Floer Homology and the Seiberg–Witten Equations (SoSe 2023)

— Homework Sheet 6 —

(due on Friday, May 19)

Exercise 6.1 (Suspension isomorphisms in Borel cohomology)**.**

Let G be a connected compact Lie group and V an orthogonal G–representation of dimension $|V| < \infty$. Write S^V for the one point compactification with G acting trivially on the point at infinity, which serves as a based point. For any based G–space X define its *V–suspension* as $\Sigma^V X = X \wedge S^V$.

(a) Show that the *reduced Borel cohomology* $\tilde{H}^*_G(X) = \tilde{H}^*_G(X, x_0)$ is isomorphic to $\tilde{H}^*(EG_+ \wedge_G X)$ where EG_+ is EG with a disjoint G–fixed base point added.

(b) Show that $(X \times V)_{hG} \to X_{hG}$ induced by the obvious projection is an oriented real vector bundle. Use this information to construct natural *suspension isomorphisms*

$$\sigma_V \colon \tilde{H}^*_G(X) \to H^{*+|V|}(\Sigma^V X).$$

Exercise 6.2 (Borel coefficient systems for the unit circle)**.**

Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle group. It is easy to see that every closed subgroup of $\mathbb{T}$ has the form $C_n = \{z \in \mathbb{T} \mid z^n = 1\}$ for some $n \geq 0$.

(a) Let $\gamma = E\mathbb{T} \times_{\mathbb{T}} \mathbb{C} \to B\mathbb{T}$ be the standard line bundle. For $n \geq 1$ identify the map $E\mathbb{T}/C_n \to E\mathbb{T}/\mathbb{T}$ induced by the inclusion $C_m \hookrightarrow \mathbb{T}$ with the unit circle bundle $S(\gamma^{\otimes n}) \xrightarrow{p_n} B\mathbb{T}$.

(b) Let $e_n \in H^2(S(\gamma^{\otimes n}))$ be the Euler class (integer coefficients used throughout). Use the *Gysin sequence*

$$\cdots \to H^{*-1}((S(\gamma^{\otimes n})) \xrightarrow{(p_n)_!} H^{*-2}(B\mathbb{T}) \xrightarrow{\cup e_n} H^*(B\mathbb{T}) \xrightarrow{p_n^*} H^*((S(\gamma^{\otimes n})) \to \cdots$$

to determine $H^*_{\mathbb{T}}(\mathbb{T}/C_n)$ as well as the maps $H^*_{\mathbb{T}}(\mathbb{T}/C_m) \to H^*_{\mathbb{T}}(\mathbb{T}/C_n)$ induced by inclusions $C_n \to C_m$.