Diplomarbeit

The L^2 Stokes Theorem on certain incomplete manifolds
(Der L^2 Satz von Stokes auf gewissen unvollständigen Mannigfaltigkeiten)

Angefertigt am
Mathematischen Institut

Vorgelegt der
Mathematisch-Naturwissenschaftlichen Fakultät der
Rheinischen Friedrich-Wilhelms-Universität Bonn

August 2009

Von

Stefan Behrens

Aus
Hannover

Contents

1 Introduction

The 20th century has revealed that one can proof remarkable results about the topology of closed manifolds by doing analysis. Prominent examples are Hodge theory and, of course, the celebrated Atiyah-Singer index theorem. Both of these are most naturally stated in the context of Sobolev spaces and elliptic pseudodifferential operators and each exhibits features of this class of operators. While the second theorem provides a deep connection between the Fredholm index of the operator and the K-theory class represented by its principal symbol, the first follows very easily from the regularity property of elliptic operators and the fact they are Fredholm operators and thus have closed range.

The thought to extend these results to more general manifolds, or even to spaces that are not manifolds but not too far away, such as algebraic varieties, suggests itself. Unfortunately, there are serious analytical obstacles that have to be overcome as soon as the manifold is non-compact or has a boundary. Although the class of elliptic pseudodifferential operators and the regularity theorem remain available, the Fredholm property breaks down for two reasons. First of all, in order to define the Sobolev spaces, it is necessary to choose various Riemannian and Hermitian[1] metrics and, unlike in the closed case, these spaces depend on the choices. Second of all, elliptic operators may not be Fredholm and even if they are, they may have different indices for different metrics.

Let (M, g) be a Riemannian manifold. Then the Riemannian metric induces a distance function d_g which gives M the structure of a metric space. The Riemannian metric g is called *complete* if (M, d_g) is a complete metric space, i.e. if every Cauchy sequence converges. It is easy to see that if M is compact, then any metric on M is complete. This leads to the guess that complete manifolds might be the next best thing. And indeed, it turns out that the analysis on complete manifolds is rather well behaved. However, on incomplete manifolds it is no exaggeration to say that everything that can go wrong, eventually will go wrong.

But as analytically unpleasent as incomplete manifolds are, they occur naturally. The most prominent examples come from (real or complex) projective varieties. Let V be such an object. It is well known that the set V_{reg} of regular points (in the sense of algebraic geometry) is open and

[1]In most cases that are of interest, the involved vector bundles are intimately related to the manifold and they carry natural Hermitian metrics induced by the choice of a Riemannian metric on the manifold. In this sense, one actually has to chose only one metric.

dense in V, and has the structure of a smooth (real or complex) manifold. Moreover, the singular locus $V_{\text{sing}} = V \setminus V_{\text{reg}}$ is an algebraic variety of strictly lower dimension. If we fix a Riemannian metric on projective space, V becomes a compact (hence complete) metric space. Hence, V_{reg} is an incomplete metric space with compact closure. Of course, the metric structure on V_{reg} is induced by the restriction of the Riemannian metric on projective space. Hence, V_{reg} is an incomplete Riemannian manifold.

The structure that we have just described is the prototype of a so called *manifold with singularities* (see Chapter 5). Further examples are metric cones, Riemannian stratified spaces and interiors of compact manifolds with boundary. The last example may sound artificial, but the point of view to consider a boundary as a singularity and to do analysis on the interior is at the heart of the b-calculus that was developed and successfully applied to boundary value problems by Richard B. Melrose.

So far we have explained how incomplete manifolds arise in geometric situations. We will now start narrowing in towards the subject of this thesis. Let (M, g) be an arbitrary Riemannian manifold. By $\Omega_c^*(M)$ we denote the smooth differential forms with compact support in the interior of M. The Riemannian metric induces a scalar product on $\Omega_c^*(M)$. We let $L^2\Omega^*(M, g)$ be its completion and consider the exterior derivative d as an unbounded operator on $L^2\Omega^*(M, g)$ with domain $\Omega_c^*(M)$. This operator is not closed but it has two natural closed extensions $d_{\min}$ and $d_{\max}$ with domains

$$\begin{aligned}
\mathcal{D}(d_{\min}) &= \left\{\omega \in L^2\Omega^*(M, g) \mid \exists\, \omega_i \in \Omega_c^*(M) \colon \omega_i \to \omega,\ d\omega_i \,\text{Cauchy}\right\} \\
\mathcal{D}(d_{\max}) &= \left\{\omega \in L^2\Omega^*(M, g) \mid d\omega \in L^2\Omega^*(M, g)\right\}.
\end{aligned}$$

All of this will be carefully explained in Chapter 2. There we will also show that these extensions are different in general. If they agree, we say that the *L^2 Stokes Theorem* holds on (M, g). In Section 4.1 we describe in detail the role played by the L^2 Stokes Theorem in proving an analogue of the Hodge theorem for the L^2 cohomology of M which is defined as

$$H_{(2)}^*(M, g) = \ker(d_{\max}) / \operatorname{ran}(d_{\max})$$

where the right hand side is to be understood in the graded sense. The L^2 cohomology can be computed from smooth forms (see Proposition 4.3) and it is thus natural to expect a relationship with the space

$$\hat{H}_{(2)}^*(M, g) := \left\{\omega \in \Omega^*(M) \cap L^2(\Lambda^* T^* M) \mid d\omega = 0, \delta\omega = 0\right\}$$

of square integrable harmonic forms. Here δ denotes the formal adjoint of the exterior derivative. The L^2 Stokes Theorem ensures that each such harmonic form represents a *unique* L^2 cohomology class (see Proposition 4.8).

The L^2 Stokes Theorem on incomplete manifolds first appeared in the work of Cheeger on manifolds with *conical singularities* ([C1, C2]). These are modeled on the *metric cone* over a closed manifold N, which es given by the manifold $(0,1) \times N$ equipped with the warped product metric

$$g_{\text{cone}} = dx^2 + x^2 g^N \tag{1.1}$$

where g^N is a Riemannian metric on N and x is the canonical coordinate in $(0,1)$. Cheeger basically proved the following theorem.

Theorem 1.1 (Cheeger)**.** *Let (M,g) be a manifold with a single conical singularity modeled on the metric cone over N.*

a) If $\dim N = 2k-1$, then the L^2 Stokes Theorem holds on M.

b) If $\dim N = 2k$ and $H^k(N;\mathbb{C}) = 0$, then the L^2 Stokes Theorem holds on M.

Later this theorem was generalized by Brüning and Lesch in [BL2] to *conformally conical singularities*. Moreover, their proof gave a complete classification of the closed extension of the exterior derivative.

Theorem 1.2 (Brüning, Lesch)**.** *In the situation of Theorem 1.1 b) we have*

$$\mathcal{D}(d_{\max})/\mathcal{D}(d_{\min}) \cong H^k(N;\mathbb{R}).$$

In [HM], Hunsicker and Mazzeo proved a result very similar to Theorem 1.1 for a different type of singularities, the so called *simple edge singularities* which are modeled as follows. Let Y be a closed manifold and suppose that Y is the total space of a fiber bundle $F \hookrightarrow Y \xrightarrow{\phi} B$. Moreover, assume that Y is equipped with a metric of the form

$$g^Y = \phi^* g^B + \kappa$$

where g^B is a Riemannian metric on B and κ restricts to a Riemannian metric on each fiber. Then the model for a simple edge singularity with edge B is the product $(0,1) \times Y$ equipped with the metric

$$g_{\text{edge}} = dx^2 + \phi^* g^B + x^2 \kappa. \tag{1.2}$$

The similarity with conical metric is obvious. Intuitively, the model space for simple edge singularities can be thought of as a fiber bundle over B with fiber a metric cone over F.

Theorem 1.3 (Mazzeo, Hunsicker)**.** *Let (M, g) be a manifold with a simple edge singularity modeled on the fiber bundle $F \hookrightarrow Y \to B$.*

a) If $\dim F = 2k - 1$, then the L^2 Stokes Theorem holds on M.

b) If $\dim F = 2k$ and $H^k(F; \mathbb{C}) = 0$, then the L^2 Stokes Theorem holds on M.

Their proof is basically an adaption of Cheeger's ad hoc calculation.

We aim to prove Theorem 1.3 with the methods developed in [BL1] and [BL2]. Our strategy is to exploit the fiber bundle structure of simple edge singularities. We will first treat the case of the trivial bundle and then use a localization principle for the L^2 Stokes Theorem in order to pass over to more general bundles.

In Chapter 2 we will review the necessary results about unbounded operators on a Hilbert space and differential operators on L^2 spaces. We define the minimal and the maximal extension of a differential operator and investigate some of their properties.

In Chapter 3 we provide a self contained introduction to the theory of Hilbert complexes as developed in [BL1] and [BL2] which is then applied to elliptic complexes. Our approach to Hilbert complexes differs slightly from the original, the difference being mostly of notational nature. The most important part of this chapter is Section 3.3 where we discuss products of Hilbert complexes and elliptic complexes.

In Chapter 4 we leave the general theory behind and study the de Rham complex of a Riemannian manifold and concentrate on the L^2 Stokes Theorem. We explain its role in the Hodge theory for L^2 cohomology and investigate its dependence on the Riemannian metric. We describe a method of proof by a localization procedure and apply this method to total spaces of fiber bundles. Here we prove our main result Theorem 4.29.

In the final Chapter 5 we turn to manifolds with singularities. We review some results on conical singularities and then apply our main theorem to simple edge singularities, giving a new proof of Theorem 1.3.

2 Differential operators as unbounded operators

We start by recalling some basic facts about unbounded operators on a Hilbert space and then show how differential operator fit in this setting. The purpose of this chapter is twofold. On one hand, it sets up our notation and on the other hand we prove some elementary results that will be used in later chapters. As general references for the theory of unbounded operators we use [RS1] and [RS2].

2.1 A review of unbounded operators

Let $\mathcal{H}$ be a separable Hilbert space. We denote the scalar product on $\mathcal{H}$ by $\langle\cdot,\cdot\rangle$ and assume it to be conjugate linear in the first variable. A *linear operator*, or just *operator* on $\mathcal{H}$ is a linear map A from a linear subspace $\mathcal{D}(A)$, called the *domain* of A, into $\mathcal{H}$. An operator A is called *closed* if $\mathcal{D}(A)$ is complete with respect to to the *graph scalar product*

$$\langle x,y\rangle_A := \langle x,y\rangle + \langle Ax,Ay\rangle\,,$$

i.e. if $(\mathcal{D}(A),\langle\cdot,\cdot\rangle_A)$ is a Hilbert space. In a way, closedness is a substitute for continuity. If A is defined on all of $\mathcal{H}$, i.e. $\mathcal{D}(A)=\mathcal{H}$, then both notions coincide. This is the content of the famous *closed graph theorem* ([RS1], Theorem III.12).

If $\mathcal{D}(A)$ is dense in $\mathcal{H}$, we say that A is *densely defined*. In this case we can define the *adjoint operator* A^*. Roughly speaking, A^* can be described as the operator with the largest domain such that the identity

$$\langle Ax,y\rangle = \langle x,A^*y\rangle$$

is satisfied for any $x\in\mathcal{D}(A)$ and $y\in\mathcal{D}(A^*)$. Explicitly, the domain of A^* is given by

$$\mathcal{D}(A^*) := \{y\in\mathcal{H}\mid \exists\,\xi\in\mathcal{H}\;\forall\,x\in\mathcal{D}(A) : \langle Ax,y\rangle = \langle x,\xi\rangle\}\,.$$

Since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, there is a unique such ξ for each $y\in\mathcal{D}(A^*)$ so that we can define $A^*y:=\xi$.

An operator B is called an *extension* of A if $\mathcal{D}(A)$ is contained in $\mathcal{D}(B)$ and the restriction of B to $\mathcal{D}(A)$ equals A. In that case, we write $A\subset B$. We say that A is *closable* if there exists a closed extension. Any closable operator A has a smallest closed extension called the *closure* of A which we denote by $\overline{A}$. For the domain of $\overline{A}$ we have

$$\mathcal{D}(\overline{A}) = \{x\in\mathcal{H}\mid \exists\,x_n\in\mathcal{D}(A) : x_n\to x,\ Ax_n\ \text{Cauchy}\}\,.$$

To be precise, for $x, y \in \mathcal{H}$ we have $x \in \mathcal{D}(\overline{A})$ and $\overline{A}x = y$ if there is a sequence $x_n \in \mathcal{D}(A)$ such that x_n converges to x and Ax_n converges to y.

For adjoints and closures, we have the following facts.

Proposition 2.1. *Let A be a densely defined operator.*

a) A^* is closed.

b) A is closable if and only if A^* is densely defined and, in that case, we have $A^{**} = \overline{A}$.

c) If A is closable then $(\overline{A})^* = A^*$.

Proof. Theorem VIII.1 in [RS1]. □

Using this we can prove that a closed operator decomposes the Hilbert space, provided the operator is densely defined.

Lemma 2.2. *Let A be a closed, densely defined operator on a Hilbert space $\mathcal{H}$. Then $\ker A$ is a closed subspace and we have an orthogonal direct sum decomposition*

$$\mathcal{H} = \ker A \oplus \overline{\operatorname{ran} A^*}. \tag{2.1}$$

Proof. According to Proposition 2.1b) $\mathcal{D}(A^*)$ is dense in $\mathcal{H}$ and we have $A = A^{**}$. Thus we have $Ax = A^{**}x = 0$ if and only if for any $y \in \mathcal{D}(A^*)$

$$\langle x, A^* y \rangle = 0.$$

This shows that $\ker A = (\operatorname{ran} A^*)^{\perp}$. Standard facts about orthogonal complem

$$(\ker A)^{\perp} = (\operatorname{ran} A^*)^{\perp\perp} = \overline{\operatorname{ran} A^*}$$

which yields the desired decomposition in (2.1). □

The following is an immediate consequence.

Corollary 2.3. *For a closed, densely defined operator we have*

$$\mathcal{D}(A) = \ker A \oplus (\mathcal{D}(A) \cap \overline{\operatorname{ran} A^*}).$$

This decomposition is orthogonal.

Let A be a closed operator. For a subspace $\mathcal{D} \subset \mathcal{D}(A)$, the restriction of A to $\mathcal{D}$ is clearly closable. If the closure of this restricted operator is equal to A, then $\mathcal{D}$ is called a *core* for A. Note that a subspace $\mathcal{D} \subset \mathcal{D}(A)$ is a core for A if and only if $\mathcal{D}$ is dense in $\mathcal{D}(A)$ with respect to the graph scalar product of A, i.e. if it is a dense subspace of the Hilbert space $(\mathcal{D}(A), \langle\cdot,\cdot\rangle_A)$. The behavior of A on any core determines A completely. As we will later see, a convenient choice of a core can simplify proofs considerably.

We say that an operator A is *symmetric*, if for any $x, y \in \mathcal{D}(A)$ we have $\langle Ax, y\rangle = \langle x, Ay\rangle$. If A is densely defined, this can be expressed as $A \subset A^*$. If equality holds, i.e. $A = A^*$, then A is called *self adjoint*. Note that, because of Proposition 2.1, any densely defined, symmetric operator is closable and any self adjoint operator is necessarily closed. If a symmetric operator A is not closed, but its closure is self adjoint, we say that A is *essentially self adjoint*. Self adjoint operators have particularly nice properties, mostly due to the *spectral theorem*:

Theorem 2.4 (Spectral theorem)**.** *Let A be a self adjoint operator on a separable Hilbert space $\mathcal{H}$. Then there exists a finite measure space (X, μ), a unitary isomorphism $U: \mathcal{H} \to L^2(X, \mu)$ and a measurable, real-valued function φ on X which is finite μ-almost everywhere such that*

$$UAU^* = M_\varphi,$$

where M_φ acts by multiplication with φ on the domain

$$\mathcal{D}(M_\varphi) = \left\{ f \in L^2(X, \mu) \mid \varphi f \in L^2(X, \mu) \right\}.$$

Proof. Theorem VIII.4 in [RS1]. □

A useful consequence is the following

Lemma 2.5. *Let A be a self adjoint operator. Then*

$$\mathcal{D}^\infty(A) := \bigcap_k^\infty \mathcal{D}(A^k) \tag{2.2}$$

is a core for A.

Proof. By the spectral theorem, we can assume that A is a multiplication operator on $L^2(X, \mu)$ with (X, μ) a finite measure space. Let $A = M_\varphi$. For $k \geq 1$ we have

$$\mathcal{D}(A^k) = \left\{ f \in L^2(X, \mu) \mid \varphi^k f \in L^2(X, \mu) \right\}. \tag{2.3}$$

Let χ_n be the characteristic function of the interval $[-n, n]$. For $f \in \mathcal{D}(A)$ we let $f_n := \chi_n(\varphi) f$. We claim that

a) $f_n \in \mathcal{D}^\infty(A)$ and

b) f_n converges to f in the graph norm of A.

For a) we observe that the function $\chi_n(\varphi(x))$ vanishes whenever $|\varphi(x)|$ is greater than n and is equal to one else. This implies

$$\|\varphi^k f_n\| = \|\varphi^k \chi_n(\varphi) f\| \le n^k \|f\| < \infty.$$

It follows from (2.3) that $f_n \in \mathcal{D}(A^k)$ for all k, hence $f_n \in \mathcal{D}^\infty(A)$.

In order to prove b) we have to check that f_n and φf_n converge in $L^2(X,\mu)$ to f and φf respectively. But this follows from Lebesgue's dominated convergence theorem and the fact that $\chi_n(\varphi)$ converges to 1 point wise. □

When performing algebraic manipulations with unbounded operators one encounters a lot of subtleties. These are caused by the fact that the operators are not defined everywhere. For two operators A and B the sum $A+B$ and the composition AB are defined on the respective domains

$$\mathcal{D}(A+B) := \mathcal{D}(A) \cap \mathcal{D}(B) \quad \text{and}$$
$$\mathcal{D}(AB) := \{x \in \mathcal{D}(B) \mid Bx \in \mathcal{D}(A)\}.$$

It is important to realize that neither $A+B$ nor AB have to be closed, closable or densely defined even if A and B have these properties. Unfortunately we will not be able to avoid these problems completely. The next Lemma collects some calculation rules that will be sufficient for our purposes.

Lemma 2.6. *Let A and B be two operators on a Hilbert space $\mathcal{H}$.*

a) If A is densely defined and $A \subset B$, then $B^* \subset A^*$.

b) If A,B, $A+B$ and AB are densely defined, then $A^* + B^* \subset (A+B)^*$ and $B^*A^* \subset (AB)^*$.

c) If A and B are closed and have orthogonal ranges, then $A+B$ is closed.

Proof. Everything but part c) is standard, so we will only prove c).

Let $x_n \in \mathcal{D}(A+B)$ such that $x_n \to x \in \mathcal{H}$ and $Ax_n + Bx_n \to y \in \mathcal{H}$. Since the ranges of A and B are orthogonal the sequences Ax_n and Bx_n must converge separately, say $Ax_n \to y'$ and $Bx_n \to y''$. But A and B are closed, so that $y' = Ax$ and $y'' = Bx$. Thus we have

$$y = y' + y'' = Ax + Bx = (A+B)x$$

and the closedness of $A+B$ follows. □

Another very useful result about the composition of certain operators is the following theorem of von Neumann. Recall that an operator A is called *non-negative* if for any $x \in \mathcal{D}(A)$ we have $\langle Ax, x \rangle \geq 0$.

Theorem 2.7 (von Neumann)**.** *Let A be a closed operator with dense domain. Then A^*A is self adjoint and non-negative.*

Proof. [RS2], Theorem X.25. □

The hardest part of the proof is to show that A^*A is densely defined. Note that the non-negativity of A^*A implies that the operator $I + A^*A$ is invertible.

Lemma 2.8. *Let A be a closed operator. Then $\mathcal{D}^\infty(A^*A)$ is a core for A.*

Proof. By Lemma 2.5 the space $\mathcal{D}^\infty := \mathcal{D}^\infty(A^*A)$ is a core for A^*A. The invertibility of $I+A^*A$ implies that $(I+A^*A)\mathcal{D}^\infty$ is dense in $\mathcal{H}$. Now assume that $x \in \mathcal{D}(A)$ is A-orthogonal to $\mathcal{D}^\infty$. Then for any $y \in \mathcal{D}^\infty$ we have

$$0 = \langle x, y \rangle + \langle Ax, Ay \rangle = \langle x, (I + A^*A)y \rangle .$$

Since $(I + A^*A)\mathcal{D}^\infty$ is dense, this implies that $x = 0$. □

Von Neumann's theorem allows us to reduce statements about closed extensions of certain operators to statements about self adjoint extensions of symmetric operators.

Definition 2.9. Let A be a densely defined operator. We say that A is *transposable* if $\mathcal{D}(A)$ is contained in $\mathcal{D}(A^*)$ and $\mathcal{D}(A)$ is invariant under A and A^*.[2] For a transposable operator A we define its *transposed operator* as $A^t := A^*|_{\mathcal{D}(A)}$.

Clearly, if A is transposable, so is A^t and we have $A^{tt} = A$.

From Proposition 2.1 it follows that a transposable operator is closable since its adjoint is densely defined. Furthermore, there are two canonical closed extensions.

Definition 2.10. Let A be a transposable operator. Then its *minimal* and *maximal extension* are the closed extensions given by

$$A_{\min} := \overline{A} = A^{**} \qquad \text{and} \qquad A_{\max} := (A^t)^*.$$

[2]i.e. $A\mathcal{D}(A) \subset \mathcal{D}(A)$ and $A^*\mathcal{D}(A) \subset \mathcal{D}(A)$.

For reasons that will become clear later we focus on the closed extensions of a transposable operator A that are contained in $A_{\max}$. In order to study these one can look at the self adjoint extensions of the symmetric, non-negative operator A^tA. Note that the transposability implies that $\mathcal{D}(A^tA) = \mathcal{D}(A)$. Hence, all three operators A, A^t and A^tA are defined on the common domain $\mathcal{D}(A)$ which we will from now on abbreviate with $\mathcal{D}$.

Lemma 2.11. *Let A be a transposable operator and B a closed extension of A which is contained in $A_{\max}$. Then B^* is a closed extension of A^t and B^*B is a self adjoint extension of A^tA.*

Proof. Let $x \in \mathcal{D}(A^t) = \mathcal{D}(A)$ and $y \in \mathcal{D}(B)$. Since be is contained in $A_{\max}$, we have

$$\langle x, By \rangle = \left\langle x, (A^t)^* y \right\rangle = \left\langle A^t x, y \right\rangle .$$

This shows that A^t is contained in B^* and we have proved the first part of the lemma.

The fact that B^* extends A^t immediately implies that B^*B is a closed, symmetric extension of A^tA which is self adjoint by Theorem 2.7. □

Next we show that two different closed extensions of A cannot induce the same self adjoint extension of A^tA.

