Endomorphism rings of generators-cogenerators

(= the artin algebras of dominant dimension at least two)

A artin algebra, e.g. finite dimensional over a field k

$M \in A$-mod is a generator-cogenerator :$\Leftrightarrow$

M is a generator: $\operatorname{add} M \ni$ all projectives

and M is a cogenerator: $\operatorname{add} M \ni$ all injectives

$B := \operatorname{End}({}_A M)$

Does B have any properties, in general?

Fix A, M, write

$$M = \Pi \oplus P \oplus I \oplus N$$

each indec summand:	projective + injective	projective, not injective	injective, not projective	neither projective nor injective

Ignore multiplicities. Then e.g. $A = \Pi \oplus P$

$\exists e = e^2 \in B : e = id_{\Pi \oplus P}$

$eBe = A$

$${}_A M_B = \mathrm{Hom}_A(A, M) = \mathrm{Hom}_A(\overline{u} \oplus P, M) = e \cdot \mathrm{Hom}_A(M, M)$$

$$= e \cdot B \quad \text{right } B\text{-projective}$$

$${}_B B = \mathrm{Hom}_A(M, M) = \underbrace{\mathrm{Hom}_A(M, \overline{u} \oplus I)}_{\text{dual of } M} \oplus \mathrm{Hom}_A(M, P \oplus N)$$

left B-projective

$\Rightarrow$ M is right B-injective

so $M = e\,B$ is right B-projective-injective

$$\left.\begin{aligned} B &= \mathrm{End}_A(M) \\ A &= \mathrm{End}_B(M) = eBe \end{aligned}\right\}$$

double centraliser property

M balanced bimodule

$A_1 \to A_0 \to {}_A M \to 0$ projective presentation

$$0 \to \underset{\underset{B}{\shortparallel}}{\mathrm{Hom}_A(M, M)} \to \mathrm{Hom}_A(A_0, M) \to \mathrm{Hom}_A(A_1, M)$$

$\in \mathrm{add}(eB)$,
since $A_0, A_1 \in \mathrm{add}(\overline{u} \oplus P)$
projective-injective over B

B has an injective resolution

$$0 \to B \to \underbrace{I_0 \to I_1}_{\text{projective}} \to \ldots$$

An algebra Λ has dominant dimension at least n

$\operatorname{domdim} \Lambda \geq n$, if and only if Λ has an injective resolution (left or right)

$$0 \to \Lambda \to I_0 \to I_1 \to \_ \to I_{n-1} \to I_n \to -$$

with $I_0, -, I_{n-1}$ projective

Example: Λ self-injective $\Rightarrow \operatorname{domdim} \Lambda = \infty$

LARGE values of domdim are desirable!

$\operatorname{domdim} B \geq 1 \iff$ B has a faithful projective-injective module eB

$\operatorname{domdim} B \geq 2 \iff$ in addition, eB is a balanced bimodule, i.e. there is a double centraliser property relating B and $A = eBe$:

$A = \operatorname{End}_B(eB)$

$B = \operatorname{End}_A(eB)$

Examples

$B = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ $\qquad A = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} = k[x]/x^2$

proj-inj

domdim $B = 2$

$$0 \to \underbrace{\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}}_{B} \to \underbrace{\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}}_{\text{proj-inj}} \to \underbrace{\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}}_{\text{proj-inj}} \to \underbrace{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}_{\text{notproj}}$$

$$B = \underbrace{\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}}_{\text{proj-inj}} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \qquad A = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}$$

domdim $B = 4$

$$0 \to \begin{smallmatrix} B \\ 3 \end{smallmatrix} \to \underbrace{\begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \to \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix}}_{\text{proj-inj}} \to \underbrace{\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}}_{\text{notproj}}$$

$$B = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 2 & & 4 \\ & 3 & \end{smallmatrix} \oplus \begin{smallmatrix} 4 \\ 3 \end{smallmatrix}$$

domdim $B = 6$

...

General theory (Morita, Tachikawa, B. Müller, ...)

