TORSIONLESS MODULES

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1. The category of torsionless modules

Let Λ be an Artin algebra and $\mathsf{mod}\text{-}\Lambda$ be the category of finitely generated right Λ-modules. We denote by $\underline{\mathsf{mod}}\text{-}\Lambda$ the stable category of $\mathsf{mod}\text{-}\Lambda$ modulo projectives and by $\overline{\mathsf{mod}}\text{-}\Lambda$ the stable category of $\mathsf{mod}\text{-}\Lambda$ modulo injectives. We denote by $\mathcal{P}_\Lambda$, resp. $\mathcal{I}_\Lambda$, the category of finitely generated projective, resp. injective, Λ-modules.

Let $\mathsf{Sub}(\Lambda)$ be the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of the submodules of the projectives and let $\mathsf{Fac}(\mathrm{D}\Lambda)$ be the full subcategory of $\mathsf{mod}\text{-}\Lambda$ consisting of the factors of the injectives.

Observation 1: There are adjoint pairs

$$(\Sigma_{\mathbf{P}}, \Omega) \ : \ \underline{\mathsf{mod}}\text{-}\Lambda \ \leftrightarrows \ \underline{\mathsf{mod}}\text{-}\Lambda \qquad \text{and} \qquad (\Sigma, \Omega_{\mathbf{I}}) \ : \ \overline{\mathsf{mod}}\text{-}\Lambda \ \leftrightarrows \ \overline{\mathsf{mod}}\text{-}\Lambda$$

where Ω is the usual syzygy functor and Σ is the usual cosyzygy functor.

Indeed $\Sigma_{\mathbf{P}}(\underline{A})$ and $\Omega_{\mathbf{I}}(\overline{A})$ are defined by the exact sequences:

$$A \xrightarrow{f^A} P_0^A \to \Sigma_{\mathbf{P}}(A) \to 0 \qquad \text{and} \qquad 0 \to \Omega_{\mathbf{I}}(A) \to I_A^0 \xrightarrow{f_A} A$$

where f^A is the minimal left $\mathcal{P}_\Lambda$-approximation of A and f_A is the minimal right $\mathcal{I}_\Lambda$-approximation of A. The above approximations exist since $\mathcal{P}_\Lambda$ and $\mathcal{I}_\Lambda$ are of finite type and can be described explicitly.

Indeed an easy computation shows that

$$\Sigma_{\mathbf{P}} = \mathrm{Tr}\Omega\mathrm{Tr} \qquad \text{and} \qquad \Omega_{\mathbf{I}} = \mathrm{D}\mathrm{Tr}\Omega\mathrm{Tr}\mathrm{D}$$

where D is the usual duality.

Observation 2: Clearly $\mathsf{Im}(f^A) = \Omega\Sigma_{\mathbf{P}}(A)$ and $\mathsf{Im}(f_A) = \Sigma\Omega_{\mathbf{I}}(A)$. Moreover:

(i) $\Omega(\underline{\mathsf{mod}}\text{-}\Lambda)$ is a reflective subcategory of $\underline{\mathsf{mod}}\text{-}\Lambda$ with reflection the map $\underline{A} \to \Omega\Sigma_{\mathbf{P}}(\underline{A})$. In particular $\underline{A}$ lies in $\Omega(\underline{\mathsf{mod}}\text{-}\Lambda)$ iff the map $\underline{A} \to \Omega\Sigma_{\mathbf{P}}(\underline{A})$ is invertible.

Note that we may choose the map $A \to \Omega\Sigma_{\mathbf{P}}(A)$ to be epic in $\mathsf{mod}\text{-}\Lambda$ and then the map

$$A \ \twoheadrightarrow \ \Omega\Sigma_{\mathbf{P}}(A)$$

is a left $\Omega(\mathsf{mod}\text{-}\Lambda)$-approximation of A.

(ii) $\Sigma(\overline{\mathsf{mod}}\text{-}\Lambda)$ is a coreflective subcategory of $\overline{\mathsf{mod}}\text{-}\Lambda$ with coreflection the map $\Sigma\Omega_{\mathbf{I}}(\overline{A}) \to \overline{A}$. In particular $\overline{A}$ lies in $\Sigma(\overline{\mathsf{mod}}\text{-}\Lambda)$ iff the map $\Sigma\Omega_{\mathbf{I}}(\overline{A}) \to \overline{A}$ is invertible.

Note that we may choose the map $\Sigma\Omega_{\mathbf{I}}(A) \to A$ to be monic in $\mathsf{mod}\text{-}\Lambda$ and then the map

$$\Sigma\Omega_{\mathbf{I}}(A) \ \rightarrowtail \ A \tag{1.1}$$

is a right $\Sigma(\mathsf{mod}\text{-}\Lambda)$-approximation of A.