Proposition 2.12. *Let A be transposable and let B and C be closed extensions of A, both contained in $A_{\max}$. If $B^*B = C^*C$, then $B = C$.*

Proof. The situation is symmetric in B and C, so we only need to show that $B \subset C$. Let $x \in \mathcal{D}(C)$. Since C^*C is densely defined, we can choose a sequence $x_n \in \mathcal{D}(C^*C)$ such that $x_n \to x$ in $\mathcal{H}$. The closedness of C implies that x_n converges to x in the graph norm of C. Using the assumption that $C^*C = B^*B$ we see that $x_n \in \mathcal{D}(B^*B)$ and

$$\langle Cx_n, Cx_n \rangle = \langle x_n, C^*Cx_n \rangle = \langle x_n, B^*Bx_n \rangle = \langle Bx_n, Bx_n \rangle .$$

This shows that the graph norms of B and C coincide on $\mathcal{D}(C^*C)$ and, in particular, that x_n converges to x in the graph norm of B. Thus $x \in \mathcal{D}(B)$ since B is closed.

It remains to show that $Bx = Cx$. Since x_n converges to x in the graph norms of B and C, the sequences Bx_n and Cx_n converges to Bx, respectively Cx. Using that B and C are contained in $A_{\max}$ we see that

$$Bx_n = A_{\max}x_n = Cx_n.$$

But $A_{\max}$ is also a closed operator. So we must have $x \in \mathcal{D}(A_{\max})$ and we can pass to the limit to obtain

$$Bx = A_{\max}x = Cx.$$

□

The last piece of general theory concerns tensor products of self adjoint operators. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. The algebraic tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is a pre-Hilbert space with scalar product

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle .$$

The *Hilbert space tensor product* $\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2$ is defined as the completion of $\mathcal{H}_1 \otimes \mathcal{H}_2$ with respect to to this scalar product.

Proposition 2.13. *For $i = 1, 2$ let A_i be self adjoint operators on $\mathcal{H}_i$ with domains of essential self adjointness $\mathcal{D}_i \subset \mathcal{H}_i$. Then the operator $A_1 \otimes A_2$ is essentially self adjoint on $\mathcal{D}_1 \otimes \mathcal{D}_2$.*

Proof. See [RS1], Theorem VIII.33. □

For notational convenience, we have restricted our presentation to operators from one Hilbert space into itself. But with minor modifications everything stays valid if the range of an operator is a different Hilbert space. We will take this for granted from now on.

2.2 Functional analytic properties of differential operators

We now turn to the special case of differential operators. For the most part we will focus on first order differential operators, although some statements are true for operators of arbitrary order.

Let (M, g) be a Riemannian manifold and let (E, h^E) be a Hermitian vector bundle over M. We denote the smooth sections of E by $\Gamma(E)$ and those who have compact support in the interior of M by $\Gamma_c(E)$. The metrics on M and E induce a scalar product on $\Gamma_c(E)$, called the L^2 *scalar product.* For $\varphi, \psi \in \Gamma_c(E)$ we let

$$\langle \varphi, \psi \rangle_{L^2(E)} := \int_M h^E(\varphi, \psi)\, d\mu_g \tag{2.4}$$

where μ_g is the measure on M induced by the Riemannian metric[3]. We will usually omit the subscript if the vector bundle is clear from the context. Let $L^2(E)$ be the Hilbert space given by the completion of $\Gamma_c(E)$ with respect to the L^2 scalar product. $L^2(E)$ is called the *space of square-integrable sections*, or just L^2 *sections* of E.

Note that for the right hand side of (2.4) to make sense it is not necessary that both φ and ψ have compact support. Indeed, it is enough if one section has compact support and smoothness it not required at all.

Let F be another Hermitian vector bundle and let $D: \Gamma(E) \to \Gamma(F)$ be a differential operator. Since D maps $\Gamma_c(E)$ into $\Gamma_c(F)$, we can consider D as an operator from $L^2(E)$ into $L^2(F)$ with domain $\Gamma_c(E)$. In general, D will not be bounded, but it is closable, as we will shortly see.

We will now recall some basic facts about differential operators and interpret them and their consequences within the abstract setting of the previous section.

With D, we can associate another differential operator

$$D^t: \Gamma(F) \to \Gamma(E),$$

called the *formal adjoint* of D, which is uniquely determined by requiring that

$$\langle D\varphi, \psi \rangle_{L^2(F)} = \langle \varphi, D^t\psi \rangle_{L^2(E)} \tag{2.5}$$

holds for any $\varphi \in \Gamma_c(E)$ and $\psi \in \Gamma_c(F)$. Note that, obviously, we have $D^{tt} = D$. If $D^t = D$ we say that D is *formally self adjoint*.

From (2.5) it follows that D^t with domain $\Gamma_c(F)$ is contained in D^*, the Hilbert space adjoint of D, which is thus densely defined. By Proposition 2.1, D is closable. In the case that $E = F$ we see that D is transposable in the sense of Definition 2.9 with transposed give by the formal adjoint. In any case, D has two closed extension

Definition 2.14. The closure of D in $L^2(E)$ is called the *minimal extension* and is denoted by $D_{\min}$. The *maximal extension* of D is the closed extension given by $D_{\max} := (D^t)^*$. We write $\mathcal{D}_{\min}(D) := \mathcal{D}(D_{\min})$ and $\mathcal{D}_{\max}(D) := \mathcal{D}(D_{\max})$.

We can describe the minimal extensions explicitly as follows. We have $\varphi \in \mathcal{D}_{\min}(D)$ and $D_{\min}\varphi = \eta \in L^2(F)$ if and only if there exists a sequence

[3] Recall that if $e_1, \dots, e_n$ is a local orthonormal frame for TM, that is $g(e_i, e_j) = \delta_{ij}$, and $\varepsilon^1, \dots, \varepsilon^n$ is the dual frame for T^*M, then the measure μ_g is locally represented by the n-form $\varepsilon^1 \wedge \dots \wedge \varepsilon^n$.

$\varphi_n \in \Gamma_c(E)$ such that $\varphi_n \to \varphi$ in $L^2(E)$ and $D\varphi_n \to \eta$ in $L^2(F)$. In order to explain the maximal extension we have to discuss the action of differential operators on distributional sections.

For the moment let

$$\tilde{\Gamma}(E) := \{\varphi: M \to E \mid \forall x \in M : \varphi(x) \in E_x\}$$

be the space of *all* sections of E (not necessarily continuous). A section $\varphi \in \tilde{\Gamma}(E)$ is called *locally integrable* if the function $|\varphi| = \sqrt{h^E(\varphi,\varphi)}$ is integrable over any compact subset of M. This implies that the expression

$$T_\varphi[\psi] := \int_M h^E(\varphi,\psi)\, d\mu_g$$

determines a complex number for any $\psi \in \Gamma_c(E)$. This construction yields a linear map $T_\varphi: \Gamma_c(E) \to \mathbb{C}$ which turns out to be continuous with respect to the natural LF-topology on $\Gamma_c(E)$. Hence, T_φ is an element of the (topological) dual space of $\Gamma_c(E)$ which we denote by $\mathcal{D}'(E)$. This space is by definition the space of *distributional sections* of E. The map $\varphi \mapsto T_\varphi$ is a (continuous) injection and we can identify the distributional section T_φ with φ.

Using the formal adjoint, we can extend the action of D to distributional sections by letting $DT[\psi] := T[D^t\psi]$. With these remarks in mind we will write $D\varphi = \psi$ for locally integrable sections φ and ψ if $DT_\varphi = T_\psi$ as distributional sections. We will also say that $D\varphi = \psi$ holds *weakly* and call ψ the *weak derivative* of φ. Note that if φ is a smooth section, then

$$DT_\varphi[\psi] = T_\varphi[D^t\psi] = \langle \varphi, D^t\psi \rangle = \langle D\varphi, \psi \rangle,$$

so that $DT_\varphi = T_{D\varphi}$, which justifies the above notation.

A consequence of the above discussion is that we can apply D to any L^2 section of E. In general, the result will not be an L^2 section of F, but considering only those sections that are mapped to $L^2(F)$ gives the following interpretation of the maximal extension.

Lemma 2.15. *For the maximal extension of D we have*

$$\mathcal{D}_{\max}(D) = \{\varphi \in L^2(E) \mid D\varphi \in L^2(F)\}$$

and $D_{\max}\varphi = D\varphi$ where $D\varphi$ is the weak derivative.

Proof. Recall that $D_{\max}$ is defined as the Hilbert space adjoint of D^t. So we have $\varphi \in \mathcal{D}_{\max}(D)$ if and only if there exists $\eta \in L^2(F)$ such that

$$\langle \eta, \psi \rangle = \langle \varphi, D^t\psi \rangle$$

for all $\psi \in \Gamma_c(F)$. But this is equivalent to $D\varphi = \eta$ in the weak sense. □

This means that $D_{\max}$ is the largest closed extension of D as an operator on $L^2(E)$ which acts as a differential operator. This also explains the name maximal extension. Note that there are indeed larger closed extensions[4] of D but these are not very interesting since they are not linked to the geometric situation.

Having defined these two closed extensions, the following questions arise.

- Are the minimal and the maximal extension the same? And if not, how different are they?
- Is the maximal extension determined by its action on smooth sections?

Unfortunately, we can only give partial answers due to the lack of our knowledge.

We will address the second question first. Let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator. The precise formulation of the question is if

$$\mathcal{D}_{\max}(D) \cap \Gamma(E) = \left\{ s \in \Gamma(E) \mid s \in L^2(E),\, Ds \in L^2(F) \right\}$$

is a core for $D_{\max}$. Surprisingly, a general answer to this question does not seem to be available. However, for the important class of *elliptic* operators the answer is positive. Recall that the *principal symbol* of D is a bundle homomorphism

$$\sigma_D \colon \pi^* E \to \pi^* F$$

where $\pi \colon T^* M \to M$ is the projection of the cotangent bundle and that D is called elliptic if σ_D is an isomorphism outside the zero section. In order to state the regularity theorem we need to recall some facts about Sobolev spaces on manifolds. For our purposes, it is enough to know the following facts.

Theorem 2.16 (Sobolev spaces)**.** *Let $D \colon \Gamma_c(E) \to \Gamma_c(F)$ be a differential operator of order k.*

a) For any $s \in \mathbb{R}$ there exists a topological vector space $H^s_{\mathrm{loc}}(E)$ of sections of E which contains the smooth sections $\Gamma(E)$ and is independent of the various metrics. Moreover, we have $H^0_{\mathrm{loc}}(E) = L^2_{\mathrm{loc}}(E)$.

b) For $t < s$ we have a continuous inclusion $H^s_{\mathrm{loc}}(E) \subset H^t_{\mathrm{loc}}(E)$. Furthermore, we have $\bigcap_{s \in \mathbb{R}} H^s_{\mathrm{loc}}(E) = \Gamma(E)$.

[4] Just pick an arbitrary element of $L^2(E)$ which is not contained in $\mathcal{D}_{\max}(D)$ and specify an arbitrary image in $L^2(F)$. This yields a closed extension which is strictly larger than $D_{\max}$.

c) For any $s \in \mathbb{R}$, D extends to a continuous linear operator

$$D: H^s_{\text{loc}}(E) \to H^{s-k}_{\text{loc}}(F).$$

For proofs we refer to the elegant exposition in [S], Chapter I.7. There one also finds the proof of the famous regularity theorem (Theorem 7.2).

Theorem 2.17 (Elliptic regularity)**.** *Let $D: \Gamma_c(E) \to \Gamma_c(F)$ be an elliptic operator of order k. If $u \in \mathcal{D}'(E)$ satisfies $Du \in H^s_{\text{loc}}(F)$, then $u \in H^{s+k}_{\text{loc}}(E)$. In particular, if Du is a smooth section, then so is u.*

Given an elliptic operator D we form the formally self adjoint operator $D^t D$. Since $\sigma_{D^t D} = \sigma_D{}^* \sigma_D$ this is also an elliptic operator. As in Lemma 2.11 the closed extension $D_{\max}$ induces a self adjoint extension of $D^t D$. To be precise, this extension is given by $D_{\max}{}^* D_{\max} = (D^t)_{\min} D_{\max}$ and is clearly contained in $(D^t D)_{\max}$ so that it acts as a differential operator. By Lemma 2.8 $\mathcal{D}^\infty(D_{\max}{}^* D_{\max})$ is a core for $D_{\max}$. We have thus proved the first half of

Proposition 2.18. *Let D be a differential operator. Then the domain $\mathcal{D}^\infty(D_{\max}{}^* D_{\max})$ is a core for $D_{\max}$. If D is elliptic, then $\mathcal{D}^\infty(D_{\max}{}^* D_{\max})$ consists solely of smooth sections.*

Proof. Let $T := D_{\max}{}^* D_{\max}$. For $u \in \mathcal{D}^\infty(T)$ we have $u \in \mathcal{D}(T^k)$ and $T^k u \in L^2(E) \subset L^2_{\text{loc}}(E)$ for all $k \geq 1$. By the elliptic regularity theorem, we have $u \in H^{2kl}_{\text{loc}}(E)$ for all k where l is the order of D, and Theorem 2.16b) implies $u \in \Gamma(E)$. □

An immediate consequence is that $\mathcal{D}_{\max}(D) \cap \Gamma(E)$ is a core for $D_{\max}$ if D is elliptic. Later we will prove a generalization for operators that appear in elliptic complexes (see Section 3.2).

We now come to the first of the questions asked on page 14. The following example shows that the minimal and maximal extensions are different in general.

Example 2.19. Consider the open unit interval $(0,1) \subset \mathbb{R}$ and the differential operator $D = -i\frac{d}{dx}$ acting on $L^2(0,1)$ with domain $C^\infty_c(0,1)$. Integration by parts shows that D is formally self adjoint. Let $f \in \mathcal{D}_{\min}(D) \cap C^\infty(0,1)$ and let $g \in C^\infty(0,1)$ be a bounded real-valued function such that $g(0) = 1$ and $g(1) = 0$ (for example $g(x) = 1 - x$). Clearly, $g \in \mathcal{D}_{\max}(D)$ and since

$D_{\min} = D^*_{\max}$ we have

$$\begin{aligned} 0 &= \langle g, D_{\min} f\rangle - \langle D_{\max} g, f\rangle \\ &= \int_0^1 \left[\overline{g(x)}(-if'(x)) - \overline{(-ig'(x))}f(x)\right] dx \\ &= -i\int_0^1 \left[g(x)f'(x) + g'(x)f(x)\right] dx \\ &= -i[g(1)f(1) - g(0)f(0)] = if(0). \end{aligned}$$

Hence, for any smooth $f \in \mathcal{D}_{\min}(D)$ we have $f(0) = 0$ which, in particular, implies that $g \notin \mathcal{D}_{\min}(D)$, since $g(0) = 1$.

This puts the emphasis on the second part of the question, namely when are the two extensions different and how does their difference manifest?

We start by reformulating the problem. More or less by definition, we have $D_{\min} = (D^t_{\max})^*$.

Lemma 2.20. *Let D be a differential operator. Then $D_{\min} = D_{\max}$ if and only if we have*

$$\left\langle D_{\max} s, t\ \right\rangle = \left\langle s, D^t_{\max} t\right\rangle$$

for any $s \in \mathcal{D}_{\max}(D)$ and $t \in \mathcal{D}_{\max}(D^t)$

In this form the problem is more tractable in some situations.

Definition 2.21. We say that *uniqueness holds* for a differential operator D if $D_{\min} = D_{\max}$.

We collect some situations in which uniqueness is known to hold.

Proposition 2.22. *Let D be an elliptic operator on a closed manifold. Then uniqueness holds for D.*

Proof. By Proposition 2.18, $D_{\max}$ admits a core consisting of smooth sections. But all smooth sections automatically have compact support. Thus $D_{\max} = D_{\min}$. □

Note that this argument works for any differential operator whose maximal extension admits a core consisting of smooth sections. So if this were true for any differential operator, then uniqueness would hold for all differential operators on closed manifolds.

As a corollary we get that for elliptic operators on closed manifolds the concepts of formal self adjointness and essential self adjointness agree.

Corollary 2.23. *Let S be a formally self adjoint elliptic operator on a closed manifold. Then S is essentially self adjoint.*

Proof. The Hilbert space adjoint S^* is a closed extension of S which agrees with the closure $\overline{S}$ by Proposition 2.20. So S is essentially self adjoint. □

Proposition 2.12 provides us with a criterion for proving uniqueness for a differential operator.

Proposition 2.24. *Let D be a differential operator. If D^tD is essentially self adjoint, then uniqueness holds for D.*

Proof. Since there is only one self adjoint extension of D^tD, there can only be one closed extension of D by Proposition 2.12. In particular, we must have $D_{\min} = D_{\max}$. □

This criterion applies to the situation of operators of Dirac type on *complete* manifolds without boundary. Recall that a first order differential operator D is said to be of Dirac type if it is formally self adjoint and the principal symbol of its square satisfies $\sigma_{D^2}(\xi) = -|\xi|^2$. In particular, operators of Dirac type are elliptic.

Proposition 2.25. *Let D be an operator of Dirac type on a complete manifold without boundary. Then D and all its powers are essentially self adjoint.*

A nice proof of this result can be found in [C3].

All these results seem to indicate that the difference between the minimal and the maximal extension can be found at infinity, i.e. in the complement of any compact set. For first order operators this can be made precise as follows.

Lemma 2.26. *Let D be a first order differential operator. If $s \in \mathcal{D}_{\max}(D)$ and $\rho \in C_c^\infty(M)$, then $\rho s \in \mathcal{D}_{\min}(D)$.*

Proof. [GL], Lemma 2.1. □

In Section 4.3 we will try to localize the question of uniqueness in the cases where D is either the exterior derivative d or its formal adjoint $\delta = d^t$. During this process we have to find conditions on a function $\rho \in C^\infty(M)$ such that multiplication with ρ maps $\mathcal{D}_{\max}(D)$ into itself, i.e $\rho s \in \mathcal{D}_{\max}(D)$ for all $s \in \mathcal{D}_{\max}(D)$.

Definition 2.27. Let (M, g) be a Riemannian manifold. A smooth function $\rho \in C^\infty(M)$ is called *C^1-bounded* if $|\rho|$ and $|d\rho|$ are uniformly bounded.

Note that the dependence on the Riemannian metric is hidden in the expression $|d\rho|$. In fact, we have

$$|d\rho|^2 = g(\operatorname{grad}_g \rho, \operatorname{grad}_g \rho).$$

Lemma 2.28. *Let (M, g) be a Riemannian manifold, $U \subset M$ an open subset and let D be either d or δ. If $\rho \in C^\infty(M)$ is C^1-bounded with* $\operatorname{supp}(\rho) \subset U$ *and $\omega \in \mathcal{D}_{\max}(D)$, then $\rho\omega \in \mathcal{D}_{\max}(D)$ and* $\operatorname{supp}(\rho\omega) \subset U$.

Proof. The statement about the support of $\rho\omega$ is trivial. Moreover, it is clear that $\rho\omega \in L^2\Omega^*(M)$ since

$$|\rho\omega| = |\rho||\omega| \le C|\omega|,$$

and analogously $\rho D\omega \in L^2\Omega^*(M)$. In order to prove that $D(\rho\omega) \in L^2\Omega^*(M)$, and hence $\rho\omega \in \mathcal{D}_{\max}(D)$, it remains to show that $D(\rho\omega) - \rho D\omega \in L^2\Omega^*(M)$. Since D is of first order we have

$$D(\rho\omega) - \rho D\omega = \sigma_D(d\rho)\omega,$$

where σ_D denotes the principal symbol of D. For the moment we have to distinguish between d and δ. It is well known that the principal symbols of these operators are given by

$$\sigma_d(d\rho)\omega = d\rho \wedge \omega \qquad \text{and} \qquad \sigma_\delta(d\rho)\omega = d\rho^\# \llcorner \omega$$

where $\llcorner$ indicates interior multiplication and $d\rho^\#$ is the vector field on M defined by $g(d\rho^\#, v) = d\rho(v)$ for any $v \in TM$.

Let $e_1, \dots, e_n$ be a local orthonormal frame for TM and let $e^1, \dots, e^n$ be the dual frame for T^*M. Then we can write

$$d\rho = \sum_i \xi_i e^i \qquad \text{and} \qquad \omega = \sum_I \omega_I e^I$$

where we use the standard multi-index notation. Observe that

$$|d\rho|^2 = \sum_i |\xi_i|^2 \qquad \text{and} \qquad |\omega|^2 = \sum_I |\omega_I|^2.$$

We compute

$$|\sigma_d(d\rho)\omega|^2 = |d\rho \wedge \omega|^2 = \sum_{i,I} |\xi_i|^2 |\omega_I|^2 \, |e^i \wedge e^I|^2 \tag{2.6}$$

and analogously using $d\rho^{\#} = \sum_i \xi_i e_i$

$$|\sigma_\delta(d\rho)\omega|^2 = |d\rho^{\#} \llcorner \omega|^2 = \sum_{i,I} |\xi_i|^2 |\omega_I|^2 \, |e_i \llcorner e^I|^2. \tag{2.7}$$

Since $\{e_i\}$ is an orthonormal frame we have

$$|e^i \wedge e^I|^2 = \begin{cases} 1 & i \notin I \\ 0 & i \in I \end{cases} \qquad \text{and} \qquad |e_i \llcorner e^I|^2 = \begin{cases} 1 & i \in I \\ 0 & i \notin I. \end{cases}$$

Treating d and δ simultaneously as D again, (2.6) and (2.7) yield

$$\begin{aligned} |\sigma_D(d\rho)\omega|^2 &\leq \sum_{i,I} |\xi_i|^2 |\omega_I|^2 \\ &= \Big(\sum_i |\xi_i|^2 \Big) \Big(\sum_I |\omega_I|^2 \Big) \\ &= |d\rho|^2 |\omega|^2. \end{aligned}$$

Since $d\rho$ is uniformly bounded, we have

$$|\sigma_D(d\rho)\omega|^2 \leq C \, |\omega|^2$$

for some $C \geq 0$ and thus $\sigma_D(d\rho)\omega \in L^2\Omega^*(M)$. This finishes the proof. $\square$

3 Hilbert complexes and elliptic complexes

In this chapter we study the framework of *Hilbert complexes* which was introduced in [BL1] as a purely functional analytic abstraction of elliptic complexes over manifolds. The theory is rather trivial in the closed case but becomes relevant on non-compact manifolds.

In Section 3.1 we introduce Hilbert complexes and study some of their features. We give complete proofs of all results that are relevant for our later applications. Although our approach follows [BL1] closely, we develop the theory in the setting of *differential graded Hilbert spaces* (see Definition 3.3) in order to emphasize the fact that, from the functional analytic point of view, Hilbert complexes are not to be thought of as a generalization of one closed operator but rather as a single operator with some additional structure. We will see that Hilbert complexes are actually much better behaved than arbitrary closed operators.

