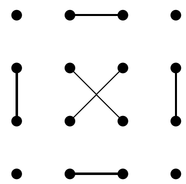
1 Classical Schur algebras

Schur algebras are certain finite-dimensional algebras linking the representation theories of general linear and symmetric groups. They were implicit in Schur's 1901 Ph.D. dissertation. In 1980 J.A. Green pointed out their relevance for modular representations. Basic references are [Gre80] and [Mar93]. For background on algebraic groups the standard reference is [Jan03].

1.1. Fix a commutative ring K. The symmetric group $\mathfrak{S}_r$ on $\{1, \ldots, r\}$ acts on tensor space $(K^n)^{\otimes r}$ by place permutation.

Definition. The *Schur algebra* $S_K(n,r)$ is the K-algebra $\mathrm{End}_{\mathfrak{S}_r}((K^n)^{\otimes r})$ of K-linear endomorphisms of $(K^n)^{\otimes r}$ commuting with the endomorphisms arising from the action of the elements of $\mathfrak{S}_r$.

The smallest non-trivial example is $S_K(2,2)$. This is isomorphic with the matrix algebra spanned by the matrices corresponding to the connected components of the following graph, as explained below.



Label the rows and columns of vertices in the graph by the multi-indices $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$ corresponding to basis elements $\mathrm{v}_1 \otimes \mathrm{v}_1$, $\mathrm{v}_1 \otimes \mathrm{v}_2$, $\mathrm{v}_2 \otimes \mathrm{v}_1$, $\mathrm{v}_2 \otimes \mathrm{v}_2$ of $(K^2) \otimes (K^2)$ where $\{\mathrm{v}_1, \mathrm{v}_2\}$ is the natural basis of K^2. This labels the sixteen vertices in the graph by pairs of multi-indices. Corresponding to a connected component in the graph is the 4×4 matrix with entries equal to 1 at the vertices of that connected component, and all other entries zero. There are 10 such matrices. Clearly each of them represents an endomorphism invariant under the action of $\mathfrak{S}_2$. These matrices in fact form a basis for $S_K(2,2)$, so $S_K(2,2)$ is free over K of rank 10.

1.2. There is an alternative way to define $S_K(n,r)$ in case K is an infinite field. The alternative approach is important because it links certain representations of the general linear group $\mathrm{GL}_n(K)$ to those of $S_K(n,r)$. The coordinate algebra $K[\mathrm{M}_n]$ of the affine variety $\mathrm{M}_n(K)$ of $n \times n$ matrices over K is a polynomial algebra $K[c_{ij} : 1 \leqslant i, j \leqslant n]$. Actually, since $\mathrm{M}_n(K)$ is an algebraic monoid under matrix multiplication, $K[\mathrm{M}_n]$ is a bialgebra with comultiplication $\Delta : K[\mathrm{M}_n] \to K[\mathrm{M}_n] \otimes K[\mathrm{M}_n]$ given by $\Delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj}$

and counit $\varepsilon : K[\mathrm{M}_n] \to K$ given by $\varepsilon(c_{ij}) = \delta_{ij}$. Set $A_K(n) = \iota^*(K[\mathrm{M}_n])$, the bialgebra of *polynomial* functions on $\mathrm{GL}_n(K)$, where $\iota : \mathrm{GL}_n(K) \to \mathrm{M}_n(K)$ is the natural inclusion and $\iota^* : K[\mathrm{M}_n] \to K[\mathrm{GL}_n]$ is the corresponding comorphism, given by restricting functions on $\mathrm{M}_n(K)$ to $\mathrm{GL}_n(K)$. It is easy to see that ι^* is injective, so it induces an isomorphism of bialgebras from $K[\mathrm{M}_n]$ onto $A_K(n)$.

The ring $A_K(n)$ is naturally graded by degree:

$$A_K(n) = \bigoplus_{r=0}^{\infty} A_K(n,r),$$

where $A_K(n,r)$ is the subspace of $A_K(n)$ consisting of the homogeneous polynomials of degree r. Note that $A_K(n,r)$ is a subcoalgebra of $A_K(n)$ and thus its linear dual $A_K(n,r)^* = \mathrm{Hom}_K(A_K(n,r), K)$ is an algebra.[1]

Proposition. *Let K be an infinite field. Then $S_K(n,r) \simeq A_K(n,r)^*$ as K-algebras.*

As a corollary we immediately obtain the fact that $\dim_K S_K(n,r) = \binom{n^2-1+r}{n^2-1} = \binom{n^2-1+r}{r}$, because this binomial coefficient counts the number of degree r monomials in n^2 variables.

