Π-points, cohomology and support spaces

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Finite Group Schemes

Definition

A finite group scheme G is a functor

$$G : \text{fin. gen. comm. } k\text{-algebras} \longrightarrow \text{groups}$$

represented by a finite dimensional commutative Hopf algebra $k[G]$.

$kG := k[G]^{\#}$ is the **group algebra** of G, a *finite dimensional, cocommutative Hopf algebra*.

Representations of G = modules over the ring kG

Examples

- Finite groups.

$$\pi \rightsquigarrow k\pi$$

- Restricted Lie algebras.
  p-restriction map $\quad [p] : g \to g$

$$u(g) = U(g)/\langle x^p - x^{[p]} \rangle$$

restricted enveloping algebra (fin. dim. cocomm. Hopf algebra).

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Frobenius Kernels

- $F : GL_n \xrightarrow{(a_{ij})\to(a_{ij}^p)} GL_n$
  $GL_{n(r)} \stackrel{\text{def}}{=} \operatorname{Ker} F^{(r)}$ is the r*-th Frobenius kernel* of GL_n.
- $\mathcal{G}$ - any group scheme. $\mathcal{G}_{(r)} = \operatorname{Ker}\{F^r : \mathcal{G} \to \mathcal{G}^{(r)}\}$
- Frobenius kernels = *infinitesimal* finite group schemes (geometrically, they only have one point).
- Connection to restricted Lie algebras:

$$u(\operatorname{Lie}\mathcal{G}) \simeq k\mathcal{G}_{(1)}$$

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Cyclic Shifted Subgroups

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Let $E = (\mathbb{Z}/p)^{\times r}$ be an elementary abelian group. Choose generators $g_1, \ldots, g_r$. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in K^r$ for some field extension K/k. A shifted cyclic subgroup $< \underline{\alpha} >$ of E corresponding to $\underline{\alpha}$ is a cyclic subgroup of KE generated by a p-unipotent element $\alpha_1(g_1 - 1) + \cdots + \alpha_r(g_r - 1) + 1$.

Theorem (Dade, 1978; Benson-Carlson-Rickard, 1996)

A kE-module M is projective if and only if the restriction of M to every cyclic shifted subgroup is projective.

Cyclic shifted subgroups are parametrized by the affine space $\mathbb{A}^n_k$.

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Varieties for elementary abelian p-groups

Definition (Rank variety)

$$V_E(M) = \{\underline{\alpha} \in \mathbb{A}^n_k : \text{ such that } M\downarrow_{\langle\alpha\rangle} \text{ is not free } \} \cup \{0\}$$

- Independent of the choice of generators of E!
- Independence follows from the “Carlson’s conjecture":

Theorem (Avrunin-Scott)

An isomorphism $V_E(k) \simeq \operatorname{Spec} \mathrm{H}^\bullet(E,k)$ restricts to

$$V_E(M) \simeq \operatorname{Spec}(\mathrm{H}^\bullet(E,k)/Ann_{\mathrm{H}^\bullet(E,k)} \operatorname{Ext}^*_E(M,M))$$

for any finite-dimensional kE-module M.

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Varieties for Lie algebras

Theorem (FP, AJ, SFB)

Let g be a restricted Lie algebra. Then $\operatorname{Spec} \mathrm{H}^*(u(g), k) \simeq \mathcal{N}^{[p]}$

where $\mathcal{N}^{[p]} = \{x \in g \mid x^{[p]} = 0\}$

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- Avrunin-Scott “type" result holds here as well.
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Definition (π-point)

A **π-point** α_K of a finite group scheme G (defined over a field extension K/k) is a map of algebras

$$\begin{tikzcd} K[t]/t^p \arrow[r, equal] & K\mathbb{Z}/p \arrow[rr, dashed, "\alpha_K"] \arrow[dr, dashed] & & KG_K \\ & & KA \arrow[ur] & \end{tikzcd}$$

which factors through some unipotent abelian subgroup scheme $A \subset G_K$.

π-points

- Examples of π-points: cyclic shifted subgroups, p-nilpotent elements in restricted Lie algebras, one parameter subgroups (with a little twist).
- Connection to cohomology:

Proposition

A π-point $\alpha_K : K[t]/t^p \to KG$ induces a non-trivial map in cohomology: $\alpha_K^* : \mathrm{H}^\bullet(G, k) \to \mathrm{H}^\bullet(K[t]/t^p, K) \simeq K[X]$.