Section 3.2 is devoted to *elliptic complexes* and their *ideal boundary conditions* as a primary source of examples for Hilbert complexes.

We go on to study products of Hilbert complexes and elliptic complexes in Section 3.3 following an unpublished note of Brï¿½ning and Lesch [BL3]. The section culminates in Proposition 3.36 which is the key result for our applications.

Finally, in Section 3.4 we make a few remarks about the cohomological aspects of Hilbert complexes and Hodge theory.

3.1 Hilbert complexes

In [BL1], Brüning and Lesch define a *Hilbert complex* to be a sequence of Hilbert spaces $\mathcal{H}_0, \ldots, \mathcal{H}_N$ together with closed, densely defined operators $D_0, \ldots, D_{N-1}$ with respective domains $\mathcal{D}_i := \mathcal{D}(D_i) \subset \mathcal{H}_i$ satisfying $D_i(\mathcal{D}_i) \subset \mathcal{D}_{i+1}$ and $D_{i+1}D_i = 0$. In other words, a Hilbert complex is a (cochain) complex of vector spaces in the sense of homological algebra

$$0 \longrightarrow \mathcal{D}_0 \xrightarrow{D_0} \mathcal{D}_1 \xrightarrow{D_1} \ldots \xrightarrow{D_{N-2}} \mathcal{D}_{N-1} \xrightarrow{D_{N-1}} \mathcal{D}_N \longrightarrow 0 \tag{3.1}$$

where we let $\mathcal{D}_N := \mathcal{H}_N$ with the additional structure that each $\mathcal{D}_i$ is a dense subspace of a Hilbert space $\mathcal{H}_i$ and each D_i is a closed operator.

In order to emphasize the fact that a Hilbert complex is determined by a single operator and to ease the notation, we will adopt a slightly different point of view. We start by introducing some terminology.

A *graded Hilbert space* is a Hilbert space $\mathcal{H}$ together with a direct sum decomposition $\mathcal{H} = \bigoplus_{i=0}^{N} \mathcal{H}_i$ where $\mathcal{H}_0, \ldots, \mathcal{H}_N$ are mutually orthogonal, closed subspaces of $\mathcal{H}$.

A subspace V of a graded Hilbert space $\mathcal{H}$ is called a *graded subspace* if $V = \bigoplus_{i=0}^{N} (V \cap \mathcal{H}_i)$. In this case, we write $V_i := V \cap \mathcal{H}_i$.

An operator A on a graded Hilbert space $\mathcal{H}$ is called a *graded operator of degree* $r \in \mathbb{Z}$ if its domain is a graded subspace, i.e. $\mathcal{D}(A) = \bigoplus_{i=0}^{N} \mathcal{D}_i(A)$ with $\mathcal{D}_i(A) \subset \mathcal{H}_i$, and $A(\mathcal{D}_i(A)) \subset \mathcal{H}_{i+r}$. If we denote by A_i the restriction of A to $\mathcal{D}_i(A)$, then we have $A = \bigoplus_{i=0}^{N} A_i$. We use the convention that $\mathcal{H}_i = \{0\}$ unless $0 \le i \le N$. Furthermore, if $i + r \notin \{0, 1, \ldots, N\}$, it is to be understood that $\mathcal{D}_i(A) = \mathcal{H}_i$ and $A_i = 0$.

Lemma 3.1. *Let A be a densely defined, graded operator of degree r on a graded Hilbert space $\mathcal{H}$. Then its adjoint A^* is a closed, graded operator of degree $-r$ and we have $(A^*)_i = (A_{i-r})^*$.*

Proof. By Proposition 2.1, A^* is closed. For the statement about the grading let $\tilde{A} := \bigoplus_{i=0}^{N} A_i{}^*$. We have to show that $\tilde{A} = A^*$. Clearly, we have $\tilde{A} \subset A^*$. Let $x = (x_{,0}, \ldots, x_N) \in \mathcal{D}(A^*)$ and let $y \in \mathcal{D}_{i-r}(A)$. Then

$$\langle (A^*x)_{i-r}, y \rangle = \langle A^*x, y \rangle = \langle x, Ay \rangle = \langle x_i, A_{i-r}y \rangle .$$

But this means that $x_i \in \mathcal{D}\left((A_{i-r})^*\right)$ and $(A_{i-r})^* x_i = (A^*x)_{i-r}$. This finishes the proof. □

Corollary 3.2. *Let A be a densely defined, closable, graded operator of degree r on a graded Hilbert space $\mathcal{H}$. Then its closure $\overline{A}$ is a graded operator of degree r and we have $(\overline{A})_i = \overline{A_i}$.*

Proof. By Proposition 2.1 we have $\overline{A} = A^{**}$. Hence, the preceding Lemma implies that $\overline{A}$ is graded of degree r with

$$(\overline{A})_i = (A^{**})_i = ((A^*)_{i-r})^* = (A_{i-r+r})^{**} = \overline{A_i}.$$

□

Definition 3.3. A *differential graded Hilbert space* is a graded Hilbert space $\mathcal{H}$ together with a graded operator D of degree ± 1 with dense domain $\mathcal{D} \subset \mathcal{H}$ such that $D(\mathcal{D}) \subset \mathcal{D}$ and $D^2 = 0$. The operator D is called the *differential*. A differential graded Hilbert space is called *positive* (*negative*) if its differential has degree $+1$ (-1).

We will usually denote a differential graded Hilbert space by $(\mathcal{D}, D)$. When the surrounding Hilbert space is not clear from the context, we write $(\mathcal{H}; \mathcal{D}, D)$.

Comparing this with Brüning and Lesch's definition of Hilbert complexes, we see:

Lemma 3.4. *Every Hilbert complex determines a closed, positive differential graded Hilbert space and vice versa. Furthermore, the correspondence is bijective.*

Proof. Starting with a Hilbert complex as in (3.1) we let $\mathcal{H} := \bigoplus_{i=0}^N \mathcal{H}_i$, $\mathcal{D} := \bigoplus_{i=0}^N \mathcal{D}_i$ and $D := \bigoplus_{i=0}^N D_i$ where $\mathcal{D}_N = \mathcal{H}_N$ and $D_N = 0$. Then $(\mathcal{H}; \mathcal{D}, D)$ is a closed, positive differential graded Hilbert space. The reverse construction is even more obvious and it is clear that both are inverse to each other. $\square$

Using this observation, we can give an alternative definition for the notion of a Hilbert complex.

Definition 3.5. A positive differential graded Hilbert space $(\mathcal{D}, D)$ is called a *pre-Hilbert complex* if its differential D is closable. It is called a *Hilbert complex* if D is closed. A Hilbert complex $(\mathcal{D}, D)$ is called an *ideal boundary condition* for a pre-Hilbert complex $(\tilde{\mathcal{D}}, \tilde{D})$ if D is a closed extension of $\tilde{D}$.

Let $(\mathcal{D}, D)$ be a pre-Hilbert complex. Since the differential is densely defined it has an adjoint. We let $\mathcal{D}^* := \mathcal{D}(D^*)$ and we claim that $(\mathcal{D}^*, D^*)$ is a closed, negative differential graded Hilbert space. By Lemma 3.1, D^* is closed and graded of degree -1. In order to see that D^* has the properties of a differential, we observe that for $x \in \mathcal{D}^*$ and $y \in \mathcal{D}$ we have

$$\langle D^* x, Dy \rangle = \langle x, DDy \rangle = 0 = \langle 0, y \rangle ,$$

so that $D^* x \in \mathcal{D}^*$ and $D^* D^* x = 0$.

Definition 3.6. The differential graded Hilbert space $(\mathcal{D}^*, D^*)$ is called the *dual complex* of $(\mathcal{D}, D)$ and is denoted by $(\mathcal{D}, D)^*$.

Dualizing a pre-Hilbert complex $(\mathcal{D}, D)$ twice yields a Hilbert complex $(\mathcal{D}, D)^{**}$ which is clearly an ideal boundary condition for $(\mathcal{D}, D)$. Moreover, the differential of $(\mathcal{D}, D)^{**}$ is given by $D^{**} = \overline{D}$. This proves

Lemma 3.7. *Any pre-Hilbert complex has an ideal boundary condition provided by the closure of its differential.*

For the time being, we will leave pre-Hilbert complexes and their ideal boundary conditions aside and further investigate the structure of Hilbert complexes.

Definition 3.8. Let $(\mathcal{D}, D)$ be a Hilbert complex. The vector space

$$\hat{H}(\mathcal{D}, D) := \ker D \cap \ker D^* \tag{3.2}$$

is called the *space of harmonic elements* of $(\mathcal{D}, D)$.

Note that $\hat{H}(\mathcal{D}, D)$ is a graded subspace of $\mathcal{H}$ with grading

$$\hat{H}^i(\mathcal{D}, D) = \ker D_i \cap \ker D_{i-1}{}^*.$$

We will come back to this space in Section 3.4. Its importance is indicated by its appearance in the next

Proposition 3.9 (Weak Hodge decomposition)**.** *Let $(\mathcal{H}; \mathcal{D}, D)$ be a Hilbert complex. Then we have an orthogonal direct sum decomposition*

$$\mathcal{H} = \hat{H}(\mathcal{D}, D) \oplus \overline{\operatorname{ran} D} \oplus \overline{\operatorname{ran} D^*}. \tag{3.3}$$

Because of the similarity with the Hodge decomposition of an elliptic complex over a compact manifold, (3.3) is called the *weak Hodge decomposition*.

Proof. By applying Lemma 2.2 to the closed operators D and D^*, we obtain two decompositions

$$\mathcal{H} = \ker D \oplus \overline{\operatorname{ran} D^*} \qquad \text{and} \qquad \mathcal{H} = \ker D^* \oplus \overline{\operatorname{ran} D}. \tag{3.4}$$

We observe that $\overline{\operatorname{ran} D} \subset \ker D$ since $D^2 = 0$ and $\ker D$ is closed. If $x \in \ker D$ is orthogonal to $\overline{\operatorname{ran} D}$, then the second equation in (3.4) implies that $x \in \ker D \cap \ker D^* = \hat{H}(\mathcal{D}, D)$. Combining this with the first equation in (3.4) implies (3.3). $\square$

The study of Hilbert complexes is simplified with the introduction of the following two operators.

Definition 3.10. Let $(\mathcal{D}, D)$ be a Hilbert complex. We define the *Gauss-Bonnet operator* as

$$D_{GB} := D + D^*$$

and the *Laplace operator* (or *Laplacian*) as

$$\Delta(\mathcal{D}, D) := D^* D + D D^* = D_{GB}^2.$$

If only one Hilbert complex is involved we will usually abbreviate the Laplacian by $\Delta = \Delta(\mathcal{D}, D)$.

It is clear that the Gauss-Bonnet operator and the Laplacian are symmetric. But even more is true.

Lemma 3.11. *The operators D_{GB} and Δ are self adjoint.*

Proof. It is enough to show that D_{GB} is self adjoint since in that case we have $\Delta = D_{GB}^2 = D_{GB}^* D_{GB}$ and Δ is self adjoint by von Neumann's theorem (2.7).

In order to see that D_{GB} is self adjoint we first have to ensure that it is densely defined. Recall that the domain is given by $\mathcal{D} \cap \mathcal{D}^*$. From the weak Hodge decomposition we get

$$\mathcal{D} = \ker D \cap \ker D^* \oplus \overline{\operatorname{ran} D} \oplus (\overline{\operatorname{ran} D^*} \cap \mathcal{D})$$

and

$$\mathcal{D}^* = \ker D \cap \ker D^* \oplus (\overline{\operatorname{ran} D} \cap \mathcal{D}^*) \oplus \overline{\operatorname{ran} D^*}.$$

Hence, we have

$$\mathcal{D} \cap \mathcal{D}^* = \ker D \cap \ker D^* \oplus (\overline{\operatorname{ran} D} \cap \mathcal{D}^*) \oplus (\overline{\operatorname{ran} D^*} \cap \mathcal{D}).$$

Because $\mathcal{D}$ and $\mathcal{D}^*$ are dense in $\mathcal{H}$, $\overline{\operatorname{ran} D} \cap \mathcal{D}^*$ must be dense in $\overline{\operatorname{ran} D}$ and $\overline{\operatorname{ran} D^*} \cap \mathcal{D}$ must be dense in $\overline{\operatorname{ran} D^*}$. So $\mathcal{D} \cap \mathcal{D}^*$ is also dense in $\mathcal{H}$.

To prove the self adjointness of D_{GB} it is enough to show that $\mathcal{D}(D_{GB}^*) \subset \mathcal{D}(D_{GB})$ since D_{GB} is symmetric.

Let $x \in \mathcal{D}(D_{GB}^*)$. We first show that $x \in \mathcal{D}$. Let $y \in \mathcal{D}^*$. Using the orthogonal decomposition $\mathcal{H} = \ker(D) \oplus \overline{\operatorname{ran}(D^*)}$ we can write $y = \bar{y} + \check{y}$ with $\bar{y} \in \ker(D)$ and $\check{y} \in \overline{\operatorname{ran}(D^*)}$. Note that we have $\check{y} \in \ker(D^*)$. We compute

$$\begin{aligned} \langle x, D^* y \rangle &= \langle x, D^* \bar{y} \rangle \\ &= \langle x, (D + D^*) \bar{y} \rangle \\ &= \langle x, D_{GB} \bar{y} \rangle \\ &= \langle D_{GB}^* x, \bar{y} \rangle . \end{aligned}$$

Writing $D_{GB}^* x = \bar{\xi} + \check{\xi}$ with $\bar{\xi} \in \ker(D)$ and $\check{\xi} \in \overline{\operatorname{ran}(D^*)}$ we see that

$$\langle D_{GB}^* x, \bar{y} \rangle = \langle \bar{\xi}, \bar{y} \rangle = \langle \bar{\xi}, y \rangle .$$

Thus we have $\langle x, D^* y \rangle = \langle \bar{\xi}, y \rangle$ which implies that $x \in \mathcal{D}(D^{**}) = \mathcal{D}$ since D is closed.

The same computation with the roles of D and D^* interchanged shows that $x \in \mathcal{D}^*$ and thus $x \in \mathcal{D} \cap \mathcal{D}^* = \mathcal{D}(D_{GB})$. $\square$

Clearly, the space of harmonic elements $\hat{H}(\mathcal{D}, D) = \ker D \cap \ker D^*$ is contained in the kernel of either operator, D_{GB} or Δ. In fact, equality holds.

Lemma 3.12. *Let $(\mathcal{D}, D)$ be a Hilbert complex. Then we have*

$$\hat{H}(\mathcal{D}, D) = \ker(D_{GB}) = \ker \Delta(\mathcal{D}, D).$$

Proof. According to the weak Hodge decomposition, the ranges of D and D^* are orthogonal. So $D_{GB}x = 0$ implies that $Dx = 0$ and $D^*x = 0$, hence we have $x \in \hat{H}(\mathcal{D}, D)$ which show the first equality.

For the second, observe that for $x \in \ker \Delta(\mathcal{D}, D)$ we have

$$0 = \langle x, \Delta(\mathcal{D}, D)x \rangle = \|D_{GB}x\|^2 .$$

Thus $x \in \ker(D_{GB})$. The other inclusion is trivial. □

Since the Laplacian is self adjoint, Lemma 2.5 provides a core given by

$$\mathcal{D}^\infty := \mathcal{D}^\infty\big(\Delta(\mathcal{D}, D)\big) = \bigcap_{k=1}^{\infty} \mathcal{D}\big(\Delta(\mathcal{D}, D)^k\big).$$

Obviously, $\mathcal{D}^\infty$ is contained in $\mathcal{D} \cap \mathcal{D}^*$ and our next goal will be to show that it is in fact a core for both D and D^*. This will take some work.

For the moment we write the weak Hodge decomposition as

$$\mathcal{H} = \hat{H} \oplus \mathcal{R} \oplus \mathcal{R}^* := \hat{H}(\mathcal{D}, D) \oplus \overline{\operatorname{ran}(D)} \oplus \overline{\operatorname{ran}(D^*)}.$$

Furthermore, we write the Laplace operator as

$$\Delta = DD^* + D^*D =: \Delta_1 + \Delta_2.$$

by von Neumann's theorem the operators Δ_i are self adjoint and we have

$$\ker \Delta_1 = \ker D^* = \hat{H} \oplus \mathcal{R}^* \qquad \text{and} \qquad \ker \Delta_2 = \ker D = \hat{H} \oplus \mathcal{R}.$$

By Corollary 2.3 we can decompose the domain of Δ_1 as

$$\mathcal{D}(\Delta_1) = \hat{H} \oplus \mathcal{R}^* \oplus (\mathcal{D}(\Delta_1) \cap \mathcal{R}),$$

and we observe that Δ_1 leaves $\mathcal{R}$ invariant, that is $\Delta_1\big(\mathcal{D}(\Delta_1) \cap \mathcal{R}\big) \subset \mathcal{R}$. We can thus define an operator on the Hilbert space $\mathcal{R}$ by

$$\tilde{\Delta}_1 := \Delta_1|_{\mathcal{D}(\Delta_1) \cap \mathcal{R}}.$$

Analogously, we get an operator $\tilde{\Delta}_2$ on $\mathcal{R}^*$.

Lemma 3.13. *The Laplace operator leaves the spaces $\hat{H}$, $\mathcal{R}$ and $\mathcal{R}^*$ invariant and can be decomposed as*

$$\Delta = \Delta|_{\hat{H}} \oplus \tilde{\Delta}_1 \oplus \tilde{\Delta}_2.$$

Proof. Using $\Delta = \Delta_1 + \Delta_2$ and the decompositions

$$\mathcal{D}(\Delta_1) = \hat{H} \oplus (\mathcal{D}(\Delta_1) \cap \mathcal{R}) \oplus \mathcal{R}^*$$
$$\mathcal{D}(\Delta_2) = \hat{H} \oplus \mathcal{R} \oplus (\mathcal{D}(\Delta_2) \cap \mathcal{R}^*)$$

we get

$$\begin{aligned}\mathcal{D}(\Delta) &= \mathcal{D}(\Delta_1) \cap \mathcal{D}(\Delta_2) \\ &= \hat{H} \oplus (\mathcal{D}(\Delta_1) \cap \mathcal{R}) \oplus (\mathcal{D}(\Delta_2) \cap \mathcal{R}^*) \\ &= \hat{H} \oplus \mathcal{D}(\tilde{\Delta}_1) \oplus \mathcal{D}(\tilde{\Delta}_2).\end{aligned}$$

It is easy to see that Δ acts as acclaimed on these spaces. Indeed, for $x \in \mathcal{D}(\tilde{\Delta}_1) = \mathcal{D}(\Delta) \cap \mathcal{R}$ we have

$$\Delta x = DD^* x = D^* D x \in \mathcal{R}$$

since $\mathcal{R} \subset \ker D$. Also note that $\Delta x = \tilde{\Delta}_1 x$. The same reasoning applies to $\mathcal{D}(\tilde{\Delta}_2)$ and $\hat{H} = \ker \Delta$ is trivially invariant under Δ. □

Corollary 3.14. *The domain $\mathcal{D}^\infty(\Delta)$ can be decomposed as*

$$\mathcal{D}^\infty(\Delta) = \hat{H} \oplus \mathcal{D}^\infty(\tilde{\Delta}_1) \oplus \mathcal{D}^\infty(\tilde{\Delta}_2).$$

We are now in a position to prove

Proposition 3.15. *$\mathcal{D}^\infty = \mathcal{D}^\infty(\Delta)$ is a core for D and D^*.*

Proof. The situation is symmetric in D and D^* so it is enough to prove the statement for D.

Let $x \in \mathcal{D}$ be orthogonal to $\mathcal{D}^\infty$ with respect to the graph norm of D, i.e. for any $y \in \mathcal{D}^\infty$ we have

$$\begin{aligned}0 &= \langle x, y \rangle + \langle Dx, Dy \rangle \\ &= \langle x, (I + D^* D) y \rangle \\ &= \langle x, (I + \Delta_2) y \rangle\end{aligned}$$

In order to conclude that $x = 0$ we have to show that $(I + \Delta_2)\mathcal{D}^\infty$ is a dense subspace of $\mathcal{H}$. Since $I + \Delta_2$ is invertible this will follow if we can show that $\mathcal{D}^\infty$ is a core for Δ_2.

From the above discussion we know that

$$\begin{aligned} \mathcal{D}^\infty(\Delta_2) &= \ker(\Delta_2) \oplus \mathcal{D}^\infty(\tilde{\Delta}_2) \\ &= \hat{H} \oplus \mathcal{R} \oplus \mathcal{D}^\infty(\tilde{\Delta}_2). \end{aligned}$$

Comparing this with Corollary 3.14 we only need to prove that $\mathcal{D}^\infty(\tilde{\Delta}_1)$ is dense in $\mathcal{R}$ with respect to graph norm of Δ_2. But this space is contained in the kernel of Δ_2 so that the graph norm is just the norm of $\mathcal{H}$ and the statement reduces to showing that $\mathcal{D}^\infty(\tilde{\Delta}_1)$ is dense in $\mathcal{R}$ which we know to be true. □

A further property of $\mathcal{D}^\infty$ is that for any $x \in \mathcal{D}^\infty$ we have

$$\begin{aligned} \Delta D x &= (DD^* + D^*D)Dx \\ &= DD^*Dx \\ &= D(D^*D + DD^*)x \\ &= D\Delta x. \end{aligned}$$

This implies that $\mathcal{D}^\infty$ is invariant under D, i.e. $D(\mathcal{D}^\infty) \subset \mathcal{D}^\infty$. Thus $(\mathcal{D}^\infty, D)$ is a pre-Hilbert complex.

Definition 3.16. The pre-Hilbert complex $(\mathcal{D}^\infty, D)$ is called the *smooth subcomplex* of $(\mathcal{D}, D)$.

Similarly we see that $\mathcal{D}^\infty$ is invariant under D^*. This shows that D with domain $\mathcal{D}^\infty$ is transposable (see Definition 2.9). As this property turns out to be very useful and also common in examples, it deserves to be singled out.

Definition 3.17. A pre-Hilbert complex $(\mathcal{D}, D)$ is called *transposable* if its differential is transposable. The differential graded Hilbert space $(\mathcal{D}, D^t)$ is called the *transposed complex* and is denoted by $(\mathcal{D}, D)^t$.

As it turns out, all examples of pre-Hilbert complexes that are of interest are transposable. This led Brüning and Lesch to include transposability into their definition of pre-Hilbert complexes (see [BL2], Definition 5.5). We chose our alternative definition because we find it natural that Hilbert complexes should also be pre-Hilbert complexes, and Hilbert complexes are in general not transposable.

The advantage of transposable pre-Hilbert complexes is that an analog of the Laplacian can be defined which does not seem to be the case for arbitrary pre-Hilbert complexes.

