Donaldson's Diagonalizibility Theorem

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IMPRS Seminar on Heegaard Floer Homology

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Outline

1. 4-Manifolds and Their Intersection Forms
   - Inftersection forms: geometrically and algebraically
   - The topological realization problem: Freedman's theorem
   - The smooth realization problem: Rokhlin's and Donaldson's theorems
2. The Proof of Donaldson's Theorem
   - The strategy: Elkies' theorem
   - Preliminaries
   - The Heegaard-Floer part of the proof

Conventions

- 3-manifolds are closed, oriented and connected
- 4-manifolds are compact, oriented, connected and may have boundary

Intersection Forms

Intersection forms of closed 4-manifolds

Let X be a closed *topological* 4-manifold.

Definition

The **intersection form** of X is the unimodular symmetric bilinear form over $\mathbb{Z}$ (SBF)

$$Q_X : H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z}) \to \mathbb{Z}$$
$$Q_X(a,b) = \langle a^* \cup b^*, [X] \rangle$$

where $a^* \in H^2(X;\mathbb{Z})$ denotes the Poincaré dual of a.

- symmetric, bilinear: obvious
- unimodular (i.e. $\det Q_X = \pm 1$): Poincaré duality
- Q_X descents to the free abelian group $H_2(X;\mathbb{Z})/\text{torsion}$

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The realization problem

- Unimodular SBFs are a classic topic in number theory.
- We are interested in the following...

Question

Which unimodular SBFs can be realized as the intersection form of a closed, **topological or smooth** *4-manifold?*

- The answer will reveal a striking gap between the topological and smooth category in dimension 4.
- Before we address the realization problem, we will collect some definitions and facts.

Intersection forms of 4-manifolds with boundary

- Let X be a compact *topological* 4-manifold with $\partial X \neq \emptyset$.
- We can still define a SBF, which is not necessarily unimodular

Definition

The *intersection form* of X is the SBF is defined as

$$Q_X(a, b) = \langle a^* \cup b^*, [X, \partial X] \rangle$$

where $a^*, b^* \in H^2(X, \partial X; \mathbb{Z})$ are Poincaré dual to $a, b \in H_2(X; \mathbb{Z})$.

Lemma

If $H_1(X; \mathbb{Z}) = 0$, then Q_X is unimodular if and only if ∂X is disjoint union of homology 3-spheres (i.e. $H_*(\dots; \mathbb{Z}) \cong H_*(S^3; \mathbb{Z})$).

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Where are the intersections?

Lemma

For any $a \in H_2(X;\mathbb{Z})$ there is a closed, oriented, embedded surface $\Sigma \subset X \setminus \partial X$ such that $a = [\Sigma]$. Furthermore, for two such surfaces Σ, Σ' we have

$$Q_X([\Sigma],[\Sigma']) = \textit{algebraic intersection number of } \Sigma \textit{ and } \Sigma'.$$

Algebraic intersection number means the following:

- Perturb Σ and Σ' to intersect transversely
- $\rightsquigarrow$ $\Sigma \cap \Sigma'$ is a finite set of points
- Count points in $\Sigma \cap \Sigma'$ with sign

Algebraic invariants

- The **rank**: $\operatorname{rk}(Q_X) := \operatorname{rk} H_2(X;\mathbb{Z}) = b_2(X)$
- The **signature**:

$$\sigma(Q_X) := \#\{\text{pos. eig. val. }\} - \#\{\text{neg. eig. val. }\} =: b_2^+(X) - b_2^-(X)$$

- The **type**:
  Q_X is called *even* if

$$Q_X(a,a) \in 2\mathbb{Z} \quad \forall\, a \in H_2(X;\mathbb{Z})$$

  and *odd* otherwise.
- Q_X is called **definite** if $|\sigma(Q_X)| = \operatorname{rk}(Q_X)$.

Algebraic Results

Theorem (Serre)

Indefinite unimodular SBFs are classified (up to isomorphism) by their rank, signature and type.

Theorem

The number of definite unimodular SBFs of a fixed rank is finite.

