Smooth 4–Manifolds and Surface Diagrams

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Goals for today

1. What is a surface diagram?
   - Where do surface diagrams come from?
   - The definition and some examples
2. How do surface diagrams describe 4–manifolds?
   - Building 4–manifolds from surface diagrams
   - Drawing Kirby diagrams
3. What can we do with surface diagrams?
   - Substitutions
   - Extracting homotopy information

Without further notice

- **All manifolds are. . .**
  . . . smooth, connected, compact, oriented
- **All maps are. . .**
  . . . smooth, surjective
- $H_*(\quad) = H_*(\quad; \mathbb{Z})$

What is a surface diagram?

A short history of the subject

Broad context: Study of maps $X^4 \to S^2$

- **Lefschetz fibrations** on symplectic 4–manifolds
  (Donaldson, Gompf '99)
- **Singular Lefschetz fibrations** on near–symplectic 4–manifolds
  (Auroux–Donaldson–Katzarkov '05)
- ~~Singular~~ **Broken Lefschetz fibrations** on all 4–manifolds
  (Gay–Kirby '07, Baykur '08, Lekili '09, Akbulut–Karakurt '08, Saeki '06)
- **Simple wrinkled fibrations** on all 4–manifolds
  (Williams '10)

$\rightsquigarrow$ Combinatorial description: **surface diagrams**

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Williams' existence theorem

Theorem (Williams '10)

Any map $X \to S^2$ is homotopic to a simple wrinkled fibration *(with arbitrarily high fiber genus)*.

Proposition (Williams '10)

Simple wrinkled fibrations *(with fiber genus at least three)* are determined by surface diagrams.

Corollary (Williams '10)

All closed 4–manifolds can be described by surface diagrams *(of arbitrarily high genus)*.

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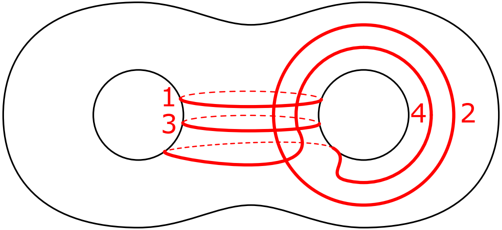
Recent related developments

- **Gay–Kirby:** Morse 2–functions, trisections
- **Baykur–Saeki:** new approach to existence of BLFs

Definition (Surface diagrams)

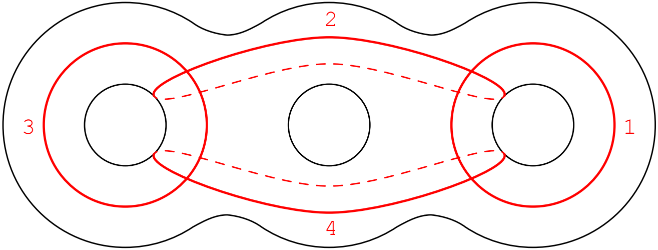
A **surface diagram** $\mathfrak{S} = (\Sigma; c_1, \ldots, c_l)$ consists of

- Σ : closed, oriented surface *("fiber")* *(genus $g \geq 1$)*
- $c_i \subset \Sigma$: simple closed curves *("vanishing cycles")* *($l \geq 2$)*
- $\#(c_i \pitchfork c_{i+1}) = 1$ *($l + 1 = 1$, "cyclic ordering")*

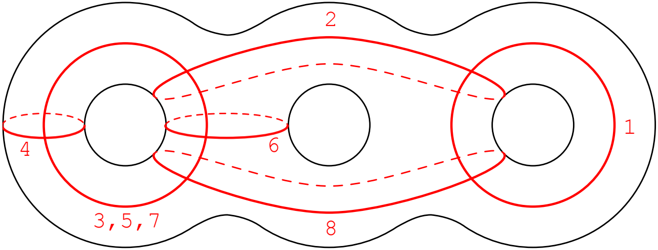


- **Note:** Intersection of non-consecutive curves can be arbitrary

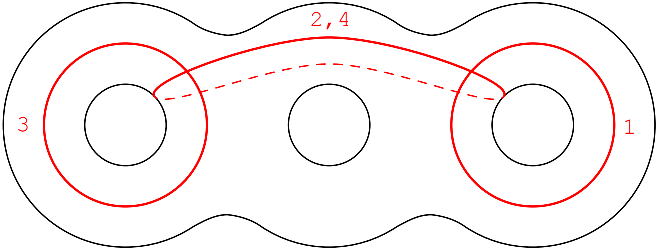
A surface diagram of $2S^1 \times S^3$ (due to Hayano)



A surface diagram of S^4 (also due to Hayano)



A surface diagram of $T^2 \times S^2 \# S^1 \times S^3$



How do surface diagrams describe 4–manifolds?

