Maximal π-points and modules of constant Jordan type

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April 30, 2007

Objectives:

The purpose of this talk is two-fold:

- To introduce a "new, improved" support variety

$$M \mapsto \Gamma(G)_M.$$

- To begin our discussion of the curious class $\mathcal{C}(kG)$ of modules of constant Jordan type

This is part of our effort to introduce new constructions and new invariants for the stable module category $stmod(kG)$.

Notation

k - field of char $p > 0$.

G - finite group scheme defined over k

kG -group algebra (finite dim'l cocommutative Hopf algebra over k)

$k[G] = kG^{\#}$ - coordinate algebra of underlying k-scheme

M a kG-module, K/k field extension, then consider:

$M_K = M \otimes_k K$ as a $G_K = G \times_{\mathsf{Spec}\, k} \mathsf{Spec}\, K$-module.

$\mathrm{JType}(\alpha_K^*(M_K))$ of M at the π-point $\alpha_K : K[t]/t^p \to KG$.

Maximal π-points for M

Definition. *(Dominance ordering) Let* $\underline{\lambda} = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$, $\underline{\nu} = \nu_1 \geq \nu_2 \geq \ldots \geq \nu_N$ *be partitions of* N. *We say that* $\underline{\lambda} \geq \underline{\nu}$ *iff* $\sum_1^i \lambda_i \geq \sum_1^i \nu_i$ *for any* i.

Definition. *Let* M *be a* kG*-module. A* π*-point* $\alpha_K : K[t]/t^p \to KG_K$ *is said to be maximal for* M *if there does* not *exist any* π*-point* $\beta_L : L[t]/t^p \to LG_L$ *such that* $\mathrm{JType}(\beta_L^*(M_L)) > \mathrm{JType}(\alpha_K^*(M_K))$.

Remark: There is no obvious relationship between the dominance ordering of π-points and the relation of specialization $\alpha_K \downarrow \beta_L$ which is defined in terms of "local projectivity" of kG-modules.

However, we do have the analogue for maximal Jordan types of the *generic Jordan type* discussed in Julia's talk.

Theorem. *(Friedlander-Pevtsova-Suslin). If M is a finite dimensional kG-module and if $\alpha_K : K[t]/t^p \to KG_K$ is a π-point maximal for M, then any π-point $\beta_L : L[t]/t^p \to LG_L$ $equivalent$ to α_K is also maximal for M.*

Idea of proof: Reduce to case in which G is commutative. Then $\Pi(G)$ is irreducible, and use result concerning generic Jordan type.

Example: Let $G = GL(3, \mathbb{F}_p)$ with $p > 3$. $\Pi(G)$ is the union of 3 irreducible components, each line meeting the other two.

Let M be the standard 3-dimensional G-module. Then $\Pi(G)_M = \Pi(G)$.

On the other hand, the maximal type occurs on the open subset of one of these three lines obtained by removing the intersection points.

Example: Let $G = (\mathbb{Z}/p)^p \rtimes \mathbb{Z}/p$. Letting M be the kG-module obtained by inducing up the trivial module k from $(\mathbb{Z}/p)^p$, we find that the Jordan type at the generic point corresponding to this maximal elementary abelian p-subgroup is $p[1]$ which is not maximal. M restricted to the generic point of the other point is projective, so the maximal Jordan type is $[p]$.

With a bit of effort, we can construct a kG-module M' which has non-comparable Jordan types at the two generic points, so that M' has two *distinct* maximal Jordan types.

We conclude that a generic Jordan type does not have to be maximal; on the other hand, the next theorem asserts that maximality is an open condition, so that maximal does not imply generic.

$\Gamma(G)_M$, the non-maximal support variety

Theorem. *(F-Pevtsova-Suslin) Assume* $\tilde{H}^\bullet(G,k) \neq 0$. *Then the subset of* $\Pi(G)_M$ *of equivalence classes of* π*-points which are* not *maximal for a given finite dimensional* kG*-module* M,

$$\Gamma(G)_M \ \subset \ \Pi(G),$$

is a proper closed subset. We call this the non-maximal support variety of M.

Remarks:

- We conclude that any generic point with a specialization which has maximal Jordan type also has maximal Jordan type.
- $\Gamma(G)_M \subset \Pi(G)_M \subset \Pi(G)$; we have equality if and only if $\Pi(G)_M \neq \Pi(G)$.

Thus, $M \mapsto \Gamma(G)_M$ is a *finer* invariant than the usual cohomological support variety.

