S} = (S, \rho, \langle\ ,\ \rangle)$ which is irreducible and negatively oriented in the case that V has odd rank $n = 2k + 1$.

(b) Two spinor bundles $\boldsymbol{S} = (S, \rho, \langle\ ,\ \rangle)$ and $\boldsymbol{S} = (S', \rho', \langle\ ,\ \rangle')$ for V are called *isomorphic* if there is a unitary bundle isomorphism $U\colon S \xrightarrow{\cong} S'$ such that $U\rho(v) = \rho'(v)U$ for all $v \in V$.

(c) Let $\mathrm{Spin}^c(V)$ be the set of isomorphism classes of spinor bundles for V. Elements of $\mathrm{Spin}^c(V)$ are called *spin^c structures* and denoted by $\mathfrak{s} = [S, \rho, \langle\ ,\ \rangle]$.

Remark A.9. The orientation convention for spinor bundles is chosen to be compatible with [KM07]. In the case that V has rank 3 we have $\omega_V^{\mathbb{C}} = -\mathrm{vol}_V$ and the convention guarantees that $\rho(\mathrm{vol}_V) = \mathrm{id}$. However, other authors use different conventions! For example, the spinor bundles in [Sal99, Frø08] are *positively oriented*. Passing between these conventions amount to changing $(S, \rho, \langle\ ,\ \rangle)$ into $(S, -\rho, \langle\ ,\ \rangle)$, that is, the sign of Clifford multiplication is reversed. This has to be taken into account when comparing formulas!

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