Theorem: dom dim $B \geq 2 \iff B =$ endomorphism algebra of some generator-cogenerator M over some A

correspondence: algebras of dom dim ≥ 2 $\longleftrightarrow$ gen-cogens

B $\qquad\qquad\qquad\qquad$ ${}_AM$

(A is determined: $A = eBe$, eB faithful projective-injective)

for a self-contained proof see:

C. M. Ringel, Artin algebras of dominant dimension at least 2
BIREP, Selected topics, Dec 5, 2007

Particular case: A of finite representation type

$M = \bigoplus$ full set of indecomposables

$B = \mathrm{End}_A(M)$, so dom dim $B \geq 2$

moreover: gl dim $B \leq 2$

check: $f: B_1 \to B_2$, $B_1, B_2 \in \mathrm{add}(B)$, has projective kernel

$$B_1 \xrightarrow{f} B_2$$

$$M \underset{B}{\otimes} B_1 \xrightarrow{1 \otimes f} M \underset{B}{\otimes} B_2 \text{ has kernel } N \underset{\text{add}(M)}{\in} A\text{-mod}$$

$$\begin{array}{ccc} \mathrm{Hom}_A(M, M \otimes B_1) & \rightarrow & \mathrm{Hom}_A(M, M \underset{B}{\otimes} B_2) \\ \shortparallel & f & \shortparallel \\ B_1 & & B_2 \end{array}$$

has kernel $\mathrm{Hom}_A(M, N) \in \mathrm{add}\, B$

Theorem (Auslander): $\exists$ correspondence

$B = \mathrm{End}_A(\oplus$ full set of indec$)$ over A of finite type

$\Updownarrow$

$\mathrm{gldim}\, B \leq 2$ and $\mathrm{domdim}\, B \geq 2$

$\{A$ of finite type$\}/_\sim \longleftrightarrow \{B : \mathrm{gldim}\, B \leq 2, \mathrm{domdim}\, B \geq 2\}/_\sim$

Auslander algebras

given B, set $A = eBe$ (eB faithful projective-injective)

show: ${}_A eB = \oplus$ full set of indecs

let $X \in A$-mod

eB is A-generator-cogenerator ($= M$ above)

resolve X: $0 \to X \to eB_1 \to eB_2$, $eB_1, eB_2 \in \mathrm{add}(eB)$

$0 \to \mathrm{Hom}_A(eB, X) \to \underbrace{\mathrm{Hom}_A(eB, eB_1) \to \mathrm{Hom}_A(eB, eB_2)}_{\in\, \mathrm{add}(B)}$

$\mathrm{Hom}_A(eB, X)$ = kernel of map between B-projectives

$\Rightarrow \mathrm{Hom}_A(eB, X) \in \mathrm{add}\, B \Rightarrow X = \mathrm{Hom}_A(eBe, X) \in \mathrm{add}(eB)$

so: A finite type $\Leftrightarrow \mathrm{repdim}\, A \leq 2$

a homological characterisation of domdim

<u>Theorem</u> (B. Müller): Suppose domdim $B \geq 2$, eB faithful and projective-injective. Then

$$\text{domdim } B \geq 2+n \iff \operatorname{Ext}^i_{\underset{A}{\overset{\shortparallel}{eBe}}}(eB, eB) = 0 \text{ for } 1 \leq i \leq n$$

<u>Conjecture</u> (Nakayama): B self-injective $\iff$ domdim $B = \infty$

by Müller's theorem: domdim $B < \infty \Rightarrow \operatorname{Ext}^i_A(eB, eB) \neq 0$ for some i

<u>Müller's version</u> of Nakayama's conjecture:

M generator-cogenerator over A, $\operatorname{Ext}^i_A(M, M) = 0$ $\forall i > 0$
$\Rightarrow M$ projective (+injective)

<u>Tachikawa's version</u> of Müller's version of Nakayama's conjecture

(1) $\operatorname{Ext}^i_A(A\text{-inj}, A\text{-proj}) = 0$ $\forall i > 0 \Rightarrow A$ self-injective

(2) A self-injective, M generator, $\operatorname{Ext}^i_A(M, M) = 0$ $\forall i > 0$
$\Rightarrow M$ projective

Nakayama's conjecture is implied by finitistic dimension conjecture

for a survey see: K. Yamagata, Frobenius algebras. In: Handbook of algebra, Vol. I, 841-887, 1996.

Examples (joint with Ming Fang)

Schur algebras: $k = \bar{k}$, $V = k^n$, $n \geq r$

$$(K^n)^{\otimes r}$$

$$k\Sigma_r \qquad S_k(n,r) \twoheadleftarrow kGL_n(k)$$

symmetric group — Schur algebra — general linear group (ring)

$$A \qquad \begin{matrix} M \\ \parallel \\ eB \end{matrix} \qquad B$$

double centraliser property = Schur-Weyl duality

($n < r$: no faithful projective-injective module)