- π-point $\rightsquigarrow$ a point on $\mathrm{Proj}\, \mathrm{H}^\bullet(G, k)$.
- Too many π-points collapse.
- There is a purely representation-theoretic solution.

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Equivalence of π-points

Let M be a G-module, $\alpha_K : K[t]/t^p \to KG$ be a π-point.
$\alpha_K^* M$ = restriction of M_K to $K[t]/t^p$ via α_K.

Definition

Let α_K, β_L be two π-points of G.
$\alpha_K \sim \beta_L \iff$ for any finite-dimensional G–module M, $\alpha_K^* M$ is free if and only if $\beta_L^* M$ is free.
$\alpha_K \downarrow \beta_L \iff$ for any finite dimensional kG-module M, $\beta_L^*(M)$ being free implies that $\alpha_K^*(M)$ is free.

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Support Space

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Support space of a finite group scheme G:

$$\Pi(G) =< \pi\text{-points} > / \sim$$

Support space of a G-module M:

$$\Pi(G)_M =< [\alpha_K] : K[t]/t^p \to KG : \alpha_K^* M \text{ is not free} >$$

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Isomorphism

- Topology: Closed sets = $\Pi(G)_M$ for all finite dimensional modules M. This defines (Zariski) topology on $\Pi(G)$.
- Moreover, $\Pi(G)$ can be made into a scheme still using representation theory *only*.
- Define $\Psi : \Pi(G) \to \operatorname{Proj} \mathrm{H}^\bullet(G, K)$ via
$$\{\alpha_K : K[t]/t^p \to KG\} \mapsto$$
$$\operatorname{Ker}\{\alpha_K^* : \mathrm{H}^\bullet(G, k) \to \mathrm{H}^\bullet(K[t]/t^p, K)\} \cap \mathrm{H}^\bullet(G, k)$$

Theorem (Friedlander-P.)

$$\Psi : \Pi(G) \to \operatorname{Proj} \mathrm{H}^\bullet(G, k)$$

is a homeomorphism which sends $\Pi(G)_M$ to $\operatorname{Proj} |G|_M$, where $|G|_M$ is the "cohomological support scheme" of the module M.

Proof ingredients

- Ψ is well-defined on equivalence classes of π-points: We use Carlson's modules L_ξ here.
- Well-defined onto Proj: "non-trivial map" in cohomology proposition.
- Injectivity:step by step reduction to the abelian case.
- Surjectivity: Detection modulo nilpotents, Suslin's generalization of the Quillen's original theorem for finite groups.

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Theorem (Detection of nilpotents, Suslin)

Let G be a finite group scheme, Λ be a unital associative G-algebra, and $\zeta \in \mathrm{H}^\bullet(G, \Lambda)$ be a homogeneous cohomology class of even degree. Then ζ is nilpotent if and only if ζ_K restricts to a nilpotent class in the cohomology of every subgroup scheme of $\mathcal{E}_K$ of G_K, where $\mathcal{E}_K \simeq \mathbb{G}_{a(r)} \times (\mathbb{Z}/p)^{\times n}$

Theorem (Detection of Projectivity)

Let G be a finite group scheme, M be a any G-module. Then M is projective if and only if $\Pi(G)_M \neq 0$.

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Examples

- $E = (\mathbb{Z}/p)^{\times r}$. Then $\Pi(E) \simeq \mathbb{P}^{r-1} \simeq \text{Proj}\, V_E(k)$.
- π-finite group. Then $\Pi(\pi)$ is given by the Quillen stratification theorem.
- g-restricted Lie algebra. Then $\Pi(g) \simeq \text{Proj}\, \mathcal{N}^{[p]}(g)$.
- $GL_{n(r)}$. Here, the Π-space is the variety of r-tuples of pair-wise commuting p-nilpotent matrices.

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Local Jordan type

- An equivalent way to look at the $\Pi(G)_M$:
  - The structure of $K[t]/t^p$-module $\alpha_K^* M$ is determined by the p-nilpotent operator $\alpha_K(t) \in End(M_K)$.
  - $[\alpha_K] \in \Pi(G)_M \leftrightarrow \alpha_K^*(M)$ is not projective $\leftrightarrow$ the Jordan form of $\alpha_K(t)$ does not consist only of blocks of size p.
- Good thing: being projective does not depend on a representative of a π-point. But for lots of modules $\Pi(G)_M$ is the entire space $\Pi(G)$.
- Look for "finer" information. Fix a module M. Attach $\alpha_K^*(M)$ to any π-point α_K.
- Problem: this DOES depend on a representative.