1.3. The preceding proposition may be deduced from the double centralizer property sometimes known as "Schur–Weyl duality", which is the following statement (first proved in [Sch27] in case $K = \mathbb{C}$, and extended to the general case in [Gre80, DP76]).

Proposition. *Let K be an infinite field. Consider the commuting actions of $\mathrm{GL}_n(K)$ and $\mathfrak{S}_r$ on $(K^n)^{\otimes r}$. The subalgebra of $\mathrm{End}_K((K^n)^{\otimes r})$ generated by the endomorphisms coming from each action equals the centralizer algebra for the other action.*

In other words, $\mathrm{End}_{\mathfrak{S}_r}((K^n)^{\otimes r}) = \rho(K\mathrm{GL}_n(K))$ and $\mathrm{End}_{\mathrm{GL}_n(K)}((K^n)^{\otimes r}) = \rho'(K\mathfrak{S}_r)$, where ρ, ρ' are the representations corresponding to the commuting actions, extended to the group algebra. To deduce Proposition 1.2 from the first equality, it remains only to show that the linear dual of $\rho(K\mathrm{GL}_n(K))$ is isomorphic with $A_K(n,r)$. To see this, observe that the map $\rho : \mathrm{GL}_n(K) \to \mathrm{End}_K((K^n)^{\otimes r})$ dualizes to give a linear map $\rho^* : \mathrm{End}_K((K^n)^{\otimes r})^* \to K^{\mathrm{GL}_n(K)}$ the image of which is isomorphic with the coefficient space of the representation ρ. But that coefficient space is easily seen to coincide with $A_K(n,r)$.

[1] This is a general fact: if C is any coalgebra over K with comultiplication Δ then $C^* = \mathrm{Hom}_K(C,K)$ is an algebra. Given $f, f' \in C^*$, the product ff' is defined by $m \circ (f \otimes f') \circ \Delta$, where $m : K \otimes K \to K$ is the multiplication.

1.4. The importance of Schur algebras stems from the next result. A representation of $\mathrm{GL}_n(K)$ is said to *polynomial* if its coefficients lie in $A_K(n)$, and *homogeneous* of degree r is its coefficients lie in $A_K(n, r)$.

Proposition. *Let K be an infinite field.*
(a) Every polynomial representation of $\mathrm{GL}_n(K)$ is a direct sum of homogeneous ones.
(b) The category of $S_K(n, r)$-modules is equivalent with the category of homogeneous polynomial representations of degree r.

Since every finite-dimensional rational representation of $\mathrm{GL}_n(K)$ is obtainable from a polynomial one by tensoring with some power of the determinant, it makes sense to focus on the polynomial representations. The proposition says that polynomial representations of $\mathrm{GL}_n(K)$ are completely determined by the Schur algebras $S_K(n, r)$ as r varies. Thus the representation theory of $\mathrm{GL}_n(K)$ may be studied via finite dimensional algebras.

Remark. In the preceding results, the assumption that K is an infinite field may be avoided by replacing the group $\mathrm{GL}_n(K)$ by the corresponding group scheme GL_n over K. Doing so allows K to be replaced by a general field (or even an integral domain).

1.5. There is a second reason for the importance of Schur algebras. That is the existence of a functor $\mathrm{Hom}_{S_K(n,r)}((K^n)^{\otimes r}, -)$ from $S_K(n, r)$-modules to $K\mathfrak{S}_r$-modules. This functor is known as the "Schur functor" and if $n \geqslant r$ it is given by an idempotent $e \in S_K(n, r)$:

$$\mathrm{Hom}_{S_K(n,r)}((K^n)^{\otimes r}, N) \simeq eN$$

as $eS_K(n, r)e$-modules. The idempotent e is just projection onto the ω-weight space of N and $eS_K(n, r)e \simeq K\mathfrak{S}_r$. Here $\omega = (1, \dots, 1, 0, \dots, 0)$, with r ones and $n - r$ zeros. The Schur functor has been used to relate polynomial representations of general linear groups with representations of symmetric groups. Some of the deepest results in the modular representation theory of $\mathfrak{S}_r$ were obtained by exploiting this connection. In particular, we have the following.