Definition 3.18. Let $(\tilde{\mathcal{D}}, \tilde{D})$ be a transposable pre-Hilbert complex. Then the symmetric, nonnegative operator

$$\Delta(\tilde{\mathcal{D}}, \tilde{D}) := (\tilde{D} + \tilde{D}^t)^2 = \tilde{D}\tilde{D}^t + \tilde{D}^t\tilde{D}$$

is called the *Laplacian* of $(\tilde{\mathcal{D}}, \tilde{D})$.

As a consequence of Lemma 2.11 and Proposition 2.12 we get

Proposition 3.19. *Let $(\tilde{\mathcal{D}}, \tilde{D})$ be a transposable pre-Hilbert complex. Then any ideal boundary condition of $(\tilde{\mathcal{D}}, \tilde{D})$ induces a self adjoint extension of $\Delta(\tilde{\mathcal{D}}, \tilde{D})$.*

If two ideal boundary conditions induce the same self adjoint extension, then they agree. In particular, if $\Delta(\tilde{\mathcal{D}}, \tilde{D})$ is essentially self adjoint, then $(\tilde{\mathcal{D}}, \tilde{D})$ has a unique ideal boundary condition.

3.2 Elliptic complexes

Let M be a Riemannian manifold and consider a complex of differential operators

$$0 \longrightarrow \Gamma_c(E_0) \xrightarrow{d_0} \Gamma_c(E_1) \xrightarrow{d_1} \dots \xrightarrow{d_{N-2}} \Gamma_c(E_{N-1}) \xrightarrow{d_{N-1}} \Gamma_c(E_N) \longrightarrow 0$$

where $E_0, \dots, E_N$ are Hermitian vector bundles over M. As in the previous section, we will take the direct sum $E := \bigoplus_{i=0}^N E_i$ and consider the differential operator $d := \bigoplus_{i=0}^N d_i$ as a graded operator of degree +1. We will denote such complexes by $\mathcal{E} := (\Gamma_c(E), d)$. We then have an orthogonal direct sum decomposition $\Gamma_c(E) = \oplus_{i=0}^N \Gamma_c(E_i)$ and $L^2(E)$ becomes a graded Hilbert space. Furthermore, we can consider d as an operator on $L^2(E)$ (see section 2.2) which is obviously graded (of degree +1). We thus obtain a pre-Hilbert complex $(L^2(E); \Gamma_c(E), d)$ which we continue to denote by $\mathcal{E}$.

Because of the existence of the formal adjoint d^t, these pre-Hilbert complexes are always transposable and we can make use of their transposed complexes $\mathcal{E}^t$ and Laplacians $\Delta(\mathcal{E})$.

The common examples of such complexes that arise in geometric situations, most notably the de Rham complex of smooth manifolds and the Dolbeault complex of complex manifolds, share an additional property.

Definition 3.20. A complex of differential operators $\mathcal{E} = (\Gamma_c(E), d)$ is called an *elliptic complex* if the *symbol sequence*

$$0 \longrightarrow \pi^* E_0 \xrightarrow{\sigma(d_0)} \pi^* E_1 \xrightarrow{\sigma(d_1)} \ldots \xrightarrow{\sigma(d_{N-2})} \pi^* E_{N-1} \xrightarrow{\sigma(d_{N-1})} \pi^* E_N \longrightarrow 0$$

is exact outside the zero section of T^*M.

Note that for a complex of length one, i.e. for a single differential operator, this is just the usual ellipticity condition. So elliptic complexes are generalizations of elliptic operators. There is yet another relationship with elliptic operators.

Proposition 3.21. *Let $\mathcal{E} = (\Gamma_c(E), d)$ be a complex of differential operators. Then $\mathcal{E}$ is an elliptic complex if and only if the associated Laplacian $\Delta(\mathcal{E})$ is elliptic. More precisely, the symbol sequence is exact at $\pi^* E_i$ if and only if $\Delta_i(\mathcal{E})$ is elliptic.*

Proof. This follows from what one could call finite dimensional Hodge theory. Consider a complex of finite dimensional Hermitian vector spaces

$$0 \longrightarrow V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \ldots \xrightarrow{f_{N-2}} V_{N-1} \xrightarrow{f_{N-1}} V_N \longrightarrow 0. \tag{3.5}$$

and let T_i be the operator on V_i defined by $T_i := f_{i-1} f_{i-1}^* + f_i^* f_i$. Then we have the following expression for the cohomology:

$$H^i(V_\bullet, f_\bullet) = \ker f_i / \operatorname{ran} f_{i-1} \cong \ker f_i \cap \ker f_{i-1}^* = \ker T_i.$$

So (3.5) is exact at V_i if and only if T_i is injective which, in the finite dimensional context, is equivalent to T_i being an isomorphism.

Now let $y \in T^*M \setminus \{0\}$. The proposition follows from the above discussion in the case of $V_i = (\pi^* E_i)_y$ and $f_i = \sigma(d_i)|_y$ and the fact that, in this situation, we have $T_i = \sigma(\Delta_i(\mathcal{E}))|_y$. □

The ellipticity condition has immediate and important consequences for ideal boundary conditions of elliptic complexes.

Corollary 3.22. *Let $\mathcal{E}$ be an elliptic complex and let $(\mathcal{D}, d)$ be an ideal boundary condition.*

a) Any harmonic element of $(\mathcal{D}, d)$ is a smooth section of E.

b) The pre-Hilbert complex $(\mathcal{D} \cap \Gamma(E), d)$ is a core complex for $(\mathcal{D}, d)$.

Proof. Let $\Delta := \Delta(\mathcal{D}, d)$. According to Lemma 3.12 we have $\hat{H}(\mathcal{D}, d) = \ker \Delta$. Elliptic regularity now implies that the right hand side consists solely of smooth sections. This proves a).

For part b) it is enough to show that $(\mathcal{D}, d)$ has a core complex that consists of smooth sections. Consider the smooth subcomplex $(\mathcal{D}^\infty, d)$ (see definition 3.16) where $\mathcal{D}^\infty = \bigcap_{n \geq 1} \mathcal{D}(\Delta^n)$. Then by elliptic regularity, any element of $\mathcal{D}^\infty$ is a smooth section, as we have seen in the discussion of Proposition 2.18. □

Part b) of the above proposition shows that in calculations within a given ideal boundary condition of an elliptic complex it is enough to work with smooth sections.

Although most of what will be said applies to arbitrary complexes of differential operators, we will focus on elliptic complexes from now on.

We introduce some concrete ideal boundary conditions of an elliptic complex $\mathcal{E} = (\Gamma_c(E), d)$. From the minimal and the maximal extensions of each d_i we get two ideal boundary conditions for $\mathcal{E}$.

Definition 3.23. We define the *relative/absolute ideal boundary condition* for an elliptic complex $\mathcal{E}$ as

$$\mathcal{E}^{\mathrm{rel/abs}} := (\mathcal{D}_{\mathrm{min/max}}(d), d_{\mathrm{min/max}}).$$

These are indeed Hilbert complexes since $\mathcal{E}^{\mathrm{rel}}$ is given by the closure of $\mathcal{E}$ and we have $\mathcal{E}^{\mathrm{abs}} = (\mathcal{E}^t)^*$. The origin of this terminology will be explain in Proposition 4.4 where we will also see that the relative and absolute ideal boundary conditions are different in general.

Definition 3.24. We say that an elliptic complex $\mathcal{E}$ has *unique ideal boundary conditions* if $\mathcal{E}^{\mathrm{rel}} = \mathcal{E}^{\mathrm{abs}}$.

The uniqueness of ideal boundary conditions will be our main focus later on. The basic criterion for uniqueness is provided by Proposition 3.19.

Proposition 3.25. *Let $\mathcal{E}$ be an elliptic complex. If $\Delta(\mathcal{E})$ is essentially self adjoint then $\mathcal{E}$ has unique ideal boundary conditions.*

As a consequence, we immediately see that an elliptic complex $\mathcal{E} = (\Gamma_c(E), d)$ over M has unique ideal boundary conditions if M is closed (Corollary 2.23) or if M is complete and $d{+}d^t$ is of Dirac type (Theorem 2.25). However, essential self adjointness of the Laplacian is a rare phenomenon in more general situations.

3.3 Products

Let $(\mathcal{H}';\mathcal{D}',D')$ and $(\mathcal{H}'';\mathcal{D}'',D'')$ be two pre-Hilbert complexes. We define another pre-Hilbert complex $(\mathcal{H};\mathcal{D},D)$ as follows. As Hilbert space we take the tensor product $\mathcal{H} := \mathcal{H}' \hat{\otimes} \mathcal{H}''$ which has a natural grading $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ where

$$\mathcal{H}_i = \bigoplus_{p+q=i} \mathcal{H}'_p \hat{\otimes} \mathcal{H}''_q.$$

Similarly, we take $\mathcal{D} := \mathcal{D}' \otimes \mathcal{D}''$ which is a dense subspace of $\mathcal{H}$ graded by

$$\mathcal{D}_i = \bigoplus_{p+q=i} \mathcal{D}'_p \otimes \mathcal{D}''_q.$$

Finally, we define the differential D on $\mathcal{D}$ as

$$D|_{\mathcal{D}'_p\otimes\mathcal{D}''_q} := D' \otimes \mathrm{id} + (-1)^p \,\mathrm{id} \otimes D''.$$

The sign ensures that we have $D^2 = 0$. In order to ease the notation we introduce an operator $\varepsilon\colon \mathcal{H}' \to \mathcal{H}'$ which is given by

$$\varepsilon|_{\mathcal{H}'_p} := (-1)^p \,\mathrm{id}\,.$$

Using this we can write

$$D = D' \otimes \mathrm{id} + \varepsilon \otimes D''.$$

Note that ε is a bounded self adjoint operator.

Definition 3.26. The pre-Hilbert complex $(\mathcal{H};\mathcal{D},D)$ defined above is called the *product* of $(\mathcal{H}';\mathcal{D}',D')$ and $(\mathcal{H}'';\mathcal{D}'',D'')$. It is denoted by $(\mathcal{H}';\mathcal{D}',D') \otimes (\mathcal{H}'';\mathcal{D}'',D'')$.

Note that this construction is just the standard tensor product of chain complexes if we forget about the surrounding Hilbert spaces.

Since it is now clear which Hilbert space surrounds the product of two pre-Hilbert complexes, we will omit them in our notation as before.

Lemma 3.27. *Let $(\mathcal{D}',D')$ and $(\mathcal{D}'',D'')$ be transposable pre-Hilbert complexes. Then their product $(\mathcal{D}',D') \otimes (\mathcal{D}'',D'')$ is transposable with*

$$\big((\mathcal{D}',D') \otimes (\mathcal{D}'',D'')\big)^t = (\mathcal{D}',D')^t \otimes (\mathcal{D}'',D'')^t. \tag{3.6}$$

Furthermore, we have

$$\Delta\big((\mathcal{D}',D') \otimes (\mathcal{D}'',D'')\big) = \Delta(\mathcal{D}',D') \otimes \mathrm{id} + \mathrm{id} \otimes \Delta(\mathcal{D}'',D''). \tag{3.7}$$

Proof. Let $s \otimes t, u \otimes v \in \mathcal{D}' \otimes \mathcal{D}''$. Then we have

$$\begin{aligned}
&\left\langle (D' \otimes \mathrm{id} + \varepsilon \otimes D'') s \otimes t, u \otimes v \right\rangle \\
&= \left\langle (D' s) \otimes t, u \otimes v \right\rangle + \left\langle (\varepsilon s) \otimes (D'' t), u \otimes v \right\rangle \\
&= \left\langle D' s, u \right\rangle \langle t, v \rangle + \langle \varepsilon s, u \rangle \left\langle D'' t, v \right\rangle \\
&= \left\langle s, D'^t u \right\rangle \langle t, v \rangle + \langle s, \varepsilon u \rangle \left\langle t, D''^t v \right\rangle \\
&= \left\langle s \otimes t, (D'^t u) \otimes v \right\rangle + \left\langle s \otimes t, (\varepsilon u) \otimes (D''^t v) \right\rangle \\
&= \left\langle s \otimes t, (D'^t \otimes \mathrm{id} + \varepsilon \otimes D''^t) u \otimes v \right\rangle .
\end{aligned}$$

This proves (3.6) and (3.7) follows from a straight forward computation. □

Considering two Hilbert complexes as pre-Hilbert complexes, their product will only be a pre-Hilbert complex which could a priori have many different ideal boundary conditions. However, this is not the case.

Lemma 3.28. *Let $(\mathcal{D}', D')$ and $(\mathcal{D}'', D'')$ be two Hilbert complexes. Then the pre-Hilbert complex $(\mathcal{D}', D') \otimes (\mathcal{D}'', D'')$ has a unique ideal boundary condition which is given by the closure of the differential.*

Proof. Consider the smooth subcomplexes $(\mathcal{D}'^{\infty}, D')$ and $(\mathcal{D}''^{\infty}, D'')$ and let $(\tilde{\mathcal{D}}, \tilde{D}) := (\mathcal{D}'^{\infty}, D') \otimes (\mathcal{D}''^{\infty}, D'')$. Since the smooth subcomplexes are transposable, so is $(\tilde{\mathcal{D}}, \tilde{D})$ by Lemma 3.27. Furthermore, $(\tilde{\mathcal{D}}, \tilde{D})$ is a subcomplex of $(\mathcal{D}, D) := (\mathcal{D}', D') \otimes (\mathcal{D}'', D'')$. Hence, if $(\tilde{\mathcal{D}}, \tilde{D})$ has a unique ideal boundary condition, then the same holds for $(\mathcal{D}, D)$.

By (3.7) we have

$$\Delta(\tilde{\mathcal{D}}, \tilde{D}) = \Delta(\mathcal{D}'^{\infty}, D') \otimes \mathrm{id} + \mathrm{id} \otimes \Delta(\mathcal{D}''^{\infty}, D'').$$

Since the Laplacians of the smooth subcomplexes are essentially self adjoint, Theorem 2.13 implies that $\Delta(\tilde{\mathcal{D}}, \tilde{D})$ is essentially self adjoint, so that $(\tilde{\mathcal{D}}, \tilde{D})$ has a unique ideal boundary condition by Proposition 3.19. □

Definition 3.29. We define the *product* of two Hilbert complexes $(\mathcal{D}', D')$ and $(\mathcal{D}'', D'')$ to be the unique ideal boundary condition of the pre-Hilbert complex $(\mathcal{D}', D') \otimes (\mathcal{D}'', D'')$ and denote it by $(\mathcal{D}', D') \hat{\otimes} (\mathcal{D}'', D'')$.

Proposition 3.30. *Let $(\mathcal{D}', D')$ and $(\mathcal{D}'', D'')$ be two Hilbert complexes. Then we have*

$$\left((\mathcal{D}', D') \hat{\otimes} (\mathcal{D}'', D'') \right)^* = (\mathcal{D}', D')^* \hat{\otimes} (\mathcal{D}'', D'')^*. \tag{3.8}$$

Proof. Consider the pre-Hilbert complex $(\mathcal{D}'^{\infty}, D'^{*}) \otimes (\mathcal{D}''^{\infty}, D''^{*})$. This is a transposable subcomplex of both sides of (3.8) with essentially self adjoint Laplacian. The same reasoning as in the preceding proof (of Lemma 3.28) yields uniqueness of ideal boundary conditions. Since both sides of (3.8) provide such ideal boundary conditions, they must agree. □

We will now develop the theory of products for elliptic complexes. Let M and N be two Riemannian manifolds and $E \to M$ and $F \to N$ Hermitian vector bundles. We will usually omit the various metrics in our notation. Then the *exterior tensor product* of E and F is defined as the vector bundle

$$E \boxtimes F := pr_M^* E \otimes pr_N^* F \to M \times N.$$

The fiber over $(x, y) \in M \times N$ is just $E_x \otimes F_y$ and it is easily checked that for $s, u \in E_x$ and $t, v \in F_y$ the expression

$$h^{E \boxtimes F}(s \otimes t, u \otimes v) := h^E(s, u) h^F(t, v)$$

defines a Hermitian metric on $E \boxtimes F$.

Since we have $T(M \times N) = TM \boxtimes TN$, we can apply the above construction to obtain a Riemannian metric on $M \times N$.

Definition 3.31. Let (M, g^M) and (N, g^N) be two Riemannian manifolds. Then the Riemannian metric on $M \times N$ given by

$$g^M \times g^N := \mathrm{pr}_M^* g^M + \mathrm{pr}_N^* g^N$$

is called the *product metric* of g^M and g^N.

Unless stated otherwise all products of manifolds will be equipped with the corresponding product metrics.

Our first task is to relate the sections of $E \boxtimes F$ to those of E and F. As before we denote the space of *all* sections of E by

$$\tilde{\Gamma}(E) := \{ s : M \to E \mid s(x) =: s_x \in E_x \}.$$

For any $s \in \tilde{\Gamma}(E)$ and $t \in \tilde{\Gamma}(F)$ we can define a section $s \otimes t \in \tilde{\Gamma}(E \boxtimes F)$ in the obvious way by letting

$$(s \otimes t)_{(x,y)} := s_x \otimes t_y.$$

This association is clearly injective and we will use it to identify $\tilde{\Gamma}(E) \otimes \tilde{\Gamma}(F)$ with its image in $\tilde{\Gamma}(E \boxtimes F)$.

Lemma 3.32. *Let $E \to M$ and $F \to N$ be Hermitian vector bundles.*

a) $\Gamma_c(E) \otimes \Gamma_c(F)$ *is sequentially dense in* $\Gamma_c(E \boxtimes F)$ *with respect to the natural LF-topology.*

b) $L^2(E) \otimes L^2(F)$ *is dense in* $L^2(E \boxtimes F)$.

As a direct consequence we get

Corollary 3.33. $L^2(E) \hat{\otimes} L^2(F) \cong L^2(E \boxtimes F)$.

Proof. We will use that any section $s \in \Gamma_c(E \boxtimes F)$ can be written as a finite sum

$$s(x,y) = \sum_i \varphi_i(x,y)\, u_i(x) \otimes v_i(y) \tag{3.9}$$

where $u_i \in \Gamma_c(E)$, $v_i \in \Gamma_c(F)$ and $\varphi_i \in C_c^\infty(M \times N)$. Such a representation can be obtained by choosing bundle atlases for E and F and using a partition of unity.

Obviously, the set $\Gamma_c(E) \otimes \Gamma_c(F)$ is contained in $\Gamma_c(E \boxtimes F)$. In the light of (3.9), for part a) it suffices to show that $C_c^\infty(M) \otimes C_c^\infty(N)$ is sequentially dense in $C_c^\infty(M \times N)$. This is easily reduced to the case where M and N are open subsets of Euclidean space, in which the result is well known (see [T], Theorem 39.2).

For part b) we first show that the scalar products in $L^2(E) \otimes L^2(F)$ and $L^2(E \boxtimes F)$ agree on $\Gamma_c^\infty(E) \otimes \Gamma_c^\infty(F)$. Let $s, u \in \Gamma_c^\infty(E)$ and $t, v \in \Gamma_c^\infty(F)$. To shorten the notation we write $s(x) =: s_x$, etc. Then we have

$$\begin{aligned}\langle s \otimes t, u \otimes v \rangle_{L^2(E \boxtimes F)} &= \int_{M \times N} h^{E \boxtimes F}(s_x \otimes t_y, u_x \otimes v_y)\, d\mu_{M \times N}(x,y) \\ &= \int_{M \times N} h^E(s_x, u_x) h^F(t_y, v_y)\, d\mu_{M \times N}(x,y).\end{aligned}$$

Since $M \times N$ is equipped with the product metric, it is easy to see that the measure $\mu_{M \times N}$ is the product measure of μ_M and μ_N. Hence, we can apply Fubini's theorem to get

$$\begin{aligned}\langle s \otimes t, u \otimes v \rangle_{L^2(E \boxtimes F)} &= \int_{M \times N} h^E(s_x, u_x) h^F(t_y, v_y)\, d\mu_{M \times N}(x,y) \\ &= \left(\int_M h^E(s_x, u_x)\, d\mu_M(x) \right) \left(\int_N h^F(t_y, v_y)\, d\mu_N(y) \right) \\ &= \langle s, u \rangle_{L^2(E)} \langle t, v \rangle_{L^2(F)}.\end{aligned}$$

This shows that $L^2(E) \otimes L^2(F)$ injects isometrically into $L^2(E \boxtimes F)$ and it remains to show that its image is dense.

Let $\{e_i\}$ and $\{f_j\}$ be orthonormal bases for $L^2(E)$ and $L^2(F)$ respectively, both consisting of smooth sections with compact supports. Then $\{e_i \otimes f_j\}$ is clearly an orthonormal set in $L^2(E \boxtimes F)$. To show that it is an orthonormal basis let $s \in \Gamma_c(E \boxtimes F)$ be orthogonal to $e_i \otimes f_j$ for all i, j. We will show that, given any fixed $(x_0, y_0) \in M \times N$, we must have $s(x_0, y_0) = 0$. Hence, s must vanish identically.

Using a bundle atlas of F such that a neighborhood of $y_0 \in N$ is contained in only one single chart, we can obtain a representation

$$s(x,y) = \sum_k \varphi_k(x,y)\, u_k(x) \otimes v_k(y)$$

as in (3.9) such that those v_k with $\varphi_k(x_0,y_0)v_k(y_0) \neq 0$ are linearly independent in a neighborhood of y_0. Using Fubini's theorem again we get

$$\begin{aligned}
0 &= \langle s, e_i \otimes f_j \rangle_{L^2(E \boxtimes F)} \\
&= \int_{M \times N} \sum_k \varphi_k\, h^E(u_k, e_i) h^F(v_k, f_j)\, d\mu_{M\times N} \\
&= \int_N h^F\Big(\sum_k \Big[\int_M \varphi_k\, h^E(u_k, e_i)\, d\mu_M \Big] v_k, f_j \Big)\, d\mu_N \\
&= \Big\langle \sum_k \langle \varphi_k u_k, e_i \rangle_{L^2(E)}\, v_k, f_j \Big\rangle_{L^2(F)}.
\end{aligned}$$

Since $\{f_j\}$ is an orthonormal basis of $L^2(F)$ this implies that

$$0 = \sum_k \langle \varphi_k u_k, e_i \rangle_{L^2(E)}\, v_k \in \Gamma_c(F).$$

By assumption the v_k are linearly independent at y_0 and we can conclude that

$$\langle \varphi_k(\cdot, y_0) u_k, e_i \rangle_{L^2(E)} = 0 \quad \text{for all } k.$$

Using that $\{e_i\}$ is an orthonormal basis of $L^2(E)$ we get, in particular, that $\varphi_k(x_0,y_0)u_k(x_0) = 0$ for all k and thus $s(x_0,y_0) = 0$ as desired. $\square$

Now let $\mathcal{E} = (\Gamma_c(E), d^E)$ and $\mathcal{F} = (\Gamma_c(F), d^F)$ be two elliptic complexes over M and N respectively. When we consider them as pre-Hilbert complexes we can form the product $\mathcal{E} \otimes \mathcal{F}$. Because of Lemma 3.33 this pre-Hilbert complex will be surrounded by $L^2(E \boxtimes F)$.

