Floer Homology and the Seiberg–Witten Equations (SoSe 2023)

— Homework Sheet 7 —

(due on Friday, May 26)

Exercise 7.1 (The gauge group of a spin^c manifold)**.**

Let (S, ρ) be a spinor bundle on an oriented Riemannian manifold M. The *gauge group* $\mathcal{G}(M) = C^\infty(M, U_1)$ acts on S by fiberwise scalar multiplication. Show that the resulting group homomorphism $\mathcal{G}(M) \to \mathrm{End}_{\mathbb{C}}(S)$ maps isomorphically onto the subgroup $U_\rho(S) \subset \mathrm{End}_{\mathbb{C}}(S)$ of fiberwise unitary isomorphisms that commute with Clifford multiplication, that is,

$$U_\rho(S) = \{U \in \mathrm{End}_{\mathbb{C}}(S) \,|\, U^*U = \mathrm{id}_S \text{ and } U\rho(a) = \rho(a)U \text{ for all } a \in T^*M\}\,.$$

Exercise 7.2 (The vector representation of $\mathrm{Spin}^c(V)$)**.**

Let V be a real inner product space. The complex Clifford algebra $\mathbb{C}\mathrm{l}(V)$ is the unital $\mathbb{C}$–algebra generated by all vectors $v \in V$ subject to the *Clifford relations* $v^2 = -|v|^2$. Define an involution on $\mathbb{C}\mathrm{l}(V)$ by $(z \cdot v_1 \cdots v_k) = \bar{z} \cdot v_k \cdots v_1$. Let $\mathrm{Spin}^c(V)$ be the multiplicative subgroup of $\mathbb{C}\mathrm{l}(V)$ generated by products zvw where $v, w \in V$, $z \in \mathbb{C}$, and $|v| = |w| = |z| = 1$. For $x \in \mathbb{C}\mathrm{l}(V)$ and $v \in V$ define $\alpha(x)v = xvx^t$. Show that this defines a surjective homomorphism $\alpha\colon \mathrm{Spin}^c(V) \to SO(V)$. *(Hint: For $e \in V$ with $|e| = 1$ show that $\rho(e)$ maps V to V and agrees with the reflection along the orthogonal complement $e^\perp \subset V$. The group $SO(V)$ is known to be generated by products of two reflections.)*