Triangulated categories without models

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May 2, 2008 / Bielefeld

Main result

Let $\mathcal{F}(\mathbb{Z}/4)$ denote the category of finitely generated free modules over $\mathbb{Z}/4$.

Theorem

The category $\mathcal{F}(\mathbb{Z}/4)$ has a unique triangulation with the identity shift functor and such that the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact. The triangulated category $\mathcal{F}(\mathbb{Z}/4)$ is neither algebraic nor topological.

$\mathbb{Z}/4$ can be replaced by any commutative local ring $(R, \mathfrak{m})$ with $\mathfrak{m} = (2) \neq 0$ and $\mathfrak{m}^2 = 0$.

Examples: $W_2(k)$ for k a perfect field k of characteristic 2 or the localization of $\mathbb{Z}/4[t]$ at the prime ideal (2).

Contents

Outline of the talk:

- formally define 'algebraic' and 'topological' triangulated categories and
- show by examples that the inclusions

$$\begin{pmatrix} \text{algebraic} \\ \triangle\text{'ed categories} \end{pmatrix} \subset \begin{pmatrix} \text{topological} \\ \triangle\text{'ed categories} \end{pmatrix} \subset (\triangle\text{'ed categories})$$

  are strict:
- the Spanier-Whitehead category $\mathcal{SW}$ is topological, but not algebraic
- the category $\mathcal{F}(\mathbb{Z}/4)$ is neither algebraic nor topological.

Algebraic triangulated categories

Definition

A triangulated category is algebraic if it embeds into the homotopy category $\mathcal{K}(\mathcal{A})$ of an additive category $\mathcal{A}$.

Examples

- The category of k-vector spaces, for a field k, with identity shift functor and exact triangles

  the triangles $$U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} U$$ which are exact.

- $\mathcal{D}(R)$ for a ring R (or dg ring, or dg category)
- $\mathcal{D}$(quasi-coh. $\mathcal{O}_X$-mod) for a scheme X
- Stmod(L) for a Frobenius ring L
- $K_{(p)}$-local stable homotopy category, p odd prime (Franke)

$n \cdot X/n = 0$ for algebraic $\mathcal{T}$

Notation

- $\mathcal{T}$: triangulated category
- $n \cdot X$: n-fold multiple of identity morphism of X
- X/n : cone of $n \cdot X$, part of distinguished triangle

$$X \xrightarrow{n\cdot} X \longrightarrow X/n \longrightarrow X[1]$$

Lemma

If $\mathcal{T}$ is algebraic, then $n \cdot X/n = 0$.

Proof.

- Can assume that $\mathcal{T} = K(\mathcal{A})$ for an additive category $\mathcal{A}$.
- For a chain complex X, the object X/n is given by
  $$(X/n)_k = X_k \oplus X_{k-1}\ , \quad d(x,y) = (dx + ny,\ -dy)\ .$$
- A nullhomotopy $s : (X/n)_k \longrightarrow (X/n)_{k+1}$ of $n \cdot X/n$ is given by $s(x,y) = (0,x)$. □

Topological example: Spanier-Whitehead category

In a general triangulated category, we can have $n \cdot X/n \neq 0$ for suitable X and n...

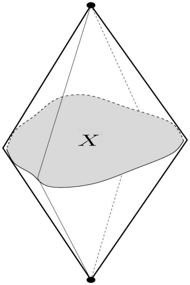
Definition

The Spanier-Whitehead category $\mathcal{SW}$ has

objects: (X, n) with X finite pointed CW-complex, $n \in \mathbb{Z}$

morphisms:

$$\mathcal{SW}((X,n),(Y,m)) = \operatorname{colim}_{k\to\infty}[\Sigma^{k+n}X, \Sigma^{k+m}Y]$$



$[-,-]$: pointed homotopy classes

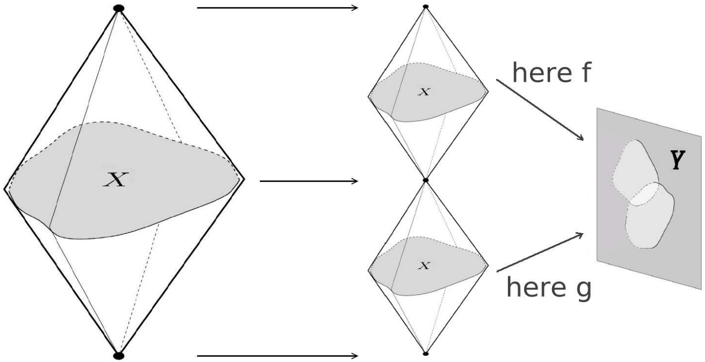
$$\Sigma X = \frac{X \times [0,1]}{X \times \{0,1\} \cup \{*\} \times [0,1]}$$

reduced suspension

Triangulation of Spanier-Whitehead category

$\mathcal{SW}$ is triangulated by:

- shift: $(X, n)[1] = (X, n+1) \cong (\Sigma X, n)$
- addition:

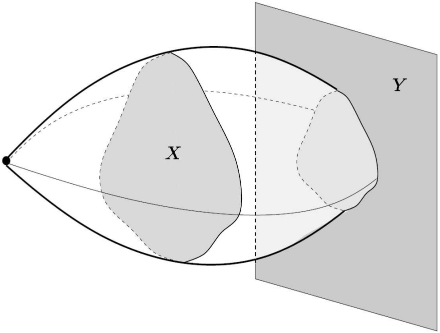


sum in $[\Sigma X, Y]$ of $f, g : \Sigma X \longrightarrow Y$

Triangulation of Spanier-Whitehead category

- exact triangles: mapping cone sequences

$$X \xrightarrow{f} Y \xrightarrow{\text{incl.}} \mathrm{Cone}(f) \xrightarrow{\text{proj.}} \Sigma X$$



mapping cone

$$\mathrm{Cone}(f) = \frac{X \times [0,1] \cup_{X \times \{1\}} Y}{X \times \{0\} \cup \{x_0\} \times [0,1]}$$

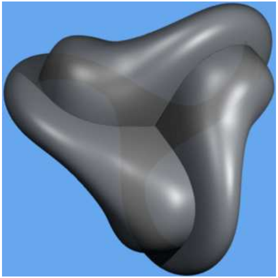
mod-n Moore spectra

Definition

The mod-n Moore spectrum S/n is a cone of multiplication by n on the sphere spectrum $S = (S^0, 0)$ in $\mathcal{SW}$.

More concretely:

$$S/n = (S^1 \cup_n D^2, -1)$$
$$S/2 = (\mathbb{R}P^2, -1)$$



Proposition

The morphism $2 \cdot S/2$ is nonzero in $\mathcal{SW}$.

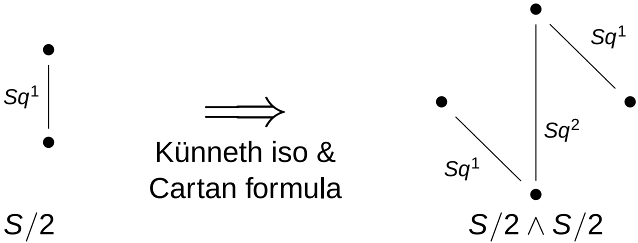
Thus the Spanier-Whitehead category is not algebraic.

[Skip proof]

Proof of $2 \cdot S/2 \neq 0$

Proof by contradiction (folklore).

- Suppose $2 \cdot S/2 = 0$.
  Smash the triangle $S \xrightarrow{2\cdot} S \longrightarrow S/2 \longrightarrow S[1]$ with $S/2$ to obtain a splitting $S/2 \wedge S/2 \cong S/2 \oplus S/2[1]$.
- Use mod-2 Steenrod operations
  $$Sq^i : H^*(X, \mathbb{F}_2) \longrightarrow H^{*+i}(X, \mathbb{F}_2)$$
  to show that $S/2 \wedge S/2$ is indecomposable:



□

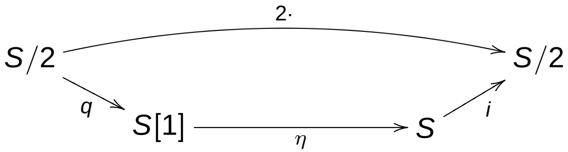
Factorization of $2 \cdot S/2$

Addendum

Given a distinguished triangle

$$S \xrightarrow{2\cdot} S \xrightarrow{i} S/2 \xrightarrow{q} S[1]$$

then the morphism $2 \cdot S/2$ factors as the composite



where η is the stable homotopy class of the Hopf map
$\eta : S^3 \longrightarrow S^2$.

Topological triangulated categories

Definition

A triangulated category is topological if it embeds
into the homotopy category of a stable model category.

stable model category: pointed Quillen model category
with Σ invertible up to homotopy

Examples

- stable homotopy category of spectra
- Spanier-Whitehead category $\mathcal{SW}$
- equivariant, motivic or localized stable homotopy category
- modules over a ring spectrum
- all algebraic triangulated categories are also topological

The ‘exotic’ triangulated category

$\mathcal{F}(\mathbb{Z}/4)$: finitely generated free modules over $\mathbb{Z}/4$

Theorem

The category $\mathcal{F}(\mathbb{Z}/4)$ has a unique triangulation with the identity shift functor and such that the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is exact. The triangulated category $\mathcal{F}(\mathbb{Z}/4)$ is neither algebraic nor topological.