Observation 3: The functors and $\Omega\mathrm{Tr} : \underline{\mathsf{mod}}\text{-}\Lambda \to \underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}}$ and $\Omega\mathrm{Tr} : \underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}} \to \underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}}$ induce a duality:

$$\Omega\mathrm{Tr} \ : \ \Omega(\underline{\mathsf{mod}}\text{-}\Lambda)^{\mathsf{op}} \ \xrightarrow{\approx} \ \Omega(\underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}})$$

with quasi-inverse the functor $\Omega\mathrm{Tr} : \Omega(\underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}})^{\mathsf{op}} \to \Omega(\underline{\mathsf{mod}}\text{-}\Lambda)$. Indeed this follows from **Observation 2**(i):

$\underline{A}$ lies in $\Omega(\underline{\mathsf{mod}}\text{-}\Lambda)$ or $\Omega(\underline{\mathsf{mod}}\text{-}\Lambda^{\mathsf{op}})$ iff the map $\underline{A} \to \Omega\Sigma_{\mathbf{P}}(\underline{A})$ is invertible iff the map $\underline{A} \to \Omega\mathrm{Tr}\Omega\mathrm{Tr}(\underline{A})$ is invertible.

The duality $\Omega\mathrm{Tr}$ is the functor η constructed in [1].

Observation 4: Since clearly $\underline{\mathsf{Sub}(\Lambda)} = \Omega(\underline{\mathsf{mod}}\text{-}\Lambda)$ and $\overline{\mathsf{Fac}(\mathrm{D}(\Lambda))} = \Sigma(\overline{\mathsf{mod}}\text{-}\Lambda)$, composing the above duality with D, we obtain an equivalence:

$$\mathrm{D}\Omega\mathrm{Tr} \ : \ \underline{\mathsf{Sub}(\Lambda)} \ \xrightarrow{\approx} \ \overline{\mathsf{Fac}(\mathrm{D}(\Lambda))}$$

with quasi-inverse the functor $\Omega\mathrm{Tr}\mathrm{D}$.

Let A be in $\mathsf{mod}\text{-}\Lambda$ and consider the following exact sequences

$$0 \ \longrightarrow \ A \ \longrightarrow I \ \longrightarrow \ \Sigma(A) \ \longrightarrow \ 0 \quad \text{and} \quad 0 \ \longrightarrow \ \Omega(I) \ \longrightarrow P \ \longrightarrow \ I \ \longrightarrow \ 0$$

where $A \to I$ is the injective envelope of A and $P \to I$ is the projective cover of I. The above exact sequences induce the following exact sequence:

$$0 \ \longrightarrow \ \Omega(I) \ \longrightarrow \Omega\Sigma(A) \ \longrightarrow \ A \ \longrightarrow \ 0$$

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and it is easy to see that the epimorphism

$$\Omega\Sigma(A) \twoheadrightarrow A \tag{1.2}$$

is a right $\Omega(\mathsf{mod}\text{-}\Lambda) = \mathsf{Sub}(\Lambda)$-approximation of A.

Now consider the full subcategory

$$\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda)) := \mathsf{add}\left\{X \oplus Y \in \mathsf{mod}\text{-}\Lambda \mid X \in \mathsf{Sub}(\Lambda),\ \ Y \in \mathsf{Fac}(\mathrm{D}(\Lambda)\right\}$$

which is a generating and cogenerating subcategory of $\mathsf{mod}\text{-}\Lambda$.

Also consider the pull-back of the maps (1.1) and (1.2):

$$\begin{tikzcd}
0 \arrow[r] & \Omega(I) \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \Sigma\Omega_{\mathbf{I}}(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega(I) \arrow[r] & \Omega\Sigma(A) \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Then the map $X \to \Omega\Sigma(A)$ is a monomorphism and there is induced a short exact sequence

$$0 \longrightarrow X \longrightarrow \Omega\Sigma(A) \oplus \Sigma\Omega_{\mathbf{I}}(A) \longrightarrow A \longrightarrow 0 \tag{1.3}$$

By using the above pull-back diagram it is easy to see that the map $\Omega\Sigma(A) \oplus \Sigma\Omega_{\mathbf{I}}(A) \longrightarrow A$ is a right $\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))$-approximation of A and $X \in \mathsf{Sub}(\Lambda) \subseteq \mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))$.

It follows that $\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))$ is contravariantly finite in $\mathsf{mod}\text{-}\Lambda$. This implies that the category

$$\mathsf{mod}\text{-}[\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))]$$

of coherent functors $[\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))]^{\mathsf{op}} \longrightarrow \mathscr{Ab}$ is abelian. Consider the restricted Yoneda functor

$$\mathsf{Y} : \mathsf{mod}\text{-}\Lambda \longrightarrow \mathsf{mod}\text{-}[\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))], \qquad \mathsf{Y}(A) = \mathsf{Hom}_\Lambda(-, A)|_{\mathsf{Sub}(\Lambda)\oplus\mathsf{Fac}(\mathrm{D}(\Lambda))}$$