We recall the construction of $\mathcal{E} \otimes \mathcal{F}$. On the set

$$\mathcal{D} = \bigoplus_{p,q} \Gamma_c(E_p) \otimes \Gamma_c(F_q) \subset \Gamma_c(E \boxtimes F)$$

consider the operator

$$\tilde{d}^{E\boxtimes F} := d^E \otimes \mathrm{id} + \varepsilon \otimes d^F.$$

Lemma 3.34. *The operator $\tilde{d}^{E\boxtimes F}$ determines a unique differential operator $d^{E\boxtimes F}$ on $E \boxtimes F$. The minimal extension of $d^{E\boxtimes F}$ coincides with the closure of $\tilde{d}^{E\boxtimes F}$ in $L^2(E \boxtimes F)$. Moreover, $(\Gamma_c(E \boxtimes F), d^{E\boxtimes F})$ becomes an elliptic complex.*

Proof. The differential operator $d^{E\boxtimes F}$ is constructed as follows. Let $s \in \Gamma_c(E \boxtimes F)$. By Lemma 3.32a) we can find a sequences $u_n \in \Gamma_c(E)$ and $v_n \in \Gamma_c(F)$ such that $u_n \otimes v_n$ converges to s in $\Gamma_c(E \boxtimes F)$. This means the following:

- For all n we have $\mathrm{supp}(u_n \otimes v_n) \subset \mathrm{supp}(s)$.
- $u_n \otimes v_n$ converges uniformly to s and the same holds for all (covariant) derivatives.

It can be shown that the sequence $\tilde{d}^{E\boxtimes F}(u_n \otimes v_n)$ converges in $\Gamma_c(E \boxtimes F)$ and we can define

$$d^{E\boxtimes F} s := \lim_{n\to\infty} \tilde{d}^{E\boxtimes F}(u_n \otimes v_n).$$

Clearly, the thus defined operator does not increase the support of a section, i.e. $\mathrm{supp}(d^{E\boxtimes F} s) \subset \mathrm{supp}(s)$, hence it must be a differential operator by a well known theorem of Peetre.

The statement about ellipticity as well as an alternative proof of the existence of $d^{E\boxtimes F}$ can be found in [P] (section IV.8). $\square$

For the moment, we let

$$\mathcal{E} \otimes_{\mathrm{ell}} \mathcal{F} := (\Gamma_c(E \boxtimes F), d^{E\boxtimes F}).$$

Note that the pre-Hilbert complexes $\mathcal{E}\otimes\mathcal{F}$ and $\mathcal{E}\otimes_{\mathrm{ell}}\mathcal{F}$ are different. However, we claim that they have the same closure.

Lemma 3.35. *The closures of $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{E} \otimes_{\mathrm{ell}} \mathcal{F}$ agree and are both equal to $\mathcal{E}^{\mathrm{rel}} \hat{\otimes} \mathcal{F}^{\mathrm{rel}}$.*

Proof. We denote the differential of $\mathcal{E}\otimes\mathcal{F}$ and $\mathcal{E}\otimes_{\mathrm{ell}}\mathcal{F}$ by $\tilde{d}$ and d respectively. Clearly, the closure of $\tilde{d}$ is contained in that of d. Conversely, let $s \in \Gamma_c(E \boxtimes F)$. By definition we have

$$ds = \lim \tilde{d}(u_n \otimes v_n)$$

with u_n and v_n as in the proof of Lemma 3.34 and the convergence being uniform. Since the supports of $u_n \otimes v_n$ are contained in the fixed compact set $\operatorname{supp}(s)$, we also have convergence in $L^2(E \boxtimes F)$. But this just means that d is contained in the closure of $\tilde{d}$. Hence, the closures of $\tilde{d}$ and d agree.

Moreover, it is clear that the closure of $\mathcal{E} \otimes_{\mathrm{ell}} \mathcal{F}$ determines an ideal boundary condition for the pre-Hilbert complex $\mathcal{E}^{\mathrm{rel}} \otimes \mathcal{F}^{\mathrm{rel}}$. But there is only one such ideal boundary condition, namely $\mathcal{E}^{\mathrm{rel}} \hat{\otimes} \mathcal{F}^{\mathrm{rel}}$. □

Clearly, the differential of $(\mathcal{E} \otimes_{\mathrm{ell}} \mathcal{F})^t$ is induced by the differential of $\mathcal{E}^t \otimes \mathcal{F}^t$. Because of this and Lemma 3.35 no harm is done by identifying $\mathcal{E} \otimes_{\mathrm{ell}} \mathcal{F}$ and $\mathcal{E} \otimes \mathcal{F}$.

The next observation well be crucial for our later applications.

Proposition 3.36. *Let $\mathcal{E}$ and $\mathcal{F}$ be elliptic complexes. Then we have*

$$(\mathcal{E} \otimes \mathcal{F})^{\mathrm{rel}} = \mathcal{E}^{\mathrm{rel}} \hat{\otimes} \mathcal{F}^{\mathrm{rel}} \qquad \textit{and} \qquad (\mathcal{E} \otimes \mathcal{F})^{\mathrm{abs}} = \mathcal{E}^{\mathrm{abs}} \hat{\otimes} \mathcal{F}^{\mathrm{abs}}.$$

Proof. The part about the relative ideal boundary condition is just a restatement of Lemma 3.35. For the absolute ideal boundary condition we have

$$\begin{aligned}(\mathcal{E} \otimes \mathcal{F})^{\mathrm{abs}} &= (\mathcal{E} \otimes \mathcal{F})^{t*} \\ &= (\mathcal{E}^t \otimes \mathcal{F}^t)^* \\ &= \mathcal{E}^{t*} \hat{\otimes} \mathcal{F}^{t*} \\ &= \mathcal{E}^{\mathrm{abs}} \hat{\otimes} \mathcal{F}^{\mathrm{abs}}.\end{aligned}$$

□

Corollary 3.37. *If $\mathcal{E}$ and $\mathcal{F}$ have unique ideal boundary conditions, then so does $\mathcal{E} \otimes \mathcal{F}$.*

3.4 Some remarks on cohomology and Hodge theory

The reader may have noticed that although we have been occupied with complexes we have not mentioned their cohomology. We use this section to give a brief review of the theory.

Definition 3.38. Let $(\mathcal{D}, D)$ be a Hilbert complex. Its cohomology is defined as the graded quotient $H^*(\mathcal{D}, D) := \ker D / \operatorname{ran} D$.

In Definition 3.8 we defined the space of harmonic elements

$$\hat{H}(\mathcal{D}, D) = \ker D \cap \ker D^*.$$

and we proved that

$$\ker D = \hat{H}^*(\mathcal{D}, D) \oplus \overline{\operatorname{ran} D}. \tag{3.10}$$

The following is immediate.

Lemma 3.39. *Let $(\mathcal{D}, D)$ be a Hilbert complex. Then we have*

$$H^*(\mathcal{D}, D) \cong \hat{H}^*(\mathcal{D}, D) \tag{3.11}$$

if and only if D has closed range.

In the case of (3.11) we say that the *Hodge theorem* holds for $(\mathcal{D}, D)$.

By (3.10) we can write the cohomology of $(\mathcal{D}, D)$ as

$$H^*(\mathcal{D}, D) = \hat{H}^*(\mathcal{D}, D) \oplus \overline{\operatorname{ran} D} / \operatorname{ran} D.$$

For purely functional analytic reasons the vector space $\overline{\operatorname{ran} D}/\operatorname{ran} D$ is either zero or infinite dimensional. Indeed, we can view D as a bounded operator between Hilbert spaces

$$D\colon (\mathcal{D}, \langle \cdot, \cdot \rangle_D) \to (\overline{\operatorname{ran} D}, \langle \cdot, \cdot \rangle)$$

where $\langle \cdot, \cdot \rangle_D$ is the graph scalar product of D. The claim now follows from the well known fact that if the range of a bounded operator between Banach spaces has finite codimension, then it must be closed (see [AA], Corollary 2.17 for example). As a consequence, we get

Corollary 3.40. *Let $(\mathcal{D}, D)$ be a Hilbert complex. If $H^*(\mathcal{D}, D)$ is finite dimensional, then the Hodge theorem holds.*

Indeed, a little more can be said in this situation.

Theorem 3.41. *Let $(\mathcal{D}, D)$ be a Hilbert complex. $H^*(\mathcal{D}, D)$ is finite dimensional if and only if D is a Fredholm operator.*

Proof. Theorem 2.4 in [BL1]. □

4 The L^2 Stokes Theorem

In this chapter apply the abstract results of the previous sections to the study of smooth manifolds. We consider the de Rham complex with compact supports of a Riemannian manifold without boundary (M, g)

$$0 \to \Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c(M)^n \to 0.$$

which we abbreviate by $\Omega_M := (\Omega_c^*(M), d)$. Note that, in order to obtain Hermitian vector bundles, we use differential forms with complex coefficients, that is

$$\Omega^*(M) := \Gamma(\Lambda^* T^* M \otimes \mathbb{C}).$$

It is well known that Ω_M is an elliptic complex as defined in Section 3.2. The Riemannian metric induces a bundle metric on $\Lambda^* T^* M$ which extends to a Hermitian metric on $\Lambda^* T^* M \otimes \mathbb{C}$ by letting

$$g(\lambda \otimes \omega, \mu \otimes \eta) := \overline{\lambda}\mu \, g(\omega, \eta)$$

and thus we can define the L^2 scalar product of $\omega, \eta \in \Omega_c^k(M)$ by

$$\langle \omega, \eta \rangle = \int_M g(\omega, \eta) \, d\mu_g.$$

For different values of k the spaces $\Omega_c^k(M)$ are by definition orthogonal. Let $L^2\Omega^*(M, g)$ denote the completion of $\Omega_c^*(M)$ with respect to the L^2 scalar product. The elements of $L^2\Omega^*(M, g)$ are called L^2 *forms* on M. If there is no ambiguity about the Riemannian metric, we just write $L^2\Omega^*(M)$.

We denote the formal adjoint of the exterior derivative by

$$\delta := d^t.$$

Recall that we have the following relations between the minimal and maximal extensions of d and δ.

$$\begin{aligned} d_{\min} &= (\delta_{\max})^* & (d_{\min})^* &= \delta_{\max} (= d^*) \\ d_{\max} &= (\delta_{\min})^* (= \delta^*) & (d_{\max})^* &= \delta_{\min} \end{aligned}$$

Since Ω_M is an elliptic complex, any ideal boundary condition will have a core of smooth sections (Corollary 3.22). This will allow us to assume that forms are smooth in computations.

4.1 The L^2 Stokes Theorem and L^2 cohomology

As before, let (M,g) be a Riemannian manifold without boundary and Ω_M its de Rham complex with compact supports. We call the relative (absolute) ideal boundary condition of Ω_M the *relative (absolute) de Rham complex* of (M,g).

It will follow from Proposition 4.4 below that Ω_M^{rel} and Ω_M^{abs} are different in general, but we have encountered situations in which they agree. For instance, this is true if M is closed or complete. The case of complete manifolds was first treated by Gaffney in [G] and this article is probably the reason for the following definition.

Definition 4.1. We say that the L^2 *Stokes Theorem* holds on (M,g) if we have $\Omega_M^{\mathrm{rel}} = \Omega_M^{\mathrm{abs}}$ or, equivalently, if we have

$$\langle d_{\max}\omega, \eta\rangle = \langle \omega, \delta_{\max}\eta\rangle \tag{4.1}$$

for all $\omega \in \mathcal{D}_{\max}(d)$ and $\eta \in \mathcal{D}_{\max}(\delta)$.

The L^2 Stokes Theorem plays a role in an analog of Hodge theory for non-compact manifolds.

Definition 4.2. Let (M,g) be a Riemannian manifold. The cohomology of the absolute de Rham complex

$$H^*_{(2)}(M,g) := H^*(\Omega_M^{\mathrm{abs}}) = \ker d_{\max} / \operatorname{ran} d_{\max}$$

is called the L^2 *cohomology* of (M,g).

It turns out that the L^2 cohomology can be computed using smooth forms only.

Proposition 4.3. *The L^2 cohomology of (M,g) is isomorphic to the cohomology of the complex $\mathcal{E} := (\mathcal{D}_{\max}(d) \cap \Omega^*(M), d)$, i.e.*

$$H^*_{(2)}(M,g) \cong \frac{\{\omega \in \Omega^*(M) \mid d\omega = 0, \omega \in L^2\Omega^*(M)\}}{\{d\omega \mid \omega, d\omega \in L^2\Omega^*(M) \cap \Omega^*(M)\}}.$$

Proof. See Theorem 3.5 in [BL1] for a proof in the case of arbitrary ideal boundary conditions of arbitrary elliptic complexes. □

Hence, for closed manifolds L^2 cohomology is isomorphic to the usual de Rham cohomology and by de Rham's theorem we have

$$H^*_{(2)}(M,g) \cong H^*(M;\mathbb{C}).$$

In particular, L^2 cohomology is independent of the Riemannian metric on M. More interesting is the next result for compact manifolds with boundary which gives an interpretation of the cohomology of the absolute and the relative de Rham complex.

Proposition 4.4. *Let Y be a compact manifold with boundary and consider its interior $M := Y \setminus \partial Y$. Then for any Riemannian metric g on Y we have isomorphisms*

$$H^*(\Omega_M^{\mathrm{abs}}) = H^*_{(2)}(M, g|_M) \cong H^*(Y; \mathbb{C})$$

and

$$H^*(\Omega_M^{\mathrm{rel}}) \cong H^*(Y, \partial Y; \mathbb{C}).$$

Proof. See [BL1], Theorem 4.1. □

As mentioned before, this result is the origin of the names relative and absolute ideal boundary condition. It is important to realize that L^2 cohomology is not a topological invariant of $M = Y \setminus \partial Y$ in this case. It is an invariant of a special class of (incomplete) Riemannian metrics on M which have the property that the corresponding metric closure of M can be identified with Y. The dependence on the metric will be further investigated in the next section.

We are looking for a generalization of Hodge theory to the context of L^2 cohomology. The most intuitive candidate for the space of *L^2 harmonic forms* is

$$\hat{H}^*_{(2)}(M, g) := \left\{\omega \in \Omega^*(M) \cap L^2\Omega^*(M) \mid d\omega = 0, \delta\omega = 0\right\},$$

which can also be expressed in terms of the closed extensions of d and δ.

Lemma 4.5. $\hat{H}^*_{(2)}(M, g) = \ker(d_{\max}) \cap \ker(\delta_{\max})$

Proof. We only have to show that the right hand side consists of smooth forms. Consider the elliptic differential operator $d + \delta$. Obviously, we have

$$\ker(d_{\max}) \cap \ker(\delta_{\max}) \subset \ker(d + \delta)_{\max}.$$

By elliptic regularity, the right hand side consists of smooth forms. □

However, there is another space of harmonic forms, namely the space of harmonic elements of the absolute de Rham complex

$$\hat{H}^*(\Omega_M^{\mathrm{abs}}) = \ker d_{\max} \cap \ker d^*_{\max} = \ker d_{\max} \cap \ker \delta_{\min}. \tag{4.2}$$

Clearly, we have $\hat{H}^*(\Omega_M^{\text{abs}}) \subset \hat{H}^*(M,g)$ by Lemma 4.5. The advantage of $\hat{H}^*(\Omega_M^{\text{abs}})$ is that it is directly linked to L^2 cohomology. From Section 3.4 we know that

$$H^*_{(2)}(M,g) \cong \hat{H}^*(\Omega_M^{\text{abs}}) \oplus \overline{\operatorname{ran} d_{\max}} / \operatorname{ran} d_{\max} \tag{4.3}$$

and so we get

Proposition 4.6. *If $d_{\max}$ has closed range, for example if $H^*_{(2)}(M,g)$ is finite dimensional, then $H^*_{(2)}(M,g) \cong \hat{H}^*(\Omega_M^{\text{abs}})$.*

Since $\overline{\operatorname{ran} d_{\max}} / \operatorname{ran} d_{\max}$ is infinite dimensional if $d_{\max}$ does not have closed range one sometimes defines the *reduced L^2 cohomology*

$$\tilde{H}^*_{(2)}(M,g) := \ker d_{\max} / \overline{\operatorname{ran} d_{\max}} \cong \hat{H}^*(\Omega_M^{\text{abs}})$$

even though this is not a cohomology theory, strictly speaking.

Let us come back to the L^2 Stokes Theorem and its consequences. We start with the most obvious one.

Lemma 4.7. *If the L^2 Stokes Theorem holds on (M,g), then $\hat{H}^*_{(2)}(M,g) = \hat{H}^*(\Omega_M^{\text{abs}})$.*

Proof. This follows directly from comparing Lemma 4.5 with (4.2). □

So if the L^2 Stokes Theorem holds, there is essentially only one notion of harmonic forms.

In any case, we have the obvious map

$$\kappa : \hat{H}^*_{(2)}(M,g) \to H^*_{(2)}(M,g)$$

sending a closed, square integrable form ω to its L^2 cohomology class. In general, this map will neither be surjective nor injective. We have seen that the surjectivity of κ is linked to $d_{\max}$ having closed range. Similarly, the injectivity of κ is related to the L^2 Stokes Theorem.

Proposition 4.8. *If the L^2 Stokes Theorem holds on (M,g), then κ is injective. In other words, each L^2 harmonic form represents a unique L^2 cohomology class.*

Proof. Let $\omega, \eta \in \hat{H}^*(M,g)$. If ω and η represent the same cohomology class, then there exists $\rho \in \mathcal{D}_{\max}(d)$ such that $\omega - \eta = d_{\max}\rho$, and for the L^2 norm of $\omega - \eta$ we have

$$\|\omega - \eta\|^2 = \langle \omega - \eta, d_{\max}\rho \rangle \,.$$

A priori, we only know that $\omega - \eta \in \ker \delta_{\max} = \ker {d_{\min}}^*$. But the L^2 Stokes Theorem implies that ${d_{\min}}^* = {d_{\max}}^* = \delta_{\min}$. In particular, we have $\omega - \eta \in \ker \delta_{\min}$ and thus obtain

$$\|\omega - \eta\|^2 = \langle {d_{\max}}^*(\omega - \eta), \rho \rangle = \langle \delta_{\min}(\omega - \eta), \rho \rangle = 0.$$

Thus we have $\omega = \eta$ which shows that κ is injective. □

Another consequence of the L^2 Stokes Theorem is an analog of Poincarï¿½ duality for L^2 cohomology on the level of harmonic forms. Let (M,g) be an n-dimensional oriented Riemannian manifold. Then we have the *Hodge operator*

$$*\colon \Lambda^k T^*M \to \Lambda^{n-k} T^*M.$$

which has the property that

$$*^2 = (-1)^{k(n-k)}. \tag{4.4}$$

In particular, $*$ is an isomorphism. Indeed, it is easy to see that it is an isometry with respect to bundle metrics induces by g.

The complex linear extension of the Hodge operator induces a linear map

$$*\colon \Omega^k(M) \to \Omega^{n-k}(M)$$

which we also call the Hodge operator and denote by the same symbol.

The Hodge operator yields the useful formula

$$\langle \omega, \eta \rangle = \int_M \overline{\omega} \wedge *\eta$$

for the L^2 scalar product on $\Omega_c^*(M)$. Together with the usual Stokes' theorem this implies that for $\omega \in \Omega_c^k(M)$ we have

$$\delta\omega = (-1)^{n(k+1)+1} * d * \omega. \tag{4.5}$$

The signs in (4.4) and (4.5) are often confusing in calculations and we will ignore them whenever the exact knowledge of the sign is not important. In this case we will write $\pm$.

Clearly, the Hodge operator acts as an isometry on $\Omega_c^*(M)$ with respect to the L^2 scalar product and thus extends to an isometry

$$*: L^2\Omega^*(M) \to L^2\Omega^*(M).$$

Now let $\omega \in L^2\Omega^{k-1}(M) \cap \mathcal{D}_{\max}(d)$. Using $d_{\max} = \delta^*$ we see that for any $\eta \in \Omega_c^{n-k}(M)$ we have

$$\langle d_{\max}\omega, *\eta\rangle = \langle \omega, \delta * \eta\rangle = \langle *\omega, *\delta * \eta\rangle\,.$$

By (4.4) and (4.5) we have

$$\langle d_{\max}\omega, *\eta\rangle = \pm\langle *\omega, d\eta\rangle \tag{4.6}$$

On the other hand, we have

$$\langle d_{\max}\omega, *\eta\rangle = \pm\langle *d_{\max}\omega, \eta\rangle = \pm\langle *d_{\max} * *\omega, \eta\rangle\,. \tag{4.7}$$

From (4.6) and (4.7) we see that

$$\langle *\omega, d\eta\rangle = \langle \pm * d_{\max} * *\omega, \eta\rangle\,.$$

But this shows that $*\omega \in \mathcal{D}(d^*) = \mathcal{D}_{\max}(\delta)$ and

$$\delta_{\max} * \omega = \pm * d_{\max} * *\omega.$$

If one works out the sign correctly, one sees that it agrees with the one in (4.5).

Proposition 4.9. *Let (M, g) be an oriented Riemannian manifold. Then we have*

$$*\mathcal{D}_{\min/\max}(d) = \mathcal{D}_{\min/\max}(\delta) \qquad \text{and} \qquad *\mathcal{D}_{\min/\max}(\delta) = \mathcal{D}_{\min/\max}(d).$$

Furthermore, (4.5) *also holds for the relative and absolute ideal boundary conditions, i.e.*

$$\delta_{\min/\max} = \pm * d_{\min/\max} *\,.$$

Proof. The case of the maximal extensions has been treated above. The statement about the minimal extensions is similar. □

After these preliminaries we return to harmonic forms.

Corollary 4.10. *Let (M, g) be an oriented Riemannian manifold. Then the Hodge operator restricts to isomorphisms*

$$*\colon \hat{H}^k(\Omega_M^{\mathrm{abs/rel}}) \to \hat{H}^{n-k}(\Omega_M^{\mathrm{rel/abs}}).$$

Proof. Recall that

$$\hat{H}^*(\Omega_M^{\mathrm{rel}}) = \ker d_{\min} \cap \ker d_{\min}^* = \ker d_{\min} \cap \ker \delta_{\max}$$

and

$$\hat{H}^*(\Omega_M^{\mathrm{abs}}) = \ker d_{\max} \cap \ker d_{\max}^* = \ker d_{\max} \cap \ker \delta_{\min}.$$

We claim that

i) $\omega \in \ker d_{\max}$ if and only if $*\omega \in \ker \delta_{\max}$ and

ii) $\omega \in \ker d_{\min}$ if and only if $*\omega \in \ker \delta_{\min}$.

