Floer Homology and the Seiberg–Witten Equations (SoSe 2023)

— Homework Sheet 8 —

(to be discussed on Tuesday, July 4)

Exercise 8.1 (The gauge group action on spin^c connections)**.**

Let (S, ρ) be a spinor bundle over an n–manifold M. For $u \in \mathcal{G}(M)$ and $A \in \mathcal{A}(S)$ define $\nabla^{uA} \colon \Gamma(S) \to \Gamma(T^*M \otimes S)$ such that the diagram below commutes:

$$\begin{array}{ccc} \Gamma(S) & \xrightarrow{\quad u \quad} & \Gamma(S) \\ {\scriptstyle \nabla^{uA}}\big\downarrow{\scriptstyle (\mathrm{def})} & & \big\downarrow{\scriptstyle \nabla^{A}} \\ \Gamma(T^*M \otimes S) & \xrightarrow{1 \otimes u} & \Gamma(T^*M \otimes S) \end{array}$$

Show that ∇^{uA} determines a spin^c connection $uA \in \mathcal{A}(S)$ and prove the formula

$$uA = A - u^{-1}du.$$

Exercise 8.2 (Spin^c structures for families of metrics)**.**

Let (S, h, ρ) be a Hermitian spinor bundle for a Riemannian 4–manifold (X, g) and $(g_t)_{t \in [0,1]}$ a smooth family of Riemannian metrics on X with $g_0 = g$.

(a) Show that there is a smooth family of maps

$$\rho_t \colon T^*M \to \mathfrak{su}(S, h) \subset \mathrm{End}_{\mathbb{C}}(S)$$

such that (S, h, ρ_t) is a spinor bundle for (X, g_t) and $\rho_0 = \rho$.

(b) Show that every spin^c connection $A_0 \in \mathcal{A}(S, \rho_0)$ and $\eta_0 \in i\Omega^2_+(X, g_0)$ can be extended to smooth families of Hermitian connections A_t on S and 2-forms $\eta_t \in i\Omega^2(X)$ such that $A_t \in \mathcal{A}(S, \rho_t)$ and $\eta_t \in i\Omega^2_+(X, g_t)$.