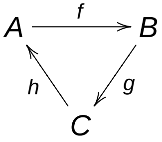
Exact triangles in $\mathcal{F}(\mathbb{Z}/4)$ allow an intrinsic characterization:

[skip characterization]

Intrinsic characterization of exact triangles

(f, g, h) is an exact triangle in $\mathcal{F}(\mathbb{Z}/4) \iff$

$(\triangle)$



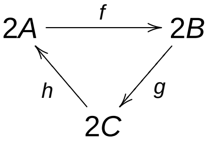
is exact and $\sigma^3 = \mathrm{Id}$.

Here $\sigma : H_*(2\triangle) \longrightarrow H_{*-1}(2\triangle)$ is the boundary homomorphism of the short exact sequence of $\mathbb{Z}/3$-graded chain complexes

$$0 \to (2\triangle) \hookrightarrow (\triangle) \overset{2}{\twoheadrightarrow} (2\triangle) \to 0$$

where $(2\triangle)$ is the chain complex

$(2\triangle)$



$\mathcal{F}(\mathbb{Z}/4)$ is triangulated

The proof that the category $\mathcal{F}(\mathbb{Z}/4)$ is triangulated as above is based on the complete control over the category $\mathcal{F}(\mathbb{Z}/4)$:

up to isomorphism, every $f : A \longrightarrow B$ in $\mathcal{F}(\mathbb{Z}/4)$ is 'diagonal'

$$f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} : A = W \oplus X \oplus Y \longrightarrow W \oplus X \oplus Z = B.$$

Then f is extended to an exact triangle by the direct sum of

$$X \xrightarrow{2} X \xrightarrow{2} X \xrightarrow{2} X$$

and the contractible triangle

$$W \oplus Y \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} W \oplus Z \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} Y \oplus Z \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} W \oplus Y.$$

Exotic objects

Definition

An object E is exotic if there exists an exact triangle

$$E \xrightarrow{2} E \xrightarrow{2} E \longrightarrow E[1]$$

for some morphism $E \longrightarrow E[1]$.

Remarks

- The class of exotic objects is closed under isomorphism, (de-)suspension and is preserved by exact functors.
- Every object of the triangulated category $\mathcal{F}(\mathbb{Z}/4)$ is exotic.
- The integer 2 is special: an exact triangle

  $$E \xrightarrow{n} E \xrightarrow{n} E \longrightarrow E[1]$$

  with $E \neq 0$ forces $n \equiv 2 \pmod 4$ and $4 \cdot 1_E = 0$.

Hopfian objects

Definition

An object A is hopfian if it admits a Hopf map, i.e., a morphism $\eta : A[1] \longrightarrow A$ which satisfies $2\eta = 0$ and such that for some (hence any) exact triangle

$$A \xrightarrow{2} A \xrightarrow{i} C \xrightarrow{q} A[1]$$

we have $i \circ \eta \circ q = 2 \cdot 1_C$.

Remark

The class of hopfian objects is closed under isomorphism, (de-)suspension and preserved by exact functors.

Proposition

Every object of a topological triangulated category is hopfian.

Proof.

In the 'universal example', the sphere spectrum in $\mathcal{SW}$, the class of the Hopf map $\eta : S^3 \longrightarrow S^2$ is a Hopf map. □

Exotic versus hopfian objects

Hopfian and exotic objects are orthogonal:

Proposition

If E is exotic and A a hopfian object, then the morphism groups $\mathcal{T}(E, A)$ and $\mathcal{T}(A, E)$ are trivial. Thus every exotic and hopfian object is a zero object.

Corollary

Every exact functor from $\mathcal{F}(\mathbb{Z}/4)$ to a topological triangulated category is trivial. Every exact functor from a topological triangulated category to $\mathcal{F}(\mathbb{Z}/4)$ is trivial.

In particular, the triangulated category $\mathcal{F}(\mathbb{Z}/4)$ has no model.

Summary

- Triangulated categories which ‘arise in nature’ are topological.
- In topological triangulated categories, all objects admit Hopf maps.
- The category $\mathcal{F}(\mathbb{Z}/4)$ has an exotic triangulation.
- ‘Exotic’ and ‘hopfian’ are orthogonal properties,
  so $\mathcal{F}(\mathbb{Z}/4)$ has no topological model.
- None of this is difficult to prove.

Open questions

Find a p-local triangulated category without a model,
for p an odd prime.
Are there rational triangulated categories without model ?

Reference:
F. Muro, S. Schwede, N. Strickland,
Triangulated categories without models.
Invent. Math. **170** (2007), 231-241.