Let $\omega \in \ker d_{\max}$. By Proposition 4.9 we have $*\omega \in \mathcal{D}_{\max}(\delta)$ and

$$\delta_{\max} * \omega = \pm * d_{\max} * *\omega = \pm * d_{\max}\omega = 0.$$

The other statements follow analogously. □

So the harmonic forms of the relative and absolute ideal boundary conditions are dual to each other.

Proposition 4.11. *Let (M, g) be an oriented Riemannian manifold. If the L^2 Stokes Theorem holds then the Hodge operator restricts to isomorphisms*

$$*\colon \hat{H}^k_{(2)}(M, g) \to \hat{H}^{n-k}_{(2)}(M, g).$$

Proof. The L^2 Stokes Theorem implies that

$$\hat{H}^*_{(2)}(M, g) = \hat{H}^*(\Omega_M^{\mathrm{abs}}) = \hat{H}^*(\Omega_M^{\mathrm{rel}}).$$

The rest follows from Corollary 4.10. □

4.2 Quasi isometry of Riemannian metrics

The set of ideal boundary conditions of and elliptic complex $\mathcal{E} = (\Gamma_c(E), d)$ depends on the involved metrics. In the general case, there is one Hermitian metric for each summand of E and, of course, the Riemannian metric on the underlying manifold, and each of those can cause trouble. In the case of the de Rham complex, all bundle metrics are induced by the Riemannian metric. This makes the dependence of the ideal boundary conditions on the metric accessible.

Definition 4.12. Let M be a smooth manifold. Two Riemannian metrics g and h on M are called *quasi isometric* if there is a constant $C \geq 1$ such that for all vector fields $X \in \Gamma(TM)$ we have

$$\frac{1}{C}\, g(X,X) \leq h(X,X) \leq C\, g(X,X). \tag{4.8}$$

More generally, if (M,g) and (N,h) are Riemannian manifolds, then a diffeomorphism $f\colon M \to N$ is called a *quasi isometry* if the Riemannian metrics g and f^*h on M are quasi isometric in the sense of (4.8).

For the remainder of this section we will focus on the first part of the definition where only one manifold is present.

Quasi isometry can be interpreted in more geometric terms. A vector field $X \in \Gamma(TM)$ is called *bounded with respect to* g if the real valued function $g(X,X)$ is bounded.

Lemma 4.13. *Two Riemannian metrics g and h are quasi isometric if they have the same bounded vector fields.*

Proof. It is obvious that quasi isometric Riemannian metrics have the same bounded vector fields. On the other hand, suppose there is a vector field $X \in \Gamma(TM)$ such that $g(X,X)$ is bounded and $h(X,X)$ is not. Then g and h cannot be quasi isometric. □

If M is compact, then any smooth function on M is bounded. In particular, any vector field is bounded with respect to any metric on M. This shows

Corollary 4.14. *Any two Riemannian metrics on a compact manifold are quasi isometric.*

As soon as M is non-compact, the situation changes drastically. In fact, if M is non-compact, then there exist smooth, positive, unbounded functions on M, and for any such $\varphi \in C^\infty(M)$ the metrics g and $\varphi\, g$ are not quasi isometric. Indeed, it is easy to see that for any $s,t > 0$ the Riemannian metrics $\varphi^s\, g$ and $\varphi^t\, g$ are quasi isometric if and only $s = t$. This shows that there are uncountably many quasi isometry classes of Riemannian metrics on M.

Lemma 4.15. *Let M be a smooth manifold. If g and h are two quasi-isometric Riemannian metrics, then their associated L^2-norms on $\Omega_c^*(M)$ are equivalent. Hence, the spaces $L^2\Omega^*(M,g)$ and $L^2\Omega^*(M,g')$ are canonically isomorphic.*

Proof. We denote the L^2 norms of g and h by $\|\cdot\|_g$ and $\|\cdot\|_h$, respectively. Recall that these norms are called equivalent if there exists a constant $\gamma \geq 1$ such that for all $\omega \in \Omega_c^*(M)$ we have

$$\frac{1}{\gamma}\|\omega\|_g \leq \|\omega\|_h \leq \gamma\|\omega\|_g. \tag{4.9}$$

Suppose that we have an estimate as in (4.8) with a constant $C \geq 1$ and consider square of the L^2 norm of h

$$\|\omega\|_h^2 = \int_M h(\omega,\omega)\, d\mu_h.$$

We start by comparing the volume densities μ_h and μ_g. Assume that M is an open subset of $\mathbb{R}^n$, where $n = \dim M$. Then g and h are given by matrix valued functions $(g_{ij}(x))$ and $(h_{ij}(x))$. Assume further that g is diagonal at a given point $x_0 \in M$. We have

$$d\mu_g = \sqrt{\det g}\, d\lambda \qquad \text{and} \qquad d\mu_h = \sqrt{\det h}\, d\lambda$$

where λ is the Lebesgue measure on $\mathbb{R}^n$. Since g is diagonal at a x_0 we have

$$\det g(x_0) = \prod_{i=1}^n g_{ii}(x_0).$$

On the other hand, by Leibniz' formula for the determinant we have

$$\det h = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}(\sigma)\prod_{i=1}^n h_{i\sigma(i)} \leq \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n |h_{i\sigma(i)}|$$

where $\mathfrak{S}_n$ denotes the symmetric group on n letters. Using the Cauchy-Schwarz inequality and the quasi isometry assumption we get

$$|h_{i\sigma(i)}| \leq \sqrt{h_{ii}\, h_{\sigma(i)\sigma(i)}} \leq C\sqrt{g_{ii}\, g_{\sigma(i)\sigma(i)}}$$

Combined with the previous equation and the formula for the determinant of g at x_0 this yields

$$\begin{aligned}\det h(x_0) \leq& C^n \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n \sqrt{g_{ii}(x_0)\, g_{\sigma(i)\sigma(i)}(x_0)}\\ =& C^n \sum_{\sigma\in\mathfrak{S}_n} \det g(x_0)\\ =& C^n n! \det g(x_0)\end{aligned}$$

For general M, we can choose coordinates around any given point such that we are in the above situation and we get the estimate

$$d\mu_h \leq C^n n!\, d\mu_g. \tag{4.10}$$

By similar arguments one can prove that for any $\omega \in \Omega^*(M)$

$$h(\omega, \omega) \leq C'\, g(\omega, \omega) \tag{4.11}$$

where the constant $C' \geq 1$ depends only on C and n.

Altogether, (4.10) and (4.11) yield

$$\|\omega\|_h \leq \gamma\, \|\omega\|_g$$

with $\gamma = n! C^n C'$. Reversing the roles of g and h the same arguments show that $\|\omega\|_g \leq \gamma\, \|\omega\|_h$ and the proof is finished. □

Corollary 4.16. *The relative and absolute ideal boundary conditions of a Riemannian manifold (M, g) depend only the quasi isometry class of g. In particular, the validity of the L^2 Stokes Theorem depends only on the quasi isometry class of g.*

Proof. Let g and h be quasi isometric. For the moment, we denote the minimal and maximal extensions of d with respect to to g by $d^g_{\text{min/max}}$ and analogously for h. By the preceding lemma, the induced L^2 spaces can be canonically identified. Hence, the two closures of any differential operator on $\Omega^*_c(M)$ agree, in particular $d^g_{\min} = d^h_{\min}$.

The situation for $d_{\max}$ is slightly different since, a priori, it might depend on derivatives of the metric. But as Ω_M is an elliptic complex, $d_{\max}$ is given as the closure of the differential operator d with domain $\mathcal{D}_{\max}(d) \cap \Omega(M)$ and we can argue as above to see that $d^g_{\max} = d^h_{\max}$. □

This result gives us some flexibility when trying to prove the L^2 Stokes Theorem.

4.3 A localization argument

Our aim in this section is to develop a method of proof for the L^2 Stokes Theorem by a localization procedure. Roughly speaking, given an open cover of a manifold such that the L^2 Stokes Theorem holds *locally* in every set of the cover we want to conclude that the L^2 Stokes Theorem hold globally on the manifold.[5] We first have to specify what it means for the L^2 Stokes Theorem to hold locally in an open subset.

Definition 4.17. Let (M,g) be a Riemannian manifold and $U \subset M$ an open subset. We say that the L^2 *Stokes Theorem holds locally in* U if for any smooth $\omega \in \mathcal{D}_{\max}(d)$ and $\eta \in \mathcal{D}_{\max}(\delta)$ with support in U we have

$$\langle d_{\max}\omega, \eta\rangle = \langle \omega, \delta_{\max}\eta\rangle .$$

Note that by restricting to smooth forms we do not lose any information (by Corollary 3.22). We could have used L^2 forms just as well but we wanted to avoid the notion of support for such forms.

Now let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover where I is at most countable. Let $\{\rho_i\}_{i\in I}$ be a smooth partition of unity subordinate to $\mathcal{U}$ and let $\{\chi_i\}_{i\in I}$ be a set of smooth cut-off functions such that

- χ_i is identically 1 on a neighborhood of $\mathrm{supp}(\rho_i)$ and
- $\mathrm{supp}(\chi_i) \subset U_i$.

Suppose that the L^2 Stokes Theorem holds locally in each U_i. In order to deduce the L^2 Stokes Theorem on the whole of M we consider the following formal calculation for smooth forms $\omega \in \mathcal{D}_{\max}(d)$ and $\eta \in \mathcal{D}_{\max}(\delta)$.

$$\begin{aligned}
\langle d_{\max}\omega, \eta\rangle &= \Big\langle \sum_{i\in I} d_{\max}(\rho_i\omega), \eta\Big\rangle \overset{!}{=} \sum_{i\in I} \langle d_{\max}(\rho_i\omega), \eta\rangle && (4.12)\\
&= \sum_{i\in I} \langle d_{\max}(\rho_i\omega), \chi_i\eta\rangle \overset{!}{=} \sum_{i\in I} \langle \rho_i\omega, \delta_{\max}(\chi_i\eta)\rangle && (4.13)\\
&= \sum_{i\in I} \langle \rho_i\omega, \delta_{\max}\eta\rangle \overset{!}{=} \Big\langle \sum_{i\in I} \rho_i\omega, \delta_{\max}\eta\Big\rangle && (4.14)\\
&= \langle \omega, \delta_{\max}\eta\rangle
\end{aligned}$$

Each step that is marked with the sign $\overset{!}{=}$ may not be valid without additional assumptions on the functions ρ_i and χ_i. Let us see what can go wrong and how it can be fixed.

We address line (4.13) first. There we tried to use that the L^2 Stokes Theorem holds locally in each U_i. In order to do that we need to know that $\rho_i\omega$ and $\chi_i\eta$ are supported in U_i, which is obviously satisfied, and that $\rho_i\omega \in \mathcal{D}_{\max}(d)$ and $\chi_i\eta \in \mathcal{D}_{\max}(\delta)$. Unfortunately, this is not true in general.

[5] A very similar argument can be found in [C2] (Lemma 4.1). The author was not aware of this until the present work had almost been finished.

One possible solution is provided by Lemma 2.28. Since d and δ are first order differential operators it is enough to assume that ρ_i and each χ_i is C^1-bounded (see Definition 2.27).

Next we observe that the lines (4.12) and (4.14) are unproblematic if I is a finite set, i.e. if $\mathcal{U}$ is a finite cover. However, if I is infinite, then convergence problems may arise.

For convenience we assume that $I = \mathbb{N}$. If we spell out the L^2 scalar products in line (4.14), we see that we have to justify the following interchange of summation and integration:

$$\sum_{i=1}^{\infty} \int_M \rho_i g(\omega, \delta\eta)\, d\mu_g = \int_M \sum_{i=1}^{\infty} \rho_i g(\omega, \delta\eta)\, d\mu_g = \int_M g(\omega, \delta\eta)\, d\mu_g.$$

We are fortunate in this situation since we have an estimate

$$\begin{aligned}\sum_{i=1}^{N} \int_M \rho_i g(\omega, \delta\eta)\, d\mu_g &= \int_M \sum_{i=1}^{N} \rho_i g(\omega, \delta\eta)\, d\mu_g \\ &\leq \int_M g(\omega, \delta\eta)\, d\mu_g < \infty\end{aligned}$$

for each $N \in \mathbb{N}$. Hence, the function $g(\omega, \delta\eta)$ provides an upper bound in $L^1(M)$ for the partial sums and we can apply Lebesgue's dominated convergence theorem.

Unfortunately, line (4.12) contains a real problem. Here we have to justify

$$\sum_{i=1}^{\infty} \int_M g(d(\rho_i\omega), \eta)\, d\mu_g = \int_M \sum_{i=1}^{\infty} g(d(\rho_i\omega), \eta)\, d\mu_g = \int_M g(d\omega, \eta)\, d\mu_g. \tag{4.15}$$

The problem is caused by the appearance of the differential of ρ_i in

$$d(\rho_i\omega) = \rho_i d\omega + d\rho_i \wedge \omega.$$

Lebesgue's theorem cannot be applied in this situation since we have no control over $d\rho_i$. In order to ensure convergence we have to require that the sequence $\left\{ \sum_{i=1}^{N} d\rho_i \right\}_{N \in \mathbb{N}}$ is uniformly bounded in $L^2\Omega^1(M, g)$.

Definition 4.18. Let (M, g) be a Riemannian manifold. A countable open cover $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ is called *admissible* if there exists a smooth partition of unity $\{\rho_i\}_{i\in\mathbb{N}}$ subordinate to $\mathcal{U}$ and a set of cut-off functions $\{\chi_i\}_{i\in\mathbb{N}}$ as above such that

a) all functions ρ_i and χ_i are C^1 bounded and

b) $\sup_{N\in\mathbb{N}} \left\| \sum_{i=1}^{N} d\rho_i \right\|_{L^2} < \infty$.

An uncountable open cover $\mathcal{U}$ is called admissible if it admits a countable, admissible refinement.

Altogether, we have seen that our formal calculation on page 49 is rigorous provided that the cover is admissible. Thus we have proved

Proposition 4.19. *Let (M, g) be a Riemannian manifold. Suppose that $\mathcal{U} = \{U_i\}_{i\in I}$ is an admissible cover such that the L^2 Stokes Theorem holds locally in each U_i. Then the L^2 Stokes Theorem holds on (M, g).*

We end this section with some examples of covers that are not admissible.

Example 4.20. This example shows how condition a) in Definition 4.18 may fail. Consider the open cover of $\mathbb{R}^2$ consisting of the two sets

$$U := \left\{(x, y) \mid y < e^{-x^2}\right\} \quad \text{and} \quad V := \left\{(x, y) \mid y > -e^{-x^2}\right\}.$$

Let ρ_U, ρ_V be a partition of unity subordinate to the cover. Then for fixed x the mean value theorem for integration yields

$$1 = \int_{-e^{-x^2}}^{e^{-x^2}} \frac{\partial}{\partial y}\rho_U(x, y)\, dy = 2e^{-x^2} \frac{\partial}{\partial y}\rho_U(x, y_0)$$

for some $y_0 \in \left(-e^{-x^2}, e^{-x^2}\right)$. This implies that the gradient of ρ_U with respect to to the Euclidean metric on $\mathbb{R}^2$ is unbounded, hence ρ_U is not C^1-bounded.

Example 4.21. Consider the open cover $U_n := \left(\frac{1}{n+2}, \frac{1}{n}\right)$, $n \in \mathbb{N}$, of the open unit interval $(0, 1)$ together with a subordinate partition of unity $\{\rho_n\}_{n\in\mathbb{N}}$ and let $S_n := \sum_{k=1}^{N} \rho_k$. Since all the intersections $U_i \cap U_j$ are relatively compact, condition a) is satisfied and condition b) means that the sequence $\{\|S_n{}'\|_{L^2}\}_{n\in\mathbb{N}}$ is bounded. In order to ease the notation we write $a_n := \frac{1}{n+1}$. By construction we have

$$S_n(x) = \begin{cases} 1, & x \geq a_n \\ 0, & x \leq a_{n+1} \end{cases},$$

thus the derivative $S_n{}'$ is supported in the interval (a_{n+1}, a_n) and satisfies

$$\int_{a_{n+1}}^{a_n} S_n{}'(x)\, dx = 1. \tag{4.16}$$

By Hï¿½lder's inequality we have

$$\int_{a_{n+1}}^{a_n} 1 \cdot {S_n}'(x)\, dx \leq \int_{a_{n+1}}^{a_n} 1^2\, dx \cdot \int_{a_{n+1}}^{a_n} ({S_n}')^2(x)\, dx$$
$$= (a_n - a_{n+1}) \left\| {S_n}' \right\|_{L^2}^2 .$$

Together with (4.16) we get

$$\left\| {S_n}' \right\|_{L^2}^2 \geq \tfrac{1}{a_n - a_{n+1}}.$$

Since the right hand side is unbounded we see that no partition of unity subordinate to the cover $\{U_n\}_{n\in\mathbb{N}}$ can satisfy condition b).

Note that condition a) can be achieved for an arbitrary open cover by passing to a countable refinement consisting of relatively compact sets. For instance, in Example 4.20 we could define

$$U_n := U \cap \{(x, y) \mid n - 1 \leq |x| \leq n + 1\}$$

and analogously V_n. We gain condition a) but by similar arguments as in Example 4.21 one can show that we lose condition b).

4.4 Total spaces of fiber bundles

Let (B, g^B) and (F, g^F) be two Riemannian manifolds. The following is an almost-direct consequence of Proposition 3.36.

Proposition 4.22. *If the L^2 Stokes Theorem holds on (B, g^B) and (F, g^F), then it also holds on $(B \times F, g^B \times g^F)$.*

Proof. It is well known that $\Omega_{B\times F}$ can be identified with $\Omega_B \otimes \Omega_F$ via the map

$$\Omega_c(B) \otimes \Omega_c(F) \to \Omega_c(B \times F)$$
$$\omega \otimes \eta \mapsto pr_B^* \omega \wedge pr_F^* \eta.$$

By Proposition 3.36 we have

$$(\Omega_B \otimes \Omega_F)^{\text{rel/abs}} = \Omega_B^{\text{rel/abs}} \hat{\otimes} \Omega_F^{\text{rel/abs}}$$

and the claim follows □

Corollary 4.23. *Let B and F be as above. Then the L^2 Stokes Theorem holds on $(B \times F, h)$ if h is quasi isometric to the product metric $g^B \times g^F$.*

Proof. This follows directly from Proposition 4.22 and Corollary 4.16. □

As our notation indicates, we want to think of the product manifold $Y := B \times F$ as the total space of the trivial fiber bundle over the base B with fiber F. Our goal is to extend Proposition 4.22 to total spaces of more general bundles. Roughly, we are aiming for a result of the type:

> *"If the L^2 Stokes Theorem holds on the base and on the fiber of a fiber bundle, then it also holds on the total space."*

This statement is, of course, too imprecise since it does not appeal to any Riemannian metrics. The Riemannian metric on the total space must be related to those on the base and the fiber in some way as the following rather trivial example shows.

Example 4.24. Consider two copies of $\mathbb{R}$ with the standard metric as the base and the fiber in the trivial bundle $\mathbb{R}^2$. If we equip $\mathbb{R}^2$ with a metric that makes it look like the open unit disk, then the L^2 Stokes Theorem will not hold on total space $\mathbb{R}^2$ (by Proposition 4.4, for example) although it holds on the base and the fiber (since $\mathbb{R}$ is complete).

Let us investigate the general situation. Let $F \hookrightarrow Y \xrightarrow{\phi} B$ be a smooth fiber bundle and suppose that we are given Riemannian metrics g^B, g^F and g^Y on the respective manifolds. Given a local trivialization

$$\tau\colon \phi^{-1}(U) \to U \times F$$

over an open subset $U \subset B$, which we will also refer to as a bundle chart over U, we can push forward the restriction of g^Y to $U \times F$. In general, there is no reason that the result will be related to the product metric on $U \times F$.

Definition 4.25. In the situation described above, τ is called a *local geometric trivialization* or *geometric bundle chart* if $\tau_*(g^Y|_{\phi^{-1}(U)})$ is quasi isometric to $(g^B|_U)\times g^F$, that is if τ is a quasi isometry between $g^Y|_{\phi^{-1}(U)}$ and $(g^B|_U)\times g^F$. The Riemannian metric g^Y is called a *local product metric* if each point $b \in B$ is contained in the domain of a geometric bundle chart. If g^Y is a local product metric, then $(F, g^F) \hookrightarrow (Y, g^Y) \xrightarrow{\phi} (B, g^B)$ is called a *locally geometrically trivial fiber bundle.*

If g^Y is a local product metric, then all fibers $(F_b := \phi^{-1}(b), g^Y|_{F_b})$, $b \in B$, are quasi isometric to (F, g^F). Hence, the fiber is a well defined quasi isometry class of Riemannian manifolds. This is not true in general.

Example 4.26. Let $Y = \mathbb{R} \times \mathbb{R}$ be equipped with the Riemannian metric

$$g^Y|_{(x,y)} := dx^2 + e^{xy} dy^2.$$

We take the projection onto the first factor to give Y the structure of a fiber bundle. Then the fiber over x is isometric to $\mathbb{R}$ with the metric $e^{xy}dy^2$. But for different values of x, these metrics are not quasi-isometric.

Proposition 4.27. *Let $(F, g^F) \hookrightarrow (Y, g^Y) \xrightarrow{\phi} (B, g^B)$ be a fiber bundle (not necessarily locally geometrically trivial) such that the L^2 Stokes Theorem holds on (B, g^B) and (F, g^F). If $\tau: \phi^{-1}(U) \to U \times F$ is a geometric bundle chart, then the L^2 Stokes Theorem holds locally in $\phi^{-1}(U)$.*

In order to prove Proposition 4.27 we need a lemma. Recall that for a diffeomorphism $f: M \to N$ the *push forward* of differential forms

$$f_*: \Omega^*(M) \to \Omega^*(N)$$

is defined as $f_* := (f^{-1})^*$.