- Definite unimodular SBFs are far from being classified because there are just too many of them.
- E.g., there are at least 10^{51} even such objects of rank 40.

Remember: Definite unimodular SBFs are mysterious!

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The Realization Problem

A first step

Proposition

Any SBF Q can be realized as the intersection form of a **smooth**, **simply connected**, *4-manifold* **with boundary**.

Proof.

- Represent Q as a matrix $(q_{ij})_{i,j=1,\ldots,n}$
- For $i = 1, \ldots, n$ let S_i be a disk bundle over S^2 with Euler number q_{ii}
- Plumb S_i and S_j together q_{ij} times
- Attach 2-handles to kill π_1
- The resulting 4-manifold P_Q does the trick

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What about closed 4-manifolds?

Recall: If Q is unimodular, then ∂P_Q is a homology 3-sphere.

Theorem (Freedman)

Every homology 3-sphere bounds a contractible, topological 4-manifold (also known as a "fake 4-ball").

This extremely deep theorem immediately implies

Corollary (Freedman)

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- ∂P_Q bounds a fake 4-ball Δ
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Freedman's work actually goes way further.

Theorem (Freedman)

Simply connected, closed, topological 4-manifolds are classified up to homeomorphism by their intersection form and Kirby-Siebenmann invariant.

- The Kirby-Siebenmann invariant is a triangulation obstruction with values in $H^4(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$.
- X simply connected $\Rightarrow \mathrm{ks}(X) = \frac{1}{8}\sigma(X)$ $(mod 2)$

So the classification of closed, simply connected, topological 4-manifolds is basically equivalent to the algebraic classification of unimodular SBFs.

What about smooth, closed manifolds?

Consider the SBF of rank 8 given by the matrix

$$E_8 := \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}$$

Lemma

The E_8-form is unimodular, even, positive definite with signature 8.

Rokhlin's theorem

- Since E_8 is even, $\mathcal{M}_{E_8}$ is spin. Actually:
  - X spin $\Rightarrow$ Q_X even
  - If $H_2(X;\mathbb{Z})$ has no 2-torsion, the converse holds

Theorem (Rokhlin)

Let X be a **smooth***, closed, spin 4-manifold. Then $\sigma(X)$ is divisible by 16.*

The bad news:

1. The smooth realization problem is more delicate then the topological one
2. $\mathcal{M}_{E_8}$ cannot have any smooth structure!

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Donaldson's theorem

- Now consider the form $E_8 \oplus E_8$. It is positive definite, even and has signature 16.

$\rightsquigarrow$ Rokhlin's theorem does not apply, but...

Theorem (Donaldson)

Let X be a closed, smooth 4-manifold with Q_X (negative) definite. Then Q_X is diagonalizable over $\mathbb{Z}$.

The good news:

1. The complexities of definite forms are not present in the smooth category!
2. Two closed, simply connected, smooth 4-manifolds are **homeomorphic** if and only if their intersection forms have the same rank, signature and type

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The Heegaard-Floer proof of Donaldson’s Theorem

The strategy: Elkies' theorem

We will use the following criterion for diagonalizability.

Definition (Characteristic elements)

Let $Q\colon V \times V \to \mathbb{Z}$ be a SBF. Then $\xi \in V$ is called a **characteristic element** of Q if for all $v \in V$

$$Q(\xi, v) \equiv Q(v, v) \ (mod 2).$$

Theorem (Elkies)

If Q is unimodular and negative definite, then Q is diagonalizable (over $\mathbb{Z}$) if and only if all characteristic elements ξ satisfy

$$Q(\xi, \xi) + \mathrm{rk}(Q) \leq 0.$$

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A reformulation of Donaldson's theorem

Let X be a closed 4-manifold.