<u>Theorem</u> (FK): $0 < p \leq r \leq n$, $p = \operatorname{char} k$, then

$$\operatorname{domdim} S(n,r) = 2(p-1)$$

example: $B = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 2 & & 4 \\ & 3 & \end{smallmatrix} \oplus \cdots \oplus \begin{smallmatrix} & p-1 & \\ p-2 & & p \\ & p-1 & \end{smallmatrix} \oplus \begin{smallmatrix} p \\ p-1 \end{smallmatrix}$

(don't know repdim $S(n,r)$, repdim $k\Sigma_r$,

do know gldim)

blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$

$\mathfrak{g}$ findim semisimple complex Lie algebra

$\mathcal{O} \subset U(\mathfrak{g})$-mod

Ψ

M finitely generated, $\mathfrak{h}$-semisimple, locally $\mathfrak{n}_+$-finite

$\mathcal{O} = \bigoplus$ blocks

fix a block, $\simeq B$-mod, B has an indecomp faithful proj-inj

eB

A $\qquad B$

$\|$

eBe $\qquad$ double centraliser property

$\cap$ $\qquad$ = Soergel's Struktursatz

coinvariant
algebra

$H^\bullet(G/B)$

observation (Fang): domdim $B = 2$ (unless B ss)

(don't know repdim B, repdim A

Sometimes do know gldim B, but this gives an upper bound for repdim A worse than the Bergh-Oppermann bound for complete intersections)

Both, Schur-Weyl duality (in general) and Soergel's Struktursatz can be shown using (relative) dominant dimension [K-Slungård-Xi, 2001]

On the use of LARGE dominant dimensions:

$B = S(n,r)$ or a block of $\mathcal{O}$

(or B quasi-hereditary with a strong duality:

$i : B \to B$ anti-automorphism fixing a complete set of primitive idempotents)

$\Delta(\lambda)$ = standard module for B

Weyl module for $S(n,r)$

Verma module for $\mathcal{O}$

$$B\text{-mod} \xrightarrow[\text{Schur functor}]{e\cdot -} A\text{-mod}$$

rarely an equivalence (but: Schur for $p=0$)

$\Delta(\lambda) \longmapsto S(\lambda)$ (Specht module for $k\Sigma_r$, trivial module for coinvariant algebra)

$$\underset{\substack{\Delta\text{-filtered}\\ \text{modules}}}{\mathcal{F}(\Delta)} \xrightarrow{e\cdot -} \underset{\substack{S\text{-filtered}\\ \text{modules}}}{\mathcal{F}(S)}$$

<u>Theorem</u> (Hemmer-Nakano, 2004): $B = S(n,r)$, $A = k\Sigma_r$, then $\mathcal{F}(\Delta) \xrightarrow{e\cdot -} \mathcal{F}(S)$ is an equivalence of exact categories **unless $p=2$ or $p=3$**

(proof relies on computations of $H^\bullet(k\Sigma_r, k)$ by Kleshchev and Nakano)

$p=2$: $\begin{matrix}1\\2\\1\end{matrix} \oplus \begin{matrix}2\\1\end{matrix}$ and $\begin{matrix}1\\1\end{matrix}$

recall: domdim $S(n,r) = 2(p-1)$

and use (for A, B as above)

Theorem [FK]: $e\cdot -$ induces isomorphisms

$$\mathrm{Ext}^i_B(\overset{\text{any}}{X}, B\text{-proj}) \simeq \mathrm{Ext}^i_A(eX, e\cdot B\text{-proj})$$

for $0 \leq i \leq \text{domdim } B - 2$

$$\mathrm{Ext}^i_B(\overset{\text{any}}{X}, \overset{\mathcal{F}(\Delta)}{\overset{\in}{Y}}) \simeq \mathrm{Ext}^i_A(eX, eY)$$

for $0 \leq i \leq \dfrac{\text{domdim } B}{2} - 2$

($p=2$: nothing, $p=3$: Ext^0)

$\dfrac{\text{domdim } B}{2}$ is an integer:

Theorem [FK]: $\text{domdim } \underset{\substack{\| \\ \text{domdim } R(B) \\ \text{Ringel dual}}}{B} = 2 \cdot \text{domdim } \overset{\substack{\text{char tilting module} \\ \in}}{T}$

[compare a theorem by Mazorchuk and Ovsienko
specialising to: $\text{gldim } B = 2 \cdot \text{projdim } T$],

proofs use counterpart of Müller's theorem:

Theorem [FK]: $\text{domdim } B \geq n \iff$

$$\mathrm{Ext}^i_B(B\text{-inj}, B\text{-proj}) = 0 \text{ for } 1 \leq i \leq n-2$$