Building a 4–manifold: from $\mathfrak{S}$ to Z

$\mathfrak{S} = (\Sigma; c_1, \ldots c_l)$ surface diagram

- Start with $\Sigma \times D^2$
- Pick $\theta_1, \ldots, \theta_l \in S^1$ ordered according to the orientation
- Attach 2–handles to $c_i \times \{\theta_i\} \subset \Sigma \times S^1$ with **fiber framing**

$\rightsquigarrow$ **4–manifold** Z **with** $\partial Z \neq \emptyset$

Note the structural similarity with Lefschetz fibrations!

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Building a 4–manifold: from Z to X

One can show the following:

- $\Sigma \times D^2 \to D^2$ extends to $w\colon Z \to D^2$ *(over a slightly larger D^2)*
- $w\colon \partial Z \to S^1$ fibration with fiber Σ' with $g(\Sigma') = g(\Sigma) - 1$

$\rightsquigarrow$ **monodromy** $\mu(\mathfrak{S}) \in \mathrm{Mod}(\Sigma')$

Closing off

Works only for $\mu(\mathfrak{S}) = 1$

$\rightsquigarrow$ $\exists$ fiber pres. diffeo. $\phi\colon \Sigma' \times S^1 \xrightarrow{\cong} \partial Z$

$\rightsquigarrow$ $X = Z \cup_\phi (\Sigma' \times D^2)$ **closed 4–manifold** + sWF $w\colon X \to S^2$

- ϕ is unique for $g(\Sigma) \geq 3$! $(\pi_1(\mathrm{Diff}(\Sigma_g)) = 1)$

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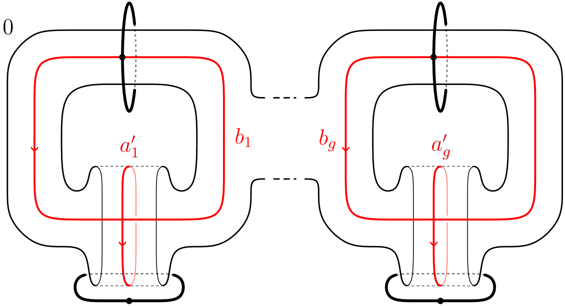
A Kirby diagram of $\Sigma \times D^2$



- S^1 direction points out of the screen
- Interval worth of fibers visible
- Fiber framing for curve on fiber: $\mathrm{fr}(\gamma) = \sum_{i=1}^{g} \langle a_i, \vec{\gamma} \rangle \langle \vec{\gamma}, b_i \rangle$

What can we do with surface diagrams?

Substitutions

Substitutions

"Take something out, put something else back in."

Substitutions

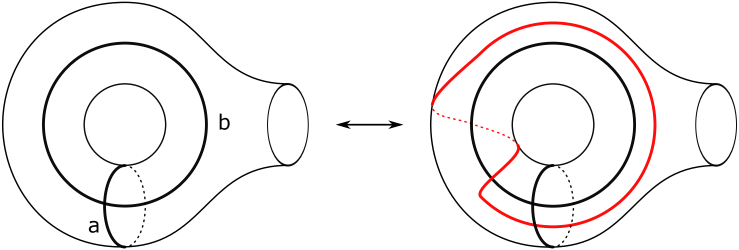
$$(\Sigma; \underbrace{c_1, c_2, \ldots, c_{r-1}, c_r}_{\text{remove } \Lambda}, \ldots) \quad \rightsquigarrow \quad (\Sigma; \underbrace{c_1, d_1, \ldots, d_k, c_r}_{\text{replace with } \Lambda'}, \ldots)$$

- Subsitutions $\rightsquigarrow$ cut-and-paste operations on 4–manifolds
- Sometimes the effect can be identified

Example 1: blowing up

Lemma (Blow-up substitution)