- $\xi \in H_2(X;\mathbb{Z})$ is characteristic for Q_X iff it is Poincaré dual to an integral lift of $w_2(X) \in H^2(X;\mathbb{Z}_2)$
- The set of integral lifts of $w_2(X)$ given by

$$\{c_1(\mathfrak{s}) \,|\, \mathfrak{s} \in \mathrm{Spin}^c(X)\}$$

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Thus, Donaldson's theorem is equivalent to

Theorem (Donaldson, reformulated)

If X admits a smooth structure and if Q_X is negative definite, then for all $\mathfrak{s} \in \mathrm{Spin}^c(X)$ we have

$$c_1(\mathfrak{s})^2 + b_2(X) \leq 0.$$

Cutting X into pieces

From now on: X^4 smooth, closed, oriented, $b_2^+(X) = 0$
Also: w.l.o.g. $b_1(X) = b_3(X) = 0$ (by surgery)

- Give X a normalized handle decomposition
- $n_i :=$ number of i-handles (normalized $\Rightarrow n_1 = n_4 = 1$)

$\rightsquigarrow$ $W := X \setminus (0\text{- and }4\text{-handle})$ cobordism from S^3 to S^3

- Decompose W into 1-,2- and 3-handles, i.e.

$$W = W_1 \cup_{Y_0} W_2 \cup_{Y_m} W_3 \quad (m = n_2)$$

- $Y_0 = \#^{n_1}(S^2 \times S^1)$, $Y_m = \#^{n_3}(S^2 \times S^1)$, $W_2 = W(\mathbb{L})$ where

$$\mathbb{L} = \mathbb{K}_1 \cup \cdots \cup \mathbb{K}_m$$

is a framed link in Y_0

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Interlude: Surgery on knots and b_1

Let $\mathbb{K} = (K, \lambda)$ be a framed knot in a 3-manifold Y and let $[K] \in H_1(Y; \mathbb{Z})$ denote its homology class.

Fact: If $[K]$ is torsion, then K has a canonical framing λ_0.

Lemma (surgery-torsion lemma)

$$b_1(Y(\mathbb{K})) = \begin{cases} b_1(Y) - 1 & \textit{if } [K] \textit{ non-torsion} \\ b_1(Y) & \textit{if } [K] \textit{ torsion, } \lambda \neq \lambda_0 \\ b_1(Y) + 1 & \textit{if } [K] \textit{ torsion, } \lambda = \lambda_0 \end{cases}$$

Consequently:

Corollary

$$b_1(Y) \leq b_1(Y(\mathbb{K})) \iff [K] \textit{ torsion}$$

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Interlude: The intersection form of $W(\mathbb{K})$

Lemma

- *If* $[K]$ *is torsion and* $\lambda \neq \lambda_0$, *then* $b_2^-(W(\mathbb{K})) = 1$ *and* $b_2^+(W(\mathbb{K})) = 0$ *(or the other way around).*
- *In any other case* $b_2^+(W(\mathbb{K})) = b_2^-(W(\mathbb{K})) = 0$.

In other words:

Lemma

If surgery on $\mathbb{K}$ *changes* b_1, *then the intersection form of* $W(\mathbb{K})$ *vanishes identically. If* b_1 *stays constant, there is exactly one non-trivial primitive class (up to sign).*

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Cutting W_2 into even smaller pieces

We further decompose $W_2 = W(\mathbb{L})$ into

$$W(\mathbb{K}_1) \cup_{Y_1} \cdots \cup_{Y_{m-1}} W(\mathbb{K}_m)$$

where $Y_i = Y_{i-1}(\mathbb{K}_i)$ $(i \geq 1)$.

Lemma (Reordering lemma)

It is possible to reorder the knots such that

- *for* $i = 1, \ldots, a,$ $b_1(Y_i)$ *is decreasing,*
- *for* $i = a+1, \ldots, b,$ $b_1(Y_i)$ *is constant,*
- *for* $i = b+1, \ldots, m,$ $b_1(Y_i)$ *is increasing.*

Proving the reordering lemma

The reordering lemma follows from:

Lemma

Let $\mathbb{L} = \mathbb{K}_1 \cup \mathbb{K}_2$ be framed link in Y^3 with $b_2^+(W(\mathbb{L})) = 0$. Let $Y_i = Y(\mathbb{K}_i)$ and $Y_{12} = Y(\mathbb{K}_1 \cup \mathbb{K}_2)$.