Example

Example

$E = \mathbb{Z}/p \times \mathbb{Z}/p$, $p > 2$, $kE = k[x, y]/(x^p, y^p)$.
$M = k[x, y]/(x^2 - y, y^p)$ - an E-module.
Define π-points $\alpha, \beta : k[t]/t^p \to kE$ by

$$\alpha(t) = x^2 - y, \text{ and } \beta(t) = y.$$

Then $\alpha \sim \beta$ because they have proportional “linear" parts.
$\alpha^* M$ is a trivial $k[t]/t^p$-module.
$\beta^* M$ has two Jordan blocks of sizes $\frac{p+1}{2}$ and $\frac{p-1}{2}$.

Generic π-points

- $\Pi(G) \simeq \operatorname{Proj} \mathrm{H}^\bullet(G, k)$.
- Let x be a generic point of $\operatorname{Proj} H^\bullet(G, k)$ (the closure of x in the Zariski topology is an irreducible component of $\operatorname{Proj} H^\bullet(G, k)$)

$$x \leftrightarrow [\alpha_K] \in \Pi(G)$$

- A π-point $\alpha_K : K[t]/t^p \to KG$ is called **generic** if its equivalence class corresponds to a generic point of $\operatorname{Proj} H^\bullet(G, k)$.
- Generic π-points are defined over large transcendental field extensions.

Example

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$E = (\mathbb{Z}/p)^{\times r}$, $kE \simeq k[t_1, \dots, t_r]/\langle t_j^p \rangle$. Let $K = K(x_1 \dots, x_r)$.

$$\alpha_K(t) = x_1 t_1 + \dots x_r t_r$$

defines a generic π-point $\alpha_K : K[t]/t^p \to KE$.

Remark

$\alpha_K(t) = x_1 t_1 + \dots x_r t_r + (t_1 + \dots t_r)^{17}$ also defines a generic point.

Jordan type of a G-module at a generic π-point DOES NOT depend on a representative!

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Definition and Main Theorem

Theorem

Let $\alpha_K, \beta_K : K[t]/t^p \to KG$ be two equivalent GENERIC π-points of G. Then for any finite-dimensional module M, we have

$$\alpha_K^* M \simeq \beta_K^* M$$

I.e. $\alpha_K(t), \beta_K(t)$ have the same Jordan type as nilpotent operators on M_K.

Definition

Let $\alpha_K : K[t]/t^p \to KG$ be a generic π-point of G. The **generic Jordan type** of a finite dimensional G-module M is the Jordan type of $\alpha_K(t)$ as a nilpotent operator on M.

Example revisited

For $G = E$, an elementary abelian p-group, the theorem takes a very explicit form:

Example

$E = (\mathbb{Z}/p)^{\times r}$, $kE \simeq k[t_1, \ldots, t_r]/\langle t_j^p \rangle$. Let $K = K(x_1 \ldots, x_r)$. Let $\alpha_K, beta : K[t]/t^p \to KE$ be two generic π-points. Then up to a scalar, $\alpha_K(t) = \beta_K(t) +$ terms of degree ≥ 2. The theorem says that $\alpha_K^*(M) \simeq \beta_K^*(M)$ for any M.

Properties

The association $M \mapsto [\alpha_K]^*(M)$ for a generic α_K can be viewed as a functor

$$[\alpha_K]^* : \quad \text{mod } G \longrightarrow \quad \text{mod } \mathbb{Z}/p$$

and has many nice properties:

- $[\alpha_K]^*(M \oplus N) \simeq [\alpha_K]^*(M) \oplus [\alpha_K]^*(N)$
- $[\alpha_K]^*(M \otimes N) \simeq [\alpha_K]^*(M) \otimes [\alpha_K]^*(N)$
- $[\alpha_K]^*(\Omega M) = \Omega([\alpha_K]^*(M))$

Alternatively,

$$[\alpha_K]^* : \text{stmod } G \longrightarrow \text{stmod } \mathbb{Z}/p$$

is a tensor-triangulated functor.

Jordan types $\sim$ partitions.

- If $\Pi(G)$ is irreducible, then the generic Jordan type of any module is maximal in the dominance ordering on partitions.
- In the reducible case, all sorts of weird things happen.

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