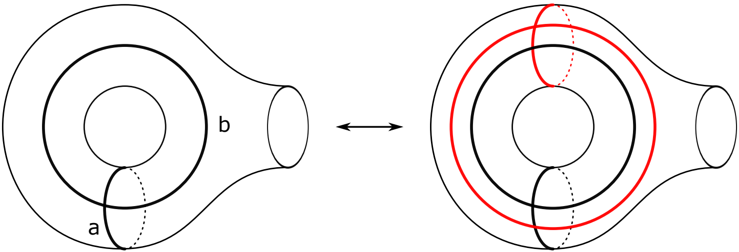
$$\begin{array}{ccc} (\Sigma; a, b, \dots) & \leftrightsquigarrow & (\Sigma; a, {\color{red}\tau_a^{\pm}(b)}, b, \dots) \\ \updownarrow & & \updownarrow \\ X & \leftrightsquigarrow & X \# \pm \mathbb{C}P^2 \end{array}$$



Example 2: stabilizing

Lemma (Stabilization substitution)

$$\begin{array}{ccc} (\Sigma; a, b, \dots) & \leftrightsquigarrow & (\Sigma; a, {\color{red} b, a}, b, \dots) \\ \updownarrow & & \updownarrow \\ X & \leftrightsquigarrow & X\#(S^2 \times S^2) \end{array}$$



Application: genus 1 classification

Theorem

X^4 closed has a genus 1 surface diagram if and only if

$$X \underset{\text{diffeo}}{\cong} k(S^2 \times S^2) \quad \text{or} \quad m\mathbb{C}\mathrm{P}^2 \# n\overline{\mathbb{C}\mathrm{P}^2}, \qquad k, m, n \geq 1$$

Note:

- S^4, $\mathbb{C}\mathrm{P}^2$, $\overline{\mathbb{C}\mathrm{P}^2}$ not in the list
- Monodromy automatically trivial $(\mathrm{Mod}(\Sigma') \cong \mathrm{Mod}(S^2) = 1)$
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Sketch of proof

- **Key 1:** $(T^2; a, b)$ describes $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$

⇝ Blow up and stabilize to realize manifolds in the list

- **Key 2:** "geometry=algebra" for curves on T^2

⇝ Enough to work in $H_1(T^2)$

Lemma

Every surface diagram on T^2 of length at least three contains a blow up or stabilization configuration.

⇝ Blow down or destabilize to reduce length to two.

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$\square$

Extracting homotopy information

From now on...

We fix the following data:

- $\mathfrak{S} = (\Sigma; c_1, \ldots, c_l)$ surface diagram with $\mu(\mathfrak{S}) = 1$
- $\vec{c}_i$ oriented version of c_i

Recall: $Z = \Sigma \times D^2 \cup \{\text{2–handles}\}$ and $X = Z \cup_\phi (\Sigma' \times D^2)$

Goal: Describe homotopy invariants of Z and X in terms of $\mathfrak{S}$!

Euler characteristic and signature

- A simple count of handles gives:

$$\chi(Z) = 2 - 2g + l \quad \text{and} \quad \chi(X) = 6 - 4g + l$$

- Also, Novikov additivity for the signature shows:

$$\sigma(X) \overset{\text{Novikov}}{=} \sigma(Z) + \sigma(\Sigma' \times D^2) = \sigma(Z)$$

- We will shortly see how $\sigma(Z)$ can be computed from $\mathfrak{S}$

Invariants of Z: π_1 and homology

- $V_{\mathfrak{S}} = \bigoplus_i \mathbb{Z}\vec{c}_i \quad \rightsquigarrow \quad \rho\colon V_{\mathfrak{S}} \longrightarrow H_1(\Sigma), \quad \rho(\vec{c}_i) = [\vec{c}_i]$
- $K_{\mathfrak{S}} = \ker(\rho) \subset V_{\mathfrak{S}}$

Lemma

- $\pi_1(Z) \cong \pi_1(\Sigma)/\langle\langle c_1, \ldots, c_l \rangle\rangle$ *(normal subgroup generated by. . .)*
- $H_1(Z) \cong H_1(\Sigma)/\langle c_1, \ldots, c_l \rangle$
- $H_2(Z) \cong K_{\mathfrak{S}} \oplus \mathbb{Z}[\Sigma]$