Lemma 4.28. *Let (M, g) and (N, h) be two Riemannian manifolds and let $f: M \to N$ be an isometry. Then we have $f_* \delta^M = \delta^N f_*$.*

Proof. It is easy to see that f_* acts isometrically with respect to the L^2 scalar products and thus extends to a unitary operator

$$f_*: L^2\Omega^*(M, g) \to L^2\Omega^*(N, h)$$

with adjoint (and inverse) f^*. In particular, for any $\omega \in \Omega_c^*(M)$ and $\eta \in \Omega_c^*(N)$ we have $\langle f_*\omega, \eta \rangle_N = \langle \omega, f^*\eta \rangle_M$ and we can compute

$$\begin{aligned}
\left\langle f_* \delta^M \omega, \eta \right\rangle_N &= \left\langle \delta^M \omega, f^* \eta \right\rangle_M = \left\langle \omega, d^M f^* \eta \right\rangle_M \\
&= \left\langle \omega, f^* d^N \eta \right\rangle_M = \left\langle f_* \omega, d^N \eta \right\rangle_N \\
&= \left\langle \delta^N f_* \omega, \eta \right\rangle_N .
\end{aligned}$$

using the well known fact that $d^M f^* \eta = f^* d^N \eta$. This shows that $f_* \delta^M \omega = \delta^N f_* \omega$. ☐

Proof of Lemma 4.27. Let $\omega \in \mathcal{D}_{\max}(d)$ and $\eta \in \mathcal{D}_{\max}(\delta)$ be smooth forms supported in $\tilde{U} := \phi^{-1}(U)$. Since the L^2 Stokes Theorem is invariant under quasi isometry we can without loss of generality assume that $\tau: \tilde{U} \to U \times F$ is an isometry where $U \times F$ is equipped with the product metric. Using this, we can transfer the situation to $B \times F$, where the L^2 Stokes Theorem is known to hold.

$$\begin{aligned}\left\langle d^Y \omega, \eta\right\rangle_Y &= \left\langle d^{\tilde{U}} \omega, \eta\right\rangle_{\tilde{U}} = \left\langle \tau_* d^{\tilde{U}} \omega, \tau_* \eta\right\rangle_{U\times F} \\ &= \left\langle d^{U\times F} \tau_* \omega, \tau_* \eta\right\rangle_{U\times F} = \left\langle d^{B\times F} \tau_* \omega, \tau_* \eta\right\rangle_{B\times F} \\ &= \left\langle \tau_* \omega, \delta^{B\times F} \tau_* \eta\right\rangle_{B\times F}\end{aligned}$$

Since τ is an isometry, we can use Lemma 4.28 and go back to Y.

$$\begin{aligned}\left\langle d^Y \omega, \eta\right\rangle_Y &= \left\langle \tau_* \omega, \delta^{B\times F} \tau_* \eta\right\rangle_{B\times F} = \left\langle \tau_* \omega, \delta^{U\times F} \tau_* \eta\right\rangle_{U\times F} \\ &= \left\langle \tau_* \omega, \tau_* \delta^{\tilde{U}} \eta\right\rangle_{U\times F} = \left\langle \omega, \delta^{\tilde{U}} \eta\right\rangle_{\tilde{U}} \\ &= \left\langle \omega, \delta^Y \eta\right\rangle_Y\end{aligned}$$

This completes the proof. □

As a next step we want to conclude that the L^2 Stokes Theorem not only holds locally in domains of certain bundle charts but globally on (Y, g^Y). Unfortunately, we pick up an extra assumption. Let $\{(U_i, \tau_i)\}_{i\in I}$ be a bundle atlas, that is an open cover $B = \bigcup_{i\in I} U_i$ together with bundle charts $\tau_i \colon \phi^{-1}(U_i) \to U_i \times F$. We say that the atlas is *admissible* if the open cover $\left\{\phi^{-1}(U_i)\right\}_{i\in I}$ of Y is admissible in the sense of Definition 4.18. Furthermore, the atlas is called *geometric* if each τ_i is a geometric bundle chart.

Theorem 4.29. *Let $(F, g^F) \hookrightarrow (Y, g^Y) \xrightarrow{\phi} (B, g^B)$ be a locally geometrically trivial fiber bundle and assume that there exists a bundle atlas which is both geometric and admissible. If the L^2 Stokes Theorem holds on (B, g^B) and on (F, g^F), then it also holds on (Y, g^Y).*

Proof. Let $\{(U_i, \tau_i)\}_{i\in I}$ be a geometric and admissible bundle atlas. By Proposition 4.27 the L^2 Stokes Theorem holds locally in $\phi^{-1}(U_i)$ for each $i \in I$. Since the open cover $\left\{\phi^{-1}(U_i)\right\}_{i\in I}$ of Y is admissible it follows from Proposition 4.19 that the L^2 Stokes Theorem holds on (Y, g^Y). □

Given a fiber bundle $F \hookrightarrow Y \xrightarrow{\phi} B$ and Riemannian metric g^B and g^F it is easy to construct a local product metric on Y by using a bundle atlas and a partition of unity. Of course, one usually meets the opposite situation where one is given the Riemannian metric g^Y and is asked if it is a local product metric for some g^B and g^F.

Let $F \hookrightarrow Y \xrightarrow{\phi} B$ be an arbitrary smooth fiber bundle again and suppose that we are given a Riemannian metric g^B on B. One usually restricts attention to a special class of metrics on Y which is adapted to the bundle structure in a certain way.

Definition 4.30. Let (M, g) and (N, h) be Riemannian manifolds. A submersion $f: M \to N$ is called a *Riemannian submersion* if the differential

$$f_*: (\ker f_*)^\perp \to TN$$

is an isometry.

So the common assumption on g^Y is that it is chosen in such a way that $\phi: Y \to B$ becomes a Riemannian submersion with respect to g^B. In this case we call g^Y a *submersion metric.*

It is easy to see that any such submersion metric g^Y has the form

$$g^Y := \phi^* g^B + \kappa$$

where κ is a symmetric 2-tensor that annihilates the orthogonal complement of $\ker \phi_*$ and restricts to a Riemannian metric in each fiber $F_b = \phi^{-1}(b)$.

Looking back at Example 4.26 where we studied $Y = \mathbb{R} \times \mathbb{R}$ equipped with the Riemannian metric

$$g^Y|_{(x,y)} := dx^2 + e^{xy} dy^2,$$

we see that this is clearly a submersion metric for the projection onto the first factor. Hence, submersion metrics are not necessarily local product metrics. In order to show that a submersion metric is a local product metric, the tensor κ has to be investigated in a given situation

The next lemma shows that admissibility of a bundle atlas can by checked on the base if the total space is equipped with a submersion metric.

Lemma 4.31. *Let $F \hookrightarrow Y \xrightarrow{\phi} B$ be a fiber bundle and let $\{(U_i, \tau_i)\}_{i \in I}$ be a bundle atlas. Assume that Y carries a submersion metric g^Y with respect to a Riemannian metric g^B on B such that the volume of the fibers is bounded. If the open cover $\{U_i\}_{i \in I}$ is admissible, then the bundle atlas $\{(U_i, \tau_i)\}_{i \in I}$ is admissible.*

Proof. We can without loss of generality assume that the bundle atlas is countable and indexed by the natural numbers, that is we can replace I with $\mathbb{N}$ in the notation.

Choose a partition of unity $\{\rho_i\}_{i \in \mathbb{N}}$ subordinate to the open cover $\{U_i\}_{i \in \mathbb{N}}$ of B and a set of cut off functions $\{\chi_i\}_{i \in \mathbb{N}}$ as in Definition 4.18, that is all functions ρ_i and χ_i are C^1-bounded and we have $\sup_{n\in\mathbb{N}} \left\| \sum_{i=1}^n d\rho_i \right\|_{L^2} < \infty$. Using the functions

$$\phi^*\rho_i := \rho_i \circ \phi \qquad \text{and} \qquad \phi^*\chi_i := \chi_i \circ \phi$$

on Y we will show that the open cover $\left\{\phi^{-1}(U_i)\right\}_{i\in\mathbb{N}}$ of Y is admissible with respect to g^Y. Clearly, $\{\phi^*\rho_i\}_{i\in\mathbb{N}}$ is a partition of unity subordinate to $\left\{\phi^{-1}(U_i)\right\}_{i\in\mathbb{N}}$ and $\phi^*\chi_i$ is identically 1 on a neighborhood of $\mathrm{supp}(\phi^*\rho_i)$ and has support in $\phi^{-1}(U_i)$. Furthermore, all these functions are constant along the fibers. Since g^Y is a submersion metric, and thus can be written as $g^Y = \phi^*g^B + \kappa$, we have at a point $y \in Y$

$$|d\phi^*\rho_i(y)|_Y = |\phi^* d\rho_i(y)|_Y = |d\rho_i(\phi(y))|_B.$$

It follows that $\phi^*\rho_i$ is C^1-bounded and the same argument applies to $\phi^*\chi_i$.

Next we consider the expression

$$\begin{aligned} \Big\| \sum_{i=1}^n d\phi^*\rho_i \Big\|_{L^2} &= \int_Y \Big| \sum_{i=1}^n d\phi^*\rho_i(y) \Big|_Y \, d\mu_Y(y) \\ &= \int_Y \Big| \sum_{i=1}^n d\rho_i(\phi(y)) \Big|_B \, d\mu_Y(y). \end{aligned}$$

The fiber bundle version of Fubini's theorem (see [M], Chapter 9.3) yields

$$\int_Y \Big| \sum_{i=1}^n d\rho_i(\phi(y)) \Big|_B \, d\mu_Y(y) = \int_B \mathrm{Vol}(F_b) \Big| \sum_{i=1}^n d\rho_i(b) \Big|_B \, d\mu_B(b)$$

and, since the volume of the fibers is assumed to be bounded, we get

$$\begin{aligned} \Big\| \sum_{i=1}^n d\phi^*\rho_i \Big\|_{L^2} &\le C \int_B \Big| \sum_{i=1}^n d\rho_i(b) \Big|_B \, d\mu_B(b) \\ &= C \, \Big\| \sum_{i=1}^n d\rho_i \Big\|_{L^2} < \infty \end{aligned}$$

for some $C > 0$ and thus

$$\sup_{n\in\mathbb{N}} \Big\| \sum_{i=1}^n d\phi^*\rho_i \Big\|_{L^2} < \infty.$$

This shows that $\left\{\phi^{-1}(U_i)\right\}_{i\in\mathbb{N}}$ is an admissible cover of Y, hence the bundle atlas $\{(U_i, \tau_i)\}_{i\in I}$ is admissible. □

5 Applications to manifolds with singularities

Consider the following situation. Let (X, d) be a complete metric space and let Σ be a proper, closed subset of X. Suppose that $M := X \setminus \Sigma$ is dense in X and has the structure of a smooth manifold. Moreover, suppose that the metric structure of the metric space $(M, d|_M)$ is induced by a Riemannian metric g on M.

Definition 5.1. In the above situation, we call X a *singular space*, Σ the *singular locus*, and M the *regular part*. The Riemannian manifold (M, g) will also be referred to as a *manifold with singularities*.

An important class of examples of such singular spaces are projective algebraic varieties (real or complex). Being defined by polynomial equations, they are closed subsets of projective space, hence compact, and they inherit a metric structure from a Riemannian metric on projective space, for example the Fubini-Studi metric. As is well known, the regular part (in the sense of algebraic geometry) is an open and dense subset which has the structure of a smooth (real or complex) manifold.

For an arbitrary singular space X the regular part (M, g) is necessarily an incomplete Riemannian manifold and its completion (as a metric space) can naturally be identified with X. Hence, all information about the singular locus Σ is encoded in the behavior of the Riemannian metric g on M *in a neighborhood of infinity*, by which we mean the complement of a compact subset. The general idea is to obtain information about a singular space X by performing analysis on its regular part. The necessity of working with incomplete metrics complicates the analysis considerably. The situation is similar to that of boundary value problems on compact manifolds where the presence of the boundary destroys much of the harmony of the analysis on closed manifolds. In fact, it is quite common to consider manifolds with boundary as singular spaces in the above sense by considering the boundary as the singular locus. Theorem 4.4 is an example where this approach is fruitful.

On manifolds with singularities, most of the general results that are known for closed (or complete) manifolds fail to hold. Elliptic operators are no longer Fredholm and operators of Dirac-type and their powers are not necessarily essentially self adjoint. In particular, elliptic complexes over manifolds with singularities will usually not have unique ideal boundary conditions and the theory developed in Chapter 3 becomes relevant.

We will only deal with compact singular spaces. In this case, the regular part will still be non-compact and incomplete, but at least it will have a compact completion. Furthermore, the complement of any open neighborhood of the singular locus will be compact.

Since the variety of singular spaces is overwhelming it is impossible to prove results for arbitrary singular spaces. Instead, one studies certain *model singularities*. Such models are described as follows. One considers a Riemannian manifold (M,g) without boundary and an open subset $U \subset M$ such that $M \setminus U$ is a compact manifold with boundary and one prescribes the behavior of the Riemannian metric on U up to quasi isometry. We will discuss two basic examples in Section 5.2.

The notion of quasi isometry is well suited for manifolds with singularities since quasi isometries map Cauchy sequences into Cauchy sequences. Hence, the completions of quasi isometric manifolds are homeomorphic. In other words, quasi isometry is restrictive enough to preserve the structure of a manifold with singularities at infinity. On the other hand, it is flexible enough to allow arbitrary changes of the metric on any compact part of the manifold.

5.1 Singularities and the L^2 Stokes Theorem

Let X be a singular space and (M,g) its regular part. The following Lemma shows that, in order to prove the L^2 Stokes Theorem for a manifold with singularities, it is enough to prove it "near" the singularities, in a precise sense.

Lemma 5.2. *Let (M,g) be a Riemannian manifold and $U \subset M$ an open subset such that $M \setminus U$ is a compact manifold with boundary. If the L^2 Stokes Theorem holds locally in U, then it holds on all of M.*

Proof. Let $Y := M \setminus U$ and choose an open neighborhood V of+Y such that the closure $\overline{V}$ is also a compact manifold with boundary. Such a neighborhood can be obtained, for example, by integrating a suitable vector field on M which is normal to the boundary of Y. Then the set $U \cap V$ is relatively compact and the open cover $M = U \cup V$ is admissible in the sense of Definition 4.18.

Since we know that the L^2 Stokes Theorem holds locally in U, we only have to show that it also holds locally in V by Proposition 4.19. But this is obvious since V is relatively compact. Hence, any smooth forms $\omega, \eta \in \Omega^*(M)$ with supports in V have compact supports and by the very definition of the operator δ we have

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle .$$

$\square$

5.2 Conical singularities and simple edge singularities

We will now consider two special types of singularities. Let N be a closed manifold. In the following we will denote the open unit interval in $\mathbb{R}$ by

$$I := (0,1) \subset \mathbb{R}.$$

Definition 5.3. Let (N,g) be a Riemannian manifold. The *metric cone* $\mathcal{C}_g N$ with *base* (N,g) is defined as the cylinder $I \times N$ equipped with the Riemannian metric $dx^2 + x^2 g$, that is

$$\mathcal{C}_g N := (I \times N, dx^2 + x^2 g), \tag{5.1}$$

where x denotes the canonical coordinate in I. Any Riemannian metric on $I \times N$ which has the form as in (5.1) as called a *conical metric.*

The following lemma shows that the quasi isometry class of a metric cone depends only on the quasi isometry class of its base.

Lemma 5.4. *Let $f\colon (M,g) \to (N,h)$ be a quasi isometry of Riemannian manifolds. Then the induced map*

$$\widetilde{f}\colon \mathcal{C}_g M \to \mathcal{C}_h N$$

given by $\widetilde{f}(x,y) := (x, f(y))$ is also a quasi isometry.

Proof. Suppose that for any vector field $X \in \Gamma(TM)$ we have

$$\frac{1}{C}\, g(X,X) \le h(f_* X, f_* X) \le C\, g(X,X) \tag{5.2}$$

for some $C \ge 1$. Since the underlying manifold of $\mathcal{C}_g M$ is just $I \times M$, we can write any vector field $V \in \Gamma(T\mathcal{C}_g M)$ as

$$V = \varphi \frac{\partial}{\partial x} + X$$

where $\varphi \in C^\infty(I \times M)$ and $X \in C^\infty(I, \Gamma(TM))$. The push forward with $\widetilde{f}$ is then given by

$$\widetilde{f}_* V = \varphi \circ \widetilde{f}^{-1} \frac{\partial}{\partial x} + f_* X.$$

Invoking (5.2) we get

$$\begin{aligned}(dx^2 + x^2h)(\widetilde{f}_*V, \widetilde{f}_*V) =&(\varphi \circ \widetilde{f}^{-1})^2 dx^2\big(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}\big) + x^2 h(f_*X, f_*X)\\ \leq&(\varphi \circ \widetilde{f}^{-1})^2 dx^2\big(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}\big) + Cx^2 g(X, X)\\ \leq&C\big[(\varphi \circ \widetilde{f}^{-1})^2 dx^2\big(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}\big) + x^2 g(X, X)\big]\\ =&C(dx^2 + x^2 g)(V, V)\end{aligned}$$

and similarly

$$(dx^2 + x^2 g)(V, V) \leq C(dx^2 + x^2 h)(\widetilde{f}_*V, \widetilde{f}_*V)$$

which shows that $\widetilde{f}: \mathcal{C}_g M \to \mathcal{C}_h N$ is a quasi isometry. □

Corollary 5.5. *Let N be a compact manifold. Then the quasi isometry class of the metric cone $\mathcal{C}N$, where $\mathcal{C}N := \mathcal{C}_g N$ for some Riemannian metric g on M, is independent of the choice of g.*

Proof. By Corollary 4.14 any two Riemannian metrics on N are quasi isometric. □

The metric cone $\mathcal{C}_g N$ is an incomplete Riemannian manifold. It can be shown that, if N is compact, then the completion of $\mathcal{C}N$ is homeomorphic to the topological cone

$$CN = [0, 1] \times N / \{0\} \times N.$$

This explains the name. For simplicity we will only consider metric cones over closed manifolds.

Definition 5.6. An incomplete manifold (M, g) is said to have a *conical singularity* modeled on N if there exists an open subset $U \subset M$ such that $M \smallsetminus U$ is a compact manifold with boundary and U is quasi isometric to the metric cone $\mathcal{C}N$.

The definition can be extended to several conical singularities in the obvious way. Conical singularities are, in a way, the simplest type of singularities. Their study began with the work of Cheeger ([C1, C2]) who proved the following result concerning the L^2 Stokes Theorem.

Theorem 5.7 (Cheeger)**.** *Let (M, g) be a manifold with a conical singularity modeled on a closed manifold N.*

a) If $\dim N = 2k - 1$, then the L^2 Stokes Theorem holds on M.

b) *If* $\dim N = 2k$ *and* $H^k(N;\mathbb{C}) = 0$*, then the* L^2 *Stokes Theorem holds on* M*.*

Proof (sketch). Let $U \subset M$ be an open subset such that $M \setminus U$ is a compact manifold with boundary and U is quasi isometric to the metric cone $\mathcal{C}N$. By Proposition 5.2 it is enough to prove that the L^2 Stokes Theorem holds locally in U and by Corollary 4.16 we can identify U with $\mathcal{C}N$ for that purpose. Note that via this identification smooth forms on M with support in U correspond to smooth forms on $\mathcal{C}N = I \times N$ that vanish in a neighborhood of $\{1\} \times N$[6].

Let $\omega, \eta \in \Omega^*(\mathcal{C}N)$ be smooth forms supported away from $\{1\} \times N$ such that $\omega \in \mathcal{D}_{\max}(d)$ and $\eta \in \mathcal{D}_{\max}(\delta)$. The L^2 Stokes Theorem holds if and only if the expression $\langle d\omega, \eta\rangle - \langle \omega, \delta\eta\rangle$ vanishes. One can show that there is a smooth family of forms $\Phi(x) \in \Omega^*(N)$, $x \in I$, such that

$$\langle d\omega, \eta\rangle - \langle \omega, \delta\eta\rangle = \lim_{x\to 0} \int_{\{x\}\times N} \Phi(x).$$

Using Hodge theory on N one can analyze the limit on the right hand side and prove that it vanishes provided that either of the conditions in the statement of the theorem is satisfied. $\square$

This result has been refined and extended by Brï¿½ning and Lesch ([BL2]) to what they call *conformally conic singularities.* The model for a conformally conic singularity is a cylinder $I \times N$ equipped with a Riemannian metric of the form

$$h(x)^2\big(dx^2 + x^2 g_N(x)\big)$$

where $h \in C^\infty(I \times N)$ and $g_N(x)$ is a family of metrics on N restricted by some technical conditions regarding the asymptotic behavior of $h(x)$ and $g_n(x)$ as $x \to 0$. For a precise statement as well as the proof of the following theorem we refer to the original source [BL2]. We only mention that the proof is based on the spectral theory of regular singular operator that was developed in [BS].

Theorem 5.8. *The conclusion of Theorem 5.7 holds for conformally conic singularities. Furthermore, if* N *is even dimensional, say* $\dim N = 2k$*, then we have*

$$\mathcal{D}_{\max}(d)/\mathcal{D}_{\min}(d) \cong H^k(N;\mathbb{C}).$$

[6] More precisely, one can identify the closure $\overline{U}$ with $(0,1] \times N$ and the statement becomes clear.

Although we will not use this generalization, it deserves to be mentioned.

We will now leave conical singularities behind and study a another, although closely related, type of singularity to which our methods developed in Section 4 can be applied. Let $F \hookrightarrow Y \xrightarrow{\phi} B$ be a smooth fiber bundle with Y closed and equipped with a submersion metric $\phi^* g^B + \kappa$ (see the discussion after Definition 4.30).

Definition 5.9. A Riemannian manifold (M, g) is said to have a *simple edge singularity* modeled on the fiber bundle $F \hookrightarrow Y \xrightarrow{\phi} B$ if there exists an open subset $U \subset M$ such that $M \setminus U$ is a compact manifold with boundary and U is quasi isometric to the cylinder $I \times Y$ equipped with the Riemannian metric $dx^2 + \phi^* g^B + x^2\kappa$.

The model singularity for simple edges

$$\left(I \times Y, dx^2 + \phi^* g^B + x^2 \kappa\right)$$

can be considered as a fiber bundle over B with fiber the metric cone $\mathcal{C}F$. Indeed, writing $\widetilde{Y} := I \times Y$ we define a map $\widetilde{\phi}: \widetilde{Y} \to B$ by $\widetilde{\phi}(x, y) := \phi(y)$. This gives $\widetilde{Y}$ the structure of a smooth fiber bundle over B and the fiber over $b \in B$ is given by

$$\widetilde{F}_b := \widetilde{\phi}^{-1}(b) = I \times \phi^{-1}(b) = I \times F_b.$$

Furthermore, the metric $g^{\widetilde{Y}} := dx^2 + \phi^* g^B + x^2 \kappa$ is a submersion metric on $\widetilde{Y}$ and the metric induced on $\widetilde{F}_b = I \times F_b$ is given by $dx^2 + x^2 \kappa|_{F_b}$. Thus $\widetilde{F}_b$ is just the metric cone over F_b and since the generic fiber F is compact, each $\widetilde{F}_b$ is quasi isometric to $\mathcal{C}F$ by Corollary 5.5.

