Floer Homology and the Seiberg–Witten Equations (SoSe 2023)

— Homework Sheet 2 —

(due on Friday, April 21)

Exercise 2.1 (Linearization of vector fields)**.**

(a) Let ξ be a vector field on a smooth manifold M and $\xi(p) = 0$ for some $p \in M$. For $v \in T_pM$ consider the map

$$D_p\xi(v) \colon C^\infty(M) \to \mathbb{R}, \quad f \mapsto v(\xi(f)).$$

Show that this construction gives a linear map $D_p\xi \colon T_pM \to T_pM$, known as the *linearization* of ξ at p.

(b) Let∇f be the gradient of a smooth function f with respect to some Riemannian metric on M. Show that $H_p(f)(v, w) = \langle D_p\nabla f(v), w\rangle$ for all $p \in \mathrm{Crit}(f)$. Conclude that $D_p\nabla f$ has only real eigenvalues and that the number of negative eigenvalues is equal to the index of $H_p(f)$.

Exercise 2.2 (Fun with $\mathbb{R}\mathrm{P}^n$)**.**

(a) For $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ show that $\tilde{f} \colon S^n \to \mathbb{R}$ given by $\tilde{f}(x_0, \ldots, x_n) = \sum_{i=0}^n \lambda_i x_i^2$ descends to a Morse function $f \colon \mathbb{R}\mathrm{P}^n \to \mathbb{R}$. Compute the critical points of f and their indices.

(b) Compute the Morse–Floer complex $C_\bullet(\mathbb{R}\mathrm{P}^2; f, \xi)$ for a suitable Morse gradient ξ.