Invariants of Z: intersection form

Lemma

$$(H_2(Z), Q_Z) \underset{\textit{isometric}}{\cong} (K_{\mathfrak{S}} \oplus \mathbb{Z}, Q_{\mathfrak{S}} \oplus (0))$$

$$Q_{\mathfrak{S}}(\xi, \eta) = \frac{1}{2} \sum_{i,j} \xi_i \eta_j \, \epsilon_{ij} \, \langle \vec{c}_i, \vec{c}_j \rangle_\Sigma \, , \quad \epsilon_{ij} = \begin{cases} +1 & \text{if } j < i \\ -1 & \text{if } j > i \end{cases}$$

where $\xi = \sum_i \xi_i \vec{c}_i$ and $\eta = \sum_j \eta_j \vec{c}_j$ are elements of $K_{\mathfrak{S}}$

- As mentioned before, we have $\sigma(X) = \sigma(Z) = \sigma(Q_{\mathfrak{S}})$
- Note that $Q_{\mathfrak{S}}$ depends only on homological data

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Invariants of Z: type (spin or not)

Lemma

1. $\mathrm{Spin}(Z) \overset{\textit{1:1}}{\longleftrightarrow} \{\xi \in \mathrm{Spin}(\Sigma) \mid \xi|_{c_i} \text{ trivial } \forall i\}$
2. *Z is spin if and only if*

$$\underbrace{[c_{i_1}]_2 + \cdots + [c_{i_r}]_2 = 0}_{\text{in } H_1(\Sigma;\mathbb{Z}_2)} \implies \sum_{j<k} \langle c_{i_j}, c_{i_k} \rangle_2 = 0$$

- Obvious adaption of an argument of Stipsicz for Lefschetz fibrations
- The second part was already proved by Hayano

Invariants of X: the subtlety of closing off

- **Recall:** $X = Z \cup_\phi (\Sigma' \times D^2)$, where $\phi \colon \Sigma' \times S^1 \xrightarrow{\cong} \partial Z$
- Consider the curve $\kappa' = \phi(* \times S^1) \subset \partial Z$
- **Fact:** κ' is freely homotopic in Z to an immersed $\kappa \subset \Sigma$.

Any such $\kappa \subset \Sigma$ is called a **closing curve**.

- X is determined by $\mathfrak{S}$ and *(a framed version of)* κ.
- If $g = g(\Sigma) \geq 3$, then κ can be determined from $\mathfrak{S}$.
(Not easy in practice!)

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Invariants of X: intersection form and type

Simplifying assumption

$[\Sigma] \in H_2(X)$ torsion $\iff$ $0 = \mathbb{Z}[\kappa] \cap \langle c_1, \dots c_l \rangle \subset H_1(\Sigma)$

- Satisfied for surface diagrams derived from constant maps

Lemma (Intersection form)

$$(H_2(X)/\mathrm{tors}, Q_X) \underset{\text{isometric}}{\cong} (K_{\mathfrak{S}}^{\mathrm{red}}, Q_{\mathfrak{S}}^{\mathrm{red}})$$

Lemma (Type)

X is spin if and only Z is spin.

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Invariants of X: π_1 and homology

Lemma

- $\pi_1(X) \cong \pi_1(Z)/\kappa \cong \pi_1(\Sigma)/\langle\langle c_1, \ldots, c_l; \kappa \rangle\rangle$
- $H_1(Z) \cong H_1(\Sigma)/\langle c_1, \ldots, c_l; \kappa \rangle$
- $H_2(X) \cong H_2(Z)/\mathcal{R}$ *($\mathcal{R} = \langle \{\phi(\gamma \times S^1)\} \rangle$: "rim tori")*

Without the fiber assumption we have a short exact sequence

$$0 \longrightarrow H_2(Z)/\mathcal{R} \longrightarrow H_2(X) \longrightarrow \mathrm{ord}(\kappa)\mathbb{Z} \longrightarrow 0$$

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A simply connected observation

- Together with Freedman and Donaldson we obtain:

Theorem/Observation

X^4 closed, $\pi_1(X) = 1$, described by $\mathfrak{S} = (\Sigma; c_1, \ldots, c_l)$ with $\Sigma \subset X$ null-homologous. The **homeomorphism type** of X is determined by the **homology classes** $[c_i] \in H_1(\Sigma)$.

- **Diffeomorphism type** depends on **isotopy classes**
- homology vs. isotopy $\rightsquigarrow$ **Torelli group** of Σ

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Thank you!