Proposition. *(a) Suppose $n \geqslant r$. Then the decomposition matrix of $K\mathfrak{S}_r$ is a submatrix of the decomposition matrix of $S_K(n, r)$.*
(b) Conversely, each decomposition number for $S(n, r)$ must appear as a decomposition number of some symmetric group $\mathfrak{S}_t$ for suitable t.

Statement (a) was first proved in [Jam80] and (b) in [Erd96].

1.6. Some further important properties of Schur algebras are listed below. Let K be a field.

(1) *The algebra $S_K(n,r)$ is quasihereditary.*

Equivalently, the category of $S_K(n,r)$-modules is a highest weight category. (This fundamental equivalence in a much more general context was proved in [CPS88, Theorem3.6].) One important consequence is that Schur algebras have finite global dimension.

(2) *(base change) $S_K(n,r) \simeq K \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n,r)$.*

Statement (1) was first observed (independently) in [Par87] and [Don87], and (2) was proved in [Gre80].

It should also be mentioned that $S(n,r)$ is an example of a cellular algebra, in the sense of [GL96]. One cellular basis is the so-called codeterminant basis of [Gre93].

1.7. Yet another approach to Schur algebras is via the algebra U_K of distributions on $\mathrm{GL}_n(K)$. Note that this algebra is often known as the "hyperalgebra" in the older literature. It is isomorphic with $K \otimes_{\mathbb{Z}} U_{\mathbb{Z}}$ where $U_{\mathbb{Z}}$ is the Kostant $\mathbb{Z}$-form used in the Chevalley groups construction. This is obtained as follows. First, consider the complex Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Let $e_{i,j} = (\delta_{k,i}\delta_{l,j})_{1 \leqslant k,l, \leqslant n}$ be the usual basis of matrix units, for $1 \leqslant i,j \leqslant n$. Define the following elements

$$e_i = e_{i,i+1}, \quad f_i = e_{i+1,i}, \quad H_j = e_{j,j}$$

(for $i = 1, \ldots, n-1$ and $j = 1, \ldots, n$) of $\mathfrak{g}$. Then $U_{\mathbb{Z}}$ is the $\mathbb{Z}$-subalgebra of $U(\mathfrak{g})$ generated by the elements

$$\frac{e_i^m}{m!}, \quad \frac{f_i^m}{m!}, \quad \binom{H_j}{m}$$

for $m \geqslant 0$. We have an action of $\mathfrak{g}$ on $(\mathbb{C}^n)^{\otimes r}$ arising from the natural action of $\mathfrak{g}$ on $\mathbb{C}^n$. (This is just the action obtained by "differentiating" the action of the Lie group $\mathrm{GL}_n(\mathbb{C})$.) One checks that the lattice $(\mathbb{Z}^n)^{\otimes r}$ is invariant under $U_{\mathbb{Z}}$. Hence we have an induced action of U_K on $K \otimes_{\mathbb{Z}} (\mathbb{Z}^n)^{\otimes r} \simeq (K^n)^{\otimes r}$. This action commutes with the place-permutation action of $\mathfrak{S}_r$ considered earlier. Most cases of the following proposition were proved in [CL74].

Proposition. *Let K be any field, or $\mathbb{Z}$. Then Schur–Weyl duality holds between the actions of U_K and $\mathfrak{S}_r$; that is, the subalgebra of $\mathrm{End}_K((K^n)^{\otimes r})$ generated by the image of each action equals the centralizer algebra for the other.*

In particular, the image of the representation $\rho : U_K \to \mathrm{End}_K((K^n)^{\otimes r})$ is the Schur algebra $S_K(n,r)$. Thanks to the base change property, Schur algebras in all characteristics may be constructed from the integral Schur algebra. Thus characteristic zero deserves special attention.