Lemma 5.10. *The bundle* $\mathcal{C}F \hookrightarrow (\widetilde{Y}, g^{\widetilde{Y}}) \xrightarrow{\widetilde{\phi}} (B, g^B)$ *is locally geometrically trivial.*

Proof. Since Y is compact, any bundle chart over $U \subset B$

$$\tau : \phi^{-1}(U) \to U \times F$$

is geometric, that is τ is a quasi isometry where the right hand side is equipped with the product metric $g^B \times g^F$ for an arbitrary choice of g^F. We get an induced bundle chart for $\widetilde{Y}$ as follows. For $y \in \phi^{-1}(U)$ we write $\tau(y) = (\tau^B(y), \tau^F(y)) \in U \times F$. Then we can define a bundle chart

$$\widetilde{\tau}: \widetilde{\phi}^{-1}(U) \to U \times (I \times F)$$

by the formula $\widetilde{\tau}(x,y) := (\tau_i^B(y); x, \tau_i^F(y))$. Now similar arguments as in the proof of Lemma 5.4 show that $\widetilde{\tau}$ is a quasi isometry if we equip the right hand side with the product metric $g^B \times (dx^2 + x^2 g^F)$. Hence, $\widetilde{\tau}$ defines a geometric bundle chart $\widetilde{\phi}^{-1}(U) \to U \times \mathcal{C}F$. $\square$

We can now give a very simple proof of the following theorem of Hunsicker and Mazzeo ([HM]).

Theorem 5.11 (Hunsicker, Mazzeo)**.** *Let (M,g) be a manifold with a simple edge singularity modeled on the fiber bundle $F \hookrightarrow Y \to B$.*

a) If $\dim F = 2k-1$, then the L^2 Stokes Theorem holds on M.

b) If $\dim F = 2k$ and $H^k(F;\mathbb{C}) = 0$, then the L^2 Stokes Theorem holds on M.

Proof. Arguing as in the beginning of the proof of Theorem 5.7 we see that it is enough to show that the L^2 Stokes Theorem holds locally in a set U which is quasi isometric to $\widetilde{Y} := I \times Y$ with a Riemannian metric as above. By Lemma 5.10 above $\widetilde{Y}$ is the total space of a locally geometrically trivial fiber bundle $\mathcal{C}F \hookrightarrow (\widetilde{Y}, g^{\widetilde{Y}}) \xrightarrow{\widetilde{\phi}} (B, g^B)$ such that the L^2 Stokes Theorem holds on the base, since B is closed, and in a suitable sense on the fiber by as seen in the proof of Theorem 5.7. We want to apply Theorem 4.29. For this we need to know that $\widetilde{Y}$ has a bundle atlas which is geometric and admissible.

As we have seen, any geometric bundle atlas of Y induces a geometric atlas for $\widetilde{Y}$. Note that since B is compact, any open cover of B is admissible. Furthermore, $g^{\widetilde{Y}}$ is a submersion metric and the volume of the fibers is bounded. Indeed, each fiber is a metric cone over a closed manifold. It is easy to see that such a cone has finite volume. Again, since B is compact, there is a uniform bound on the volume of the fibers and Lemma 4.31 implies that any admissible cover of B induces an admissible cover of $\widetilde{Y}$. So by starting with any geometric bundle atlas for Y we obtain an atlas of $\widetilde{Y}$ which is both geometric and admissible. $\square$

Note that we have used rather little of the machinery of Chapter 4. Our proof of Theorem 5.11 can easily be adapted to the situation where the fiber bundle $F \hookrightarrow Y \xrightarrow{\phi} B$ is not closed but is allowed to have non compact base or fiber as long as the following conditions are satisfied:

- Y locally geometrically trivial and the L^2 Stokes Theorem holds on B and F.

- Y carries a submersion metric such that the volume of the fibers is uniformly bounded.
- Y has a geometric bundle atlas such that the underlying open cover of B is admissible.

However, it is doubtful if this leads to interesting applications.

A Zusammenfassung

Das zentrale Thema dieser Arbeit ist eine gewisse Eigenschaft des de Rham Komplexes einer Riemannschen Mannigfaltigkeit (M, g), der L^2 *Satz von Stokes*, die im Zusammenhang zur Hodge Theorie für die L^2 Kohomologie steht. Der L^2 Satz von Stokes betrifft den de Rham Komplex mit kompakten Trägern

$$0 \to \Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c(M)^n \to 0,$$

der im Folgenden mit $\Omega_M := (\Omega_c^*(M), d)$ bezeichnet wird. Wir betrachten aus technischen Gründen komplexwertige Differentialformen, so dass unsere Notation als

$$\Omega_{(c)}^*(M) := \Gamma_{(c)}(\Lambda^* T^* M \otimes \mathbb{C})$$

zu verstehen ist. Hierbei bezeichnet $\Gamma_{(c)}(\cdot)$ die glatten Schnitte (mit kompaktem Träger) eines Vektorbündels. Die Riemannsche Metrik g induziert ein Skalarprodukt auf $\Omega_c^*(M)$, das sogenannte L^2 *Skalarprodukt*, welches für $\omega, \eta \in \Omega_c^k(M)$ gegeben ist durch

$$\langle \omega, \eta \rangle := \int_M g(\overline{\omega}, \eta)\, d\mu_g,$$

wobei μ_g das durch g induzierte Maß auf M ist. Die Vervollständigung von $\Omega_c^*(M)$ bezüglich dieses Skalarproduktes wird mit $L^2\Omega^*(M, g)$ bezeichnet. $L^2\Omega^*(M, g)$ ist somit ein Hilbertraum und die äußere Ableitung

$$d\colon \Omega_c^*(M) \to \Omega_c^*(M)$$

kann als unbeschränkter Operator in $L^2\Omega^*(M, g)$ aufgefasst werden. Dieser Operator ist nicht abgeschlossen, besitzt jedoch zwei kanonische abgeschlossene Erweiterungen, die sogenannte *minimale Erweiterung* $d_{\min}$ und die *maximale Erweiterung* $d_{\max}$, deren Definitionsbereiche wie folgt beschrieben werden können:

$$\begin{aligned}
\mathcal{D}(d_{\min}) &= \left\{\omega \in L^2\Omega^*(M, g) \mid \exists\, \omega_i \in \Omega_c^*(M)\colon \omega_i \to \omega,\ d\omega_i \,\text{Cauchy}\right\} \\
\mathcal{D}(d_{\max}) &= \left\{\omega \in L^2\Omega^*(M, g) \mid d\omega \in L^2\Omega^*(M, g)\right\}.
\end{aligned}$$

Diese Erweiterungen sind im Allgemeinen verschieden und hängen von der Riemannschen Metrik ab.

Definition A.1. Der L^2 *Satz von Stokes* gilt auf (M, g) falls die abgeschlossenen Erweiterungen $d_{\min}$ und $d_{\max}$ übereinstimmen.

Der L^2 Satz von Stokes spielt eine Rolle für die sogenannte L^2 Kohomologie

$$H^*_{(2)}(M,g) = \ker(d_{\max})/\operatorname{ran}(d_{\max}),$$

wobei die rechte Seite als graduiert zu verstehen ist. Analog zur Hodge Theorie auf geschlossenen Mannigfaltigkeiten versucht man eine Beziehung zu den L^2 *harmonischen Formen*

$$\hat{H}^*_{(2)}(M,g) := \left\{\omega \in \Omega^*(M) \cap L^2\Omega^*(M) \mid d\omega = 0, \delta\omega = 0\right\}$$

herzustellen. Offensichtlich hat man die kanonische Abbildung

$$\kappa : \hat{H}^*_{(2)}(M,g) \to H^*_{(2)}(M,g),$$

die eine L^2 harmonische Formen auf ihre L^2 Kohomologieklasse schickt. Diese Abbildung ist im Allgemeinen weder injektiv noch surjektiv. Der L^2 Satz von Stokes garantiert die Injektivität.

Eine etwas greifbarere Formulierung des L^2 Satzes von Stokes lautet wie folgt. Es bezeichne $\delta := d^t$ den zu d formal adjungierten Differentialoperator. Dieser ist durch die Forderung $\langle d\omega, \eta\rangle = \langle \omega, \delta\eta\rangle$ für alle $\omega, \eta \in \Omega^*_c(M)$ definiert. Es ist dann leicht zu sehen, dass die Identität $d_{\max} = \delta^*$ gilt, wobei δ^* den zu δ adjungierten Operator (im Sinne von Operatoren in Hilberträumen) bezeichnet. Der L^2 Satz von Stokes gilt genau dann, wenn die Gleichung $\langle d\omega, \eta\rangle = \langle \omega, \delta\eta\rangle$ für alle glatten Formen $\omega \in \mathcal{D}(d_{\max})$ und $\eta \in \mathcal{D}(\delta_{\max})$, das heißt für alle $\omega, \eta \in \Omega^*(M) \cap L^2\Omega^*(M,g)$ mit $d\omega, \delta\eta \in L^2\Omega^*(M,g)$, erfüllt ist. Die Einschränkung auf glatte Formen ist erlaubt, da der de Rham Komplex ein elliptischer Komplex ist.

Der L^2 Satz von Stokes gilt trivialerweise auf geschlossenen, jedoch nicht auf allen Riemannschen Mannigfaltigkeiten. Ein Gegenbeispiel ist durch das Innere einer kompakten Mannigfaltigkeit mit Rand gegeben. In diesem Fall entsprechen die minimale und maximale Erweiterung gerade den relativen und absoluten Randbedingungen. Es ist jedoch ein klassisches Resultat von Gaffney ([G]), dass der L^2 Satz von Stokes auf *vollständigen* Mannigfaltigkeiten ohne Rand gilt. Dabei heißt eine Riemannsche Mannigfaltigkeit (M,g) vollständig, wenn M bezüglich der Riemannschen Abstandsfunktion d_g ein vollständiger metrischer Raum ist. Dies hat zur Folge, dass der L^2 Satz von Stokes nur für unvollständige Mannigfaltigkeiten interessant ist.

Prominente Beispiele für unvollständige Mannigfaltigkeiten sind reguläre Teile von (reellen oder komplexen) projektiven Varietäten. Gerade für diese ist Hodge Theorie, und somit auch der L^2 Satz von Stokes, interessant.

In dieser Arbeit stehen jedoch andere unvollständige Mannigfaltigkeiten im Vordergrund. Bevor wir diese beschreiben ist es allerdings nötig einige weitere Begriffe einzuführen.

Zwei Riemannsche Metriken g und h auf einer glatten Mannigfaltigkeit M heißen *quasi isometrisch* wenn es eine Konstante $C \geq 1$ gibt, so dass für jedes Vektorfeld $X \in \Gamma(TM)$ gilt

$$\frac{1}{C}\, g(X,X) \leq h(X,X) \leq C\, g(X,X).$$

Allgemeiner nennt man zwei Riemannsche Mannigfaltigkeiten (M,g) und (N,h) quasi isometrisch wenn es einen Diffeomorphismus $f\colon M \to N$ gibt, so dass die Metriken f^*h und g auf M quasi isometrisch im obigen Sinn sind. Die Nützlichkeit von quasi Isometrien liegt darin, dass sie es erlauben, verschiedene L^2 Räume zu identifizieren. Für den L^2 Satz von Stokes hat dies folgende Konsequenz.

Satz A.2. *Die Gültigkeit des L^2 Satzes von Stokes ist invariant unter quasi Isometrien.*

Sei N eine geschlossene Mannigfaltigkeit. Der *metrische Kegel* über N ist definiert als der Zylinder $(0,1) \times N$ zusammen mit einer Riemannschen Metrik der Gestalt

$$g_{\text{cone}} = dx^2 + x^2 g^N,$$

wobei x die kanonische Koordinate in $(0,1)$ und g^N eine Riemannsche Metrik auf N bezeichnet. Bis auf quasi Isometrie ist der metrische Kegel über N unabhängig von der Wahl von g^N. Wir definieren $\mathcal{C}N := \big((0,1) \times N, g_{\text{cone}}\big)$.

Eine Riemannsche Mannigfaltigkeit (M,g) hat eine *kegelartige Singularität*, falls es eine offene Teilmenge $U \subset M$ gibt, so dass $M \setminus U$ eine kompakte Mannigfaltigkeit mit Rand ist und U quasi isometrisch zu dem metrischen Kegel $\mathcal{C}N$ über einer geschlossenen Mannigfaltigkeit N ist. Der L^2 Satz von Stokes auf Mannigfaltigkeiten mit kegelartigen Singularitäten wurde zuerst von Cheeger bearbeitet ([C1, C2]).

Theorem A.3. *Sei (M,g) eine Riemannsche Mannigfaltigkeit mit einer kegelartigen Singularität modelliert auf den metrischen Kegel über einer geschlossenen Mannigfaltigkeit N.*

a) Falls $\dim N = 2k-1$, so gilt der L^2 Satz von Stokes auf (M,g).

b) Falls $\dim N = 2k$ und $H^k(N;\mathbb{C}) = 0$, so gilt der L^2 Satz von Stokes auf (M,g).

Der Beweis wird durch eine explizite Rechnung auf Hodge Theorie auf N zurückgeführt.

Ein verblüffend ähnliches Resultat für eine andere Art von Singularität wird in [HM] bewiesen. Sei Y eine geschlossene Mannigfaltigkeit, die als Totalraum eines differenzierbaren Faserbündels $F \hookrightarrow Y \xrightarrow{\phi} B$ gegeben ist. Ferner sei Y ausgestattet mit einer Riemannschen Metrik der Form

$$g^Y = \phi^* g^B + \kappa,$$

wobei g^B eine Riemannsche Metrik auf B ist und κ ein symmetrischer Tensor ist, der eingeschränkt auf jede Faser $F_b = \phi^{-1}(b)$, $b \in B$, eine Riemannsche Metrik liefert.

Eine Riemannsche Mannigfaltigkeit (M, g) hat eine *einfache Ecke* modelliert auf dem Faserbündel $F \hookrightarrow Y \xrightarrow{\phi} B$, falls es eine offene Teilmenge $U \subset M$ gibt, so dass $M \setminus U$ eine kompakte Mannigfaltigkeit mit Rand ist und U quasi isometrisch zu der Riemannsche Mannigfaltigkeit

$$(\widetilde{Y}, g_{\text{edge}}) := \big((0,1) \times Y, dx^2 + \phi^* g^B + x^2 \kappa\big)$$

ist. Für einfache Ecken gilt folgendes Resultat.

Theorem A.4. *Sei (M, g) eine Riemannsche Mannigfaltigkeit mit einer einfachen Ecke modelliert auf $F \hookrightarrow Y \xrightarrow{\phi} B$.*

a) Falls $\dim F = 2k - 1$, so gilt der L^2 Satz von Stokes auf (M, g).

b) Falls $\dim F = 2k$ und $H^k(F; \mathbb{C}) = 0$, so gilt der L^2 Satz von Stokes auf (M, g).

Der in [HM] gegebene Beweis ist eine Verallgemeinerung von Cheegers Rechnung im kegelartigen Fall. Der Ausgangspunkt dieser Arbeit war es, einen alternativen Beweis für Theorem A.4 zu finden, der die Bündelstruktur von einfachen Ecken ausnutzt um das Problem auf Theorem A.3 zurück zu führen. Dieser Ansatz führt auf die Frage, unter welchen Voraussetzungen man den L^2 Satz von Stokes auf dem Totalraum eines Faserbündels folgern kann, sofern er auf sowohl Basis als auch Faser gilt.

Zunächst betrachten wir den Fall des trivialen Bündels $B \times F$, wobei wir annehmen, dass B und F Riemannsche Metriken g^B und g^F tragen. Auf $B \times F$ haben wir dann die sogenannte Produktmetrik

$$g^B \times g^F := \mathrm{pr}_B^* g^B + \mathrm{pr}_F^* g^F.$$

Satz A.5. *Falls der L^2 Satz von Stokes auf (B, g^B) und (F, g^F) gilt, so gilt er auch auf $(B \times F, g^B \times g^F)$.*

Dieser Satz folgt aus der Produkttheorie von *Hilbertkomplexen* und ist im wesentlichen in [BL1] enthalten.

Der nächste Schritt stellt eine Lokalisierung des L^2 Satzes von Stokes dar. Sei (M, g) eine Riemannsche Mannigfaltigkeit und $U \subset M$ offen. Wir sagen, dass der L^2 Satz von Stokes *lokal in U* gilt, falls die Gleichung $\langle d\omega, \eta\rangle = \langle \omega, \delta\eta\rangle$ für alle glatten Formen $\omega \in \mathcal{D}(d_{\max})$ und $\eta \in \mathcal{D}(\delta_{\max})$ mit Träger in U erfüllt ist.

Wir nennen eine offene Überdeckung $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$ von M *zulässig* falls es eine zu $\mathcal{U}$ subordinierte Zerlegung der Eins $\{\rho_i\}_{i\in\mathbb{N}}$ sowie Funktionen $\{\chi_i\}_{i\in\mathbb{N}}$ gibt, so dass folgende Bedingungen erfüllt sind:

- χ_i hat Träger in U_i und ist identisch Eins auf dem Träger von ρ_i
- ρ_i und χ_i sind C^1-beschränkt, das heißt ρ_i und χ_i sind beschränkte Funktionen mit beschränktem Gradientenfeld
- $\sup_{N\in\mathbb{N}} \left\| \sum_{i=1}^N d\rho_i \right\|_{L^2} < \infty$

Mit Hilfe einer zulässigen Überdeckung lässt sich der L^2 Satz von Stokes lokalisieren.

Satz A.6. *Sei (M, g) eine Riemannsche Mannigfaltigkeit zusammen mit einer zulässigen Überdeckung $\mathcal{U} = \{U_i\}_{i\in\mathbb{N}}$, so dass der L^2 Satz von Stokes lokal in jedem U_i gilt. Dann gilt der L^2 Satz von Stokes auf (M, g).*

Die Voraussetzungen des Satzes sind gerade so gewählt, dass die Rechnung für glatte Formen $\omega \in \mathcal{D}(D_{\max})$ und $\eta \in \mathcal{D}(\delta_{\max})$

$$\begin{aligned}
\langle d\omega, \eta\rangle &= \Big\langle \sum_{i\in I} d(\rho_i\omega), \eta \Big\rangle = \sum_{i\in I} \langle d(\rho_i\omega), \eta\rangle \\
&= \sum_{i\in I} \langle d(\rho_i\omega), \chi_i\eta\rangle = \sum_{i\in I} \langle \rho_i\omega, \delta(\chi_i\eta)\rangle \\
&= \sum_{i\in I} \langle \rho_i\omega, \delta\eta\rangle = \Big\langle \sum_{i\in I} \rho_i\omega, \delta\eta \Big\rangle \\
&= \langle \omega, \delta\eta\rangle
\end{aligned}$$

einen Beweis liefert.

Wir wenden uns nun wieder Faserbündeln zu. Sei wie oben $F \hookrightarrow Y \xrightarrow{\phi} B$ ein differenzierbares Faserbündel und g^F, g^B und g^Y seien Riemannsche

Metriken auf den jeweiligen Mannigfaltigkeiten. Zwar ist Y als Faserbündel lokal diffeomorph zu dem Produkt einer offenen Teilmenge von B mit F, jedoch sind die Diffeomorphismen im Allgemeinen keine quasi Isometrien. Eine Bündelkarte über einer offenen Menge $U \subset B$, das heißt ein fasererhaltender Diffeomorphismus

$$\tau\colon \phi^{-1}(U) \to U \times F,$$

heißt *geometrisch*, falls τ eine quasi Isometrie bezüglich der Riemannschen Metriken $g^Y|_{\phi^{-1}(U)}$ und $(g^B|_U)\times g^F$ ist. Folglich nennen wir einen Bündelatlas $\{(U_i,\tau_i)\}_{i\in\mathbb{N}}$, bestehend aus einer offenen Überdeckung $\{U_i\}_{i\in\mathbb{N}}$ von B und Bündelkarten τ_i über U_i, *geometrisch* falls jedes τ_i eine geometrische Bündelkarte ist. Ein Bündel, das einen geometrischen Atlas besitzt, heißt *lokal geometrisch trivial*.

Lemma A.7. *Sei (U,τ) eine geometrische Bündelkarte. Falls der L^2 Satz von Stokes auf (B,g^B) und (F,g^F) gilt, so gilt er lokal in $\phi^{-1}(U) \subset Y$.*

Dies folgt im wesentlichen aus den Sätzen A.5 und A.2. Zusammen mit Satz A.6 erhalten wir das folgende Resultat.

Theorem A.8. *Das Bündel Y sei lokal geometrisch trivial und besitze einen Bündelatlas, der sowohl geometrisch als auch zulässig ist. Falls der L^2 Satz von Stokes auf (B,g^B) und (F,g^F) gilt, so gilt er auch auf (Y,g^Y).*

Hiermit können wir nun Theorem A.4 auf Theorem A.3 zurückführen, indem wir Theorem A.8 auf das Bündel

$$(\widetilde{Y}, g_{\text{edge}}) = \big((0,1)\times Y, dx^2 + \phi^* g^B + x^2\kappa\big)$$

anwenden. Dieses Bündel ist lokal geometrisch trivial mit dem metrischen Kegel $\mathcal{C}F$ als Faser. Ferner ist es leicht zu sehen, dass es einen geometrischen, zulässigen Atlas besitzt. Da B geschlossen ist, gilt der L^2 Satz von Stokes auf (B,g^B) und unter den Voraussetzungen an F in Theorem A.4 folgt aus Theorem A.3, dass der L^2 Satz von Stokes auch auf $\mathcal{C}F$ gilt.

References

[AA] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory*, Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002.

[BL1] J. Brüning and M. Lesch, *Hilbert complexes*, J. Funct. Anal. **108** (1992), no. 1, 88–132.

[BL2] ———, *Kähler-Hodge theory for conformal complex cones*, Geom. Funct. Anal. **3** (1993), no. 5, 439–473.

[BL3] ———, *Some Remarks on tensor products of unbounded Operators and Hilbert complexes*, unpublished.

[BS] J. Brüning and R. Seeley, *An index theorem for first order regular singular operators*, Amer. J. Math. **110** (1988), no. 4, 659–714.

[C1] J. Cheeger, *On the spectral geometry of spaces with cone-like singularities*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 5, 2103–2106.

[C2] ———, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 91–146.

[C3] P. R. Chernoff, *Essential self-adjointness of powers of generators of hyperbolic equations*, J. Functional Analysis **12** (1973), 401–414.

[G] M. P. Gaffney, *A special Stokes's theorem for complete Riemannian manifolds*, Ann. of Math. (2) **60** (1954), 140–145.

[GL] D. Grieser and M. Lesch, *On the L^2-Stokes theorem and Hodge theory for singular algebraic varieties*, Math. Nachr. **246/247** (2002), 68–82.

[HM] E. Hunsicker and R. Mazzeo, *Harmonic forms on manifolds with edges*, Int. Math. Res. Not. **52** (2005), 3229–3272.

[M] J.-M. Morvan, *Generalized curvatures*, Geometry and Computing, vol. 2, Springer-Verlag, Berlin, 2008.

[P] R. S. Palais, *Seminar on the Atiyah-Singer index theorem*, Annals of Mathematics Studies, No. 57, Princeton University Press, Princeton, N.J., 1965.

[RS1] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York, 1972.

[RS2] ———, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975.

[S] M. A. Shubin, *Pseudodifferential operators and spectral theory*, 2nd ed., Springer-Verlag, Berlin, 2001. Translated from the 1978 Russian original by Stig I. Andersson.

[T] F. Trèves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.