Floer Homology and the Seiberg–Witten Equations (SoSe 2023)

— Homework Sheet 9 —

(to be discussed on Thursday, July 6)

Exercise 9.1 (The quadratic term on 3–manifolds)**.**

Let (S, ρ) be a spinor bundle over a 3–manifold Y. Recall that every pair of spinors $\phi, \psi \in \Gamma(S)$ determines a complex valued 1–form $\rho^{-1}(\psi\phi^*)_0 \in \Omega^1(Y; \mathbb{C})$. Show that there is a constant c such that for every $\alpha \in \Omega^1(Y; \mathbb{C})$ we have

$$\left\langle \rho^{-1}(\phi\psi^*)_0, \alpha \right\rangle_{\Lambda^1_{\mathbb{C}} Y} = c \, \langle \phi, \rho(\alpha)\psi \rangle_S$$

and determine the value of the constant.

(Hint: Equip $\operatorname{End}_C(S)$ *with the Hermitian metric* $\langle A, B \rangle = \frac{1}{2} \operatorname{tr}_{\mathbb{C}}(A^* B)$*.)*

Exercise 9.2 (The Yang–Mills equation on cylinders)**.**

Let Y be a closed, oriented Riemannian 3–manifold. The space of connections on the trivial principal SU_2–bundle $Y \times SU_2 \to Y$ is canonically identified with the space of Lie algebra valued 1–forms $\Omega^1(Y; \mathfrak{su}_2)$.

(a) Show that the *Chern–Simons functional*

$$\mathrm{cs}\colon \Omega^1(Y; \mathfrak{su}_2) \to \mathbb{R}, \quad \mathrm{cs}(a) = \frac{1}{8\pi^2} \int_Y \operatorname{tr}\left(a \wedge da + \frac{2}{3} a \wedge a \wedge a\right).$$

admits a gradient with respect to to the L^2 scalar product $(a, b)_0 = \frac{1}{2} \int_Y \operatorname{tr}(a^* b) \, \mathrm{vol}_Y$.

(b) A smooth curve $a\colon \mathbb{R} \to \Omega^1(Y; \mathfrak{su}_2)$ can be interpreted as a 1–form $\hat{a} \in \Omega^1(\mathbb{R} \times Y; \mathfrak{su}_2)$ on the cylinder $\mathbb{R} \times Y$. Show that the the gradient flow equation $\dot{a} = \nabla \mathrm{cs}(a)$ is equivalent to the *Yang–Mills equation*

$$F^+_{\hat{a}} = F_{\hat{a}} + *_4 F_{\hat{a}} = 0$$

where $*_4$ is the Hodge operator on $\mathbb{R} \times Y$ for the cylindrical metric $dt^2 + g_Y$.