Warning: Generally speaking,

- $$\Gamma(G)_{M_1 \oplus M_2} \neq \Gamma(G)_{M_1} \cup \Gamma(G)_{M_2}.,$$

- $$\Gamma(G)_{M_1 \otimes M_2} \neq \Gamma(G)_{M_1} \cap \Gamma(G)_{M_2}.$$

Specialization and strict specialization

Recall: Concept of specialization in algebraic geometry, thus in $\mathrm{Proj}\, H^\bullet(G,k)$, and the corresponding concept in $\Pi(G)$. Indeed, $\alpha_K \sim \beta_L$ if and only if $\alpha_K \downarrow \beta_L, \beta_L \downarrow \alpha_K$.

Definition. *We say that the π-point α_K of G $\mathit{strictly}$ $\mathit{specializes}$ to β_L (and write $\alpha_K \downdownarrows \beta_L$), if there exists a commutative local domain R over k with field of fractions K and residue field L, together with a map of R-algebras $\nu_R : R[t]/t^p \to RG$ such that*

$$\nu_R \otimes_R K = \alpha_K, \quad \nu_R \otimes_R L = \beta_L.$$

The following theorem clarifies the role of specialization of π-points and ordering of Jordan types.

Theorem. *Let G be a finite group scheme and let $\alpha_K,\ \beta_L$ be π-points of G with $\alpha_K \downarrow \beta_L$. Then there exist π-points $\alpha'_{K'} \sim \alpha_K$ and $\beta'_{L'} \sim \beta_L$ such that $\alpha'_{K'} \downdownarrows \beta'_{L'}$.*

Furthermore, if $\alpha_K \downdownarrows \beta_L$ and if M is any finite dimensional kG-module, then

$$\mathrm{JType}(\alpha^*_K(M_K)) \ \geq \ \mathrm{JType}(\beta^*_L(M_L)).$$

Modules of constant Jordan type

Definition. *Let G be a finite group scheme and M a finite dimensional kG-module. Then M is said to be of constant Jordan type if $\Gamma(G)_M = \emptyset$.*

Proposition. *Let G be a finite group scheme and let M be a kG-module. Then the following are equivalent:*

- *M has constant Jordan type.*
- $\mathrm{JType}(\alpha_K^*(M_K))$ *is the $same$ for every π-point α_K of G.*
- *Each point of $\Pi(G)$ has a representative π-point yielding the same Jordan type for the pull-back of M.*
- *Whenever $\alpha_K \downdownarrows \beta_L$,*

$$\mathrm{JType}(\alpha_K^*(M_K)) \ = \ \mathrm{JType}(\beta_L^*(M_L)).$$

First examples

- Trivial module k ($\mathrm{JType}(k) = [1]$).
- Heller shifts of the trivial module: $\Omega^i(k)$; $\mathrm{JType}(\Omega^i(k)) = n[p] + [1]$ if i is even, or $\mathrm{JType}(\Omega^i(k)) = n[p] + [p-1]$ if i is odd.
- $G = SL_{2(1)}$ and M any rational Sl_2-module.
- G an elementary abelian p-group and

$$M = I^m/I^n, \; n \geq m,$$

where $I \subset kG$ is the augmentation ideal.

Closure properties

Let $\mathcal{C}(kG)$ be the class of finite dimensional modules of constant Jordan type.

Proposition. *Let G be a finite group scheme and consider $M_1,\ M_2\ \in \mathcal{C}(kG)$.*

- $M_1 \oplus M_2\ \in \mathcal{C}(kG)$.
- $M_1 \otimes M_2\ \in \mathcal{C}(kG)$ *(see next page).*
- $Hom_k(M_1, M_2)\ \in \mathcal{C}(kG)$.
- $\Omega^i(M_1) \in \mathcal{C}(kG)$ *for any $i \in \mathbb{Z}$.*

Moreover, if M is an indecomposable, non-projective kG-module of constant Jordan type, then $\tau(M)$ defined by the Auslander-Reiten almost split sequence

$$0 \to \tau(M) \to E \to M \to 0$$

is of constant Jordan type if and only if $M \in \mathcal{C}(kG)$.

Tensor products

Remark: A typical π-point is not a map of Hopf algebras, so that $\alpha_K^*(M_{1,K} \otimes_K M_{2,K})$ is not necessarily isomorphic to $\alpha_K^*(M_{1,K} \otimes_K \alpha_K^*(M_{2,K})$.

Theorem. *Let G be a finite group scheme and consider finite dimensional kG-modules $M_1,\ M_2$. If the π-point α_K has maximal Jordan type on both M_1 and M_2, then*

$$\alpha_K^*(M_{1,K} \otimes_K M_{2,K}) \ \simeq \ \alpha_K^*(M_{1,K} \otimes_K \alpha_K^*(M_{2,K}).$$

Warning: There exists an example consisting of a finite group scheme G, finite dimensional kG-modules $M_1,\ M_2$, and a π-point α_K which has maximal Jordan type on both M_1 and M_2 but does not have maximal Jordan type on $M_1 \otimes M_2$.