1. $b_1(Y) < b_1(Y_1) > b_1(Y_{12}) \;\Rightarrow\; b_1(Y) > b_1(Y_2) < b_1(Y_{12})$
2. $b_1(Y) < b_1(Y_1) = b_1(Y_{12}) \;\Rightarrow\; b_1(Y) = b_1(Y_2) < b_1(Y_{12})$
3. $b_1(Y) = b_1(Y_1) > b_1(Y_{12}) \;\Rightarrow\; b_1(Y) > b_1(Y_2) = b_1(Y_{12})$

Proof.

Use surgery-torsion lemma, linking form of torsion knots and $b_2^+ = 0$ for proofs by contradiction. □

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More consequences of $b_2^+(X) = 0$

Lemma

1. *The connecting map $\delta\colon H^1(Y_i;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ in the obvious Mayer-Vietoris sequence is trivial.*
2. *For any $\mathfrak{s} \in \mathrm{Spin}^c(X)$ the class $c_1(\mathfrak{s}|_{Y_i}) \in H^2(Y_i;\mathbb{Z})$ is torsion.*

The first part will be useful soon in combination with:

Lemma

Let $W = W_1 \cup_N W_2$ be a composite cobordism and let $\mathfrak{s}_i \in \mathrm{Spin}^c(W_i)$ such that $\mathfrak{s}_1|_N = \mathfrak{s}_2|_N$. Then

$$\left\{\mathfrak{s} \in \mathrm{Spin}^c(W) \;\middle|\; \mathfrak{s}|_{W_i} = \mathfrak{s}_i\right\} \xleftrightarrow{1:1} \delta H^1(N;\mathbb{Z}).$$

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Let $W = W_1 \cup_N W_2$ *be a composite cobordism and let* $\mathfrak{s}_i \in \mathrm{Spin}^c(W_i)$ *such that* $\mathfrak{s}_1|_N = \mathfrak{s}_2|_N$. *Then*

$$\{\mathfrak{s} \in \mathrm{Spin}^c(W) \mid \mathfrak{s}|_{W_i} = \mathfrak{s}_i\} \xleftrightarrow{1:1} \delta H^1(N;\mathbb{Z}).$$

Enter Heegaard-Floer theory

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We will obtain the desired inequality by studying the **maps induced by W on HF^∞**.

- As usual, we will treat 1-, 2- and 3-handles separately
- **Recall:** $HF^\circ(Y, \mathfrak{t})$ has an action of $\Lambda^*(H_1(Y;\mathbb{Z})/\text{torsion})$
- $K \subset Y$ knot $\rightsquigarrow HF^\infty(Y, \mathfrak{t}) = \ker[K] \oplus \ker[K]^\perp$

The 1- and 3-handles

As usual, the 1- and 3-handles don't cause too much trouble

Lemma (1-handles)

$$F^{\infty}_{W_1,\mathfrak{s}_1}(HF^{\infty}(S^3)) = \ker[K_1]^{\perp} \cap \cdots \cap \ker[K_m]^{\perp} \subset HF^{\infty}(Y_0, \mathfrak{t}_0)$$

Lemma (3-handles)

$F^{\infty}_{W_3,\mathfrak{s}_3} : HF^{\infty}(Y_m, \mathfrak{t}_0) \to HF^{\infty}(S^3)$ induces an isomorphism

$$HF^{\infty}(Y_m, \mathfrak{t}_0) \supset \ker[L_1] + \cdots + \ker[L_m] \to HF^{\infty}(S^3)$$

where $L_i \subset Y_m$ is the core of the solid torus glued in during the surgery on K_i.

(Apparently, both claims follow "easily from the definition" of the maps.)

