Floer Homology and the Seiberg–Witten Equations (SoSe 2023)

— Homework Sheet 4 —

(due on Friday, May 5)

Exercise 4.1 (Another construction of index pairs)**.**

Let $\phi\colon X \times \mathbb{R} \to X$ be a flow on a locally compact metric space X denoted by $\phi(x,t) = x \cdot t$. Call $N \subset X$ *strongly isolating* if there is some finite $T > 0$ such that $x \cdot [-T,T] \subset N$ implies $x \in \operatorname{int}(N)$. Consider the following subsets of N:

$$N^T = \{x \in N \,|\, x \cdot [-T,T] \subset N\} \qquad E^T = \left\{x \in N^T \,\middle|\, x \cdot [0,T] \cap \partial N \neq \emptyset\right\}$$

It is clear from the definitions that $N^{S+T} = (N^T)^S$ for $S, T > 0$, $N^T \subset N^S$ for $S < T$ for $N \subset M$, and $N^T \subset M^T$ for $N \subset M$.

(a) Show that every strongly isolating set is isolating and that every compact isolating set is strongly isolating. Conclude that every compact isolated invariant set $S \subset X$ has a strongly isolating neighborhood.

(b) If N is a compact isolating neighborhood of $S = \operatorname{Inv}(N)$, then (N^T, E^T) is an index pair for sufficiently large $T \gg 0$.

(Hint: Assuming that $N^T \subset \operatorname{int}(N)$ show that $N^{2T} \subset \operatorname{int}(N^T)$.)

Exercise 4.2 (Identifying the Floer differential)**.**

Let (f,ξ) be a Morse–Smale pair on a smooth n–manifold M and ϕ the local flow generated by $-\xi$. Moreover let $S \subset M$ be a compact isolated invariant set and $p, q \in \operatorname{Crit}(f) \cap S$ with $\mu(p) - \mu(q) = 1$. Let $S_0 = \{p,q\} \cup (M(p,q) \cap S)$ be the union of all trajectories between p and q. This is a compact isolated invariant set (why?). It is proved in [Flo89, Lemma 3.1] that the Floer differential is given by

$$d\,\langle p\rangle = \#_2 \hat{M}(p,q)\,\langle q\rangle\,.$$

Read the argument in [Flo89] and fill in the omitted details.

References

[Flo89] Andreas Floer, *Witten's complex and infinite dimensional Morse theory*, J. Differ. Geom. **30** (1989), no. 1, 207–221.