Corollary. *Assume that* $End_k(M) \simeq k \oplus proj$ *(i.e.,* M *is endotrivial). Then* $M \in \mathcal{C}(kG)$ *and* $\mathrm{JType}(M)$ *equals either* $m[p] + [1]$ *or* $m[p] + [p-1]$*.*

Proposition. *Let* G *be a finite group scheme and* M *a* kG*-module of constant Jordan type. Then any* kG*-summand of* M *is also of constant Jordan type.*

Sketch of Proof: Assume $M \simeq M_1 \oplus M_2$ as kG-modules. Let $[\alpha_K] \in \Pi(G)$ be a generic point at which $\alpha_K^*(M_{1,K})$ is maximal. Then for any representative α'_F of $[\alpha_K]$, $(\alpha'_F)^*(M_{2,F})$ is minimal. Hence for any strict specialization $\alpha'_F \downdownarrows \beta_L$, $\mathrm{JType}((\alpha'_F)^*(M_{2,F})) = \mathrm{JType}(\beta_L^*(M_{2,L}))$. Thus, $\mathrm{JType}(\alpha_K^*(M_1)) = \mathrm{JType}(\beta_L^*(M_{2,L}))$.

Now use the fact that $\Pi(G)$ is connected.

Here is an "essentially trivial" method of constructing modules of constant Jordan type.

Proposition. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of* kG*-modules. If* M_1, M_3 *are modules of constant Jordan type and if the pull-back of this sequence via any* π*-point* α_K *is a split short exact sequence of* $K[t]/t^p$*-modules, then* M_2 *is also of constant Jordan type.*

Example: Let $\zeta \in H^{2n}(G,k)$ be nilpotent and consider the associated short exact sequence

$$0 \to k \to M \to \Omega^{2n-1}(k) \to 0.$$

Then M has constant Jordan type.

Example: Assume that $\Pi(G)$ has dimension at least 1. Let $\zeta \in \hat{H}^n(G,K)$ be an element of the Tate cohomology of G of negative degree. Consider the associated short exact sequence

$$0 \to k \to M \to \Omega^{n-1}(k) \to 0.$$

Then M has constant Jordan type.

Generalized Carlson modules

Definition. *Let G be a finite group scheme and let $\zeta_1, \ldots, \zeta_r \in H^*(G,k)$ be homogenous elements of degrees $n_1, \ldots, n_r$. We define*

$$L_{\zeta_1,\ldots,\zeta_r} \equiv \mathit{Ker}\{\sum_i \hat{\zeta}_i : \bigoplus \Omega^{n_i}(k) \to k\},$$

where $\hat{\zeta}_i$ corresponds to ζ_i.

Proposition. *If the radical of the ideal generated by $\zeta_1, \ldots, \zeta_r \in H^\bullet(G,k)$ is the augmentation ideal, then $L_{\zeta_1,\ldots,\zeta_r}$ is a module of constant Jordan type.*

Proposition. *If the radical of the ideal generated by the Bocksteins of $\zeta_1, \ldots, \zeta_r \in H^{odd}(G,k)$ is the augmentation ideal, then $L_{\zeta_1,\ldots,\zeta_r}$ is a module of constant Jordan type.*

Auslander-Reiten components

Theorem. *Let G be a finite group scheme, and let M be an indecomposable non-projective module of constant Jordan type. Let Θ be a component of the stable Auslander-Reiten quiver of kG containing the module M. Assume that one of the following conditions hold: either all vertices of Θ are absolutely indecomposable, or k is perfect. Then for any $[N] \in \Theta$, the module N has constant Jordan type.*

Sketch of proof: Use the facts that $\tau(M) \in \mathcal{C}(kG)$, that the restriction of any almost split sequence to $K[t]/t^p$ actually splits, and that any summand of a module in $\mathcal{C}(kG)$ is also in $\mathcal{C}(kG)$.

Corollary. *There exists an indecomposable module of constant Jordan type $n[1] + [\mathrm{proj}]$ for any positive interger n.*

Questions

Question: What (stable) Jordan types are of the form $\mathrm{JType}(M)$ for $M \in \mathcal{C}(G)$?

Conjecture. Assume $p > 2$, $G = (\mathbb{Z}/p)^{\times r}$, $r > 2$. There does NOT exist a G-module of constant Jordan type $[2] + [\mathrm{proj}]$.

Question: Can we formulate natural invariants which distinguish some kG-modules M_1 $M_2 \in \mathcal{C}(G)$ with $\mathrm{JType}(M_1) = \mathrm{JType}(M_2)$.

(For example, if $M \in \mathcal{C}(G)$, then

$$\mathrm{JType}(M) \;=\; \mathrm{JType}(M^{\#}) \;=\; \mathrm{JType}(\tau(M)).)$$