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Reminder: The composition law

Theorem (Composition law)

Let $W = W_1 \cup_N W_2$ be a composite cobordism. Then the maps induced on HF° satisfy

$$F^\circ_{W_2,\mathfrak{s}_2} \circ F^\circ_{W_1,\mathfrak{s}_1} = \sum_{\{\mathfrak{s}\in \mathrm{Spin}^c(W) \mid \mathfrak{s}|_{W_i}=\mathfrak{s}_i\}} F^\circ_{W,\mathfrak{s}}.$$

In our situation we get:

Corollary

$$F^\infty_{W(\mathbb{L}),\mathfrak{s}} = F^\infty_{W(\mathbb{K}_m),\mathfrak{s}_m} \circ \cdots \circ F^\infty_{W(\mathbb{K}_1),\mathfrak{s}_1}$$

This means that we can work 2-handle by 2-handle.

The technicalities, part 1

- Let $\mathbb{K}$ be a framed knot in Y^3
- Let $\mathfrak{s} \in \mathrm{Spin}^c(W(\mathbb{K}))$ such that its restrictions $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ and $\mathfrak{t}' \in \mathrm{Spin}^c(Y(\mathbb{K}))$ are torsion.
- Assume that for $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ torsion

$$HF^\infty(Y, \mathfrak{t}) \cong \Lambda^{b_1(Y)} H^1(Y; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[U, U^{-1}]$$

Proposition (b_1 constant)

Suppose that $b_2^-(W(\mathbb{K})) = 1$. Then $F^\infty_{W(\mathbb{K}),\mathfrak{s}}$ is an isomorphism.

Note: The assumption holds iff the surgery on $\mathbb{K}$ leaves b_1 constant.

The technicalities, part 2

Proposition (b_1 changes)

Suppose that $b_2^+(W(\mathbb{K}) = b_2^+(W(\mathbb{K}) = 0$.

1. *If* $[K]$ *is non-torsion, then* $F^\infty_{W(\mathbb{K}),\mathfrak{s}}$ *induces an isomorphism*

$$HF^\infty(Y,\mathfrak{t}) \supset \ker[K]^\perp \xrightarrow{\cong} HF^\infty(Y(\mathbb{K}),\mathfrak{t}').$$

2. *If* $[K]$ *is torsion, then* $F^\infty_{W(\mathbb{K}),\mathfrak{s}}$ *induces an isomorphism*

$$HF^\infty(Y,\mathfrak{t}) \xrightarrow{\cong} \ker[L] \subset HF^\infty(Y(\mathbb{K}),\mathfrak{t}')$$

where L *is the core of the glued in solid torus.*

Note: The assumption holds iff the surgery on $\mathbb{K}$ changes b_1.

Towards the conclusion

Putting all this together, we get

Proposition

For any $\mathfrak{s} \in \mathrm{Spin}^c(X)$, *the map*

$$F^{\infty}_{W,\mathfrak{s}|_W} : HF^{\infty}(S^3) \to HF^{\infty}(S^3)$$

is an isomorphism.

Corollary

$F^{+}_{W,\mathfrak{s}|_W} : HF^{+}(S^3) \to HF^{+}(S^3)$ *is surjective.*

Proof.

Exact sequence relating HF^-, HF^{∞}, HF^+ and naturality. □

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Reminder: Absolute gradings

For $\mathfrak{t} \in \mathrm{Spin}^c(Y)$ torsion, we constructed an absolute grading

$$\mathrm{gr}\colon HF^\circ(Y, \mathfrak{t}) \to \mathbb{Q}$$

with the following properties:

1. $HF^+(S^3)$ is supported in degrees ≥ 0
2. For a cobordism W betwenn torsion Spin^c 3-manifolds

$$\mathrm{gr}(F^\circ_{W,\mathfrak{s}}(\xi)) - \mathrm{gr}(\xi) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}$$

The conclusion

Corollary

$F^+_{W,\mathfrak{s}|_W} : HF^+(S^3) \to HF^+(S^3)$ is surjective.

- Pick $\xi \in HF^+(S^3)$ such that
  - $F^+_{W,\mathfrak{s}|_W}(\xi) \neq 0$
  - $\mathrm{gr}(F^+_{W,\mathfrak{s}|_W}(\xi)) = 0$
- Compute the expression

$$\mathrm{gr}(F^\circ_{W,\mathfrak{s}}(\xi)) - \mathrm{gr}(\xi)$$