Clearly any coherent functor $F : [\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))]^{\mathsf{op}} \longrightarrow \mathscr{Ab}$ admits a presentation

$$0 \longrightarrow \mathsf{Y}(A) \longrightarrow \mathsf{Y}(X_1) \longrightarrow \mathsf{Y}(X_0) \longrightarrow F \longrightarrow 0$$

where the X_i lie in $\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))$. On the other hand (1.3) induces an exact sequence

$$0 \longrightarrow \mathsf{Y}(X) \longrightarrow \mathsf{Y}\big(\Omega\Sigma(A) \oplus \Sigma\Omega_{\mathbf{I}}(A)\big) \longrightarrow \mathsf{Y}(A) \longrightarrow 0$$

which implies that $\mathsf{pd}\,\mathsf{Y}(A) \leqslant 1$. Putting things together we infer that in general:

$$\mathsf{gl.\,dim}\,\mathsf{mod}\text{-}[\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))] \leqslant 3$$

Of course if $\mathsf{Sub}(\Lambda)$ is of finite type, i.e. $\mathsf{Sub}(\Lambda) = \mathsf{add}\,X$ for some module X, or equivalently $\mathsf{Fac}(\mathrm{D}(\Lambda))$ is of finite type, i.e. $\mathsf{Fac}(\mathrm{D}(\Lambda)) = \mathsf{add}\,Y$ for some module Y, where we may choose $Y = \Sigma\mathrm{DTr}X$, then $\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda)) = \mathsf{add}(X \oplus Y)$, the module $X \oplus Y$ is a generator-cogenerator and $\mathsf{mod}\text{-}[\mathsf{Sub}(\Lambda) \oplus \mathsf{Fac}(\mathrm{D}(\Lambda))] \approx \mathsf{mod}\text{-}\,\mathsf{End}_\Lambda(X \oplus Y)$.

It follows that $\mathsf{gl.\,dim}\,\mathsf{End}_\Lambda(X \oplus Y) \leqslant 3$ and then: $\mathsf{rep.dim}\Lambda \leqslant 3$.

2. Problems

Recall that for an Artin algebra Λ, the full subcategory $\mathsf{CM}(\Lambda)$ of Cohen-Macaulay modlues, consists of all modules A such that $A \cong \mathsf{Im}(P^{-1} \to P^0)$ for an infinite acyclic complex $\cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$ of finitely generated projectives which remains exact after the application of the functor $\mathsf{Hom}_\Lambda(-\Lambda)$. An Artin algebra Λ is said to be of *finite Cohen-Macaulay type*, if $\mathsf{CM}(\Lambda)$ is of finite type.

Recall that Λ is called *Gorenstein*, if $\mathsf{id}\,\Lambda_\Lambda = \mathsf{id}\,{}_\Lambda\Lambda = d < \infty$. For such algebras we have $\Omega^d(\mathsf{mod}\text{-}\Lambda) = \mathsf{CM}(\Lambda)$. If Λ is of finite global dimension d, then Λ is Gorenstein and $\Omega^d(\mathsf{mod}\text{-}\Lambda) = \mathsf{CM}(\mathsf{mod}\text{-}\Lambda) = \mathcal{P}_\Lambda$. If Λ is self-injective, then Λ is Gorenstein and $\Omega^0(\mathsf{mod}\text{-}\Lambda) = \mathsf{CM}(\mathsf{mod}\text{-}\Lambda) = \mathsf{mod}\text{-}\Lambda$.

Remark 2.1. If Λ is a 1-Gorenstein algebra of finite Cohen-Macaulay type, then $\mathsf{rep.dim}\Lambda \leqslant 3$ since in this case $\mathsf{CM}(\Lambda) = \mathsf{Sub}(\Lambda)$. Now the results of [1] suggest to look at the following problems which generalizes the torsionless finite situation.

- **(I)** If $\Omega^d(\mathsf{mod}\text{-}\Lambda)$, for $d \geqslant 1$, is of finite type, is $\mathsf{rep.dim}\Lambda \leqslant d+2$?
  If the answer to **(I)** is yes, then it would follow that:
  - (i) If $\mathsf{gl.\,dim}\,\Lambda = d < \infty$, then $\mathsf{rep.dim}\Lambda \leqslant \mathsf{gl.\,dim}\,\Lambda + 2$.
  - (ii) More generally if Λ is d-Gorenstein of finite Cohen-Macaulay type, then $\mathsf{rep.dim}\Lambda \leqslant d+2$.
- **(II)** If the category of d-torsion-free modules in the sense of Auslander (i.e. modules A admitting an axact sequence $0 \to A \to P^0 \to P^1 \to \cdots \to P^d$, where the P^i are projective, such that the sequence $\mathsf{Hom}_\Lambda(P^d, \Lambda) \to \cdots \to \mathsf{Hom}_\Lambda(P^1, \Lambda) \to \mathsf{Hom}_\Lambda(P^0, \Lambda) \to \mathsf{Hom}_\Lambda(A, \Lambda) \to 0$ is exact), is of finite type, is $\mathsf{rep.dim}\Lambda \leqslant d+2$?

References

[1] C.M. Ringel, *The torsionless modules of an Artin algebra*, preprint, (2008).

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