1.8. Recall that Serre gave a presentation of all complex semisimple Lie algebras via generators and relations. This leads immediately to a presentation of the enveloping algebra $U = U(\mathfrak{gl}_n(\mathbb{C}))$: it is the associative algebra with 1 on generators

$$e_i,\ f_i \quad (1 \leqslant i \leqslant n-1); \qquad H_i \quad (1 \leqslant i \leqslant n)$$

subject to the relations

(R1) $H_i H_j = H_j H_i$;

(R2) $e_i f_j - f_j e_i = \delta_{i,j}(H_j - H_{j+1})$;

(R3) $H_i e_j - e_j H_i = (\delta_{i,j} - \delta_{i,j+1}) e_j, \quad H_i f_j - f_j H_i = -(\delta_{i,j} - \delta_{i,j+1}) f_j$;

(R4) $e_i^2 e_j - 2 e_i e_j e_i + e_j e_i^2 = 0$ if $|i-j| = 1$; $e_i e_j = e_j e_i$ otherwise;

(R5) $f_i^2 f_j - 2 f_i f_j f_i + f_j f_i^2 = 0$ if $|i-j| = 1$; $f_i f_j = f_j f_i$ otherwise

for all i, j for which the given relation makes sense.

A natural question is whether or not the Schur algebra $S_{\mathbb{C}}(n,r)$ may be obtained by generators and relations in a way that is compatible with Serre's presentation of U. In other words, can one find a set of generators for the kernel of the surjection $U \to S_{\mathbb{C}}(n,r)$ in terms of the Serre generators of U? The question has an affirmative answer, formulated in the following recent result from [DG02].

Proposition. *The kernel of* $U \to S_{\mathbb{C}}(n,r)$ *is generated by the elements* $H_1 + \cdots + H_n - r$ *and* $H_i(H_1 - 1) \cdots (H_i - r)$, *for* $1 \leqslant i \leqslant n$.

Thus it follows that $S_{\mathbb{C}}(n,r)$ may be defined as the associative algebra with 1 on the same generators as U, subject to the relations (R1)–(R5) defining U, along with the additional relations

(R6) $H_1 + \cdots + H_n = r$;

(R7) $H_i(H_1 - 1) \cdots (H_i - r) = 0$.

The integral Schur algebra may be obtained as the image of $U_{\mathbb{Z}}$ under the quotient map $U \to S_{\mathbb{C}}(n,r)$. In other words, it is the $\mathbb{Z}$-subalgebra of $S_{\mathbb{C}}(n,r)$ generated by the divided powers $\frac{e_i^m}{m!}$, $\frac{f_i^m}{m!}$ along with the $\binom{H_i}{m}$. One then obtains all Schur algebras $S_K(n,r)$ via change of base ring: $S_K(n,r) \simeq K \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n,r)$ for any commutative ring K.

Remark. One may replace $\mathbb{C}$ by any field of characteristic zero in the above result.

2 q-Schur algebras

The q-Schur algebras were first defined by Dipper and James [DJ89, DJ91]; they form a natural generalization of the ideas in Section 1. Jimbo [Jim86] independently considered a q-analogue of Schur–Weyl duality, and [BLM90] considered q-Schur algebras from a geometric viewpoint, in terms of relative positions of pairs of flags on a finite dimensional vector space over a finite field. General references for q-Schur algebras are the monographs [Don98, Mat99].

2.1. Fix a commutative ring K, and an invertible element $v \in K$. Put $q = v^2$. Recall the Iwahori–Hecke algebra $\mathscr{H} = \mathscr{H}_{K,v}(\mathfrak{S}_r)$, the associative algebra given by generators T_i for $i = 1, \dots, r-1$ with the relations

(H1) $(T_1 - v)(T_i + v^{-1}) = 0$;

(H2) $T_i T_j T_i = T_j T_i T_j$ if $|i - j| = 1$;

(H3) $T_i T_j = T_j T_i$ if $|i - j| > 1$.

Relations (H2), (H3) are called braid relations because they define the braid group on r strands; the algebra $\mathscr{H}$ is therefore a quotient of the group algebra of the braid group. Note that some authors replace (H1) by the relation $(T_i + 1)(T_i - q) = 0$ (with $q = v^2$); this leads to an equivalent theory although most formulas look somewhat different. Notice also that $\mathscr{H}$ is a deformation of the group algebra $K\mathfrak{S}_r$: by setting $v = 1$ we have $\mathscr{H} = K\mathfrak{S}_r$.

Let $\{\mathrm{v}_1, \dots, \mathrm{v}_n\}$ be the standard basis of K^n. We define an action of $\mathscr{H}$ on $(K^n)^{\otimes r}$ by letting the generator T_k act as the linear endomorphism defined by sending a basis element $B := \mathrm{v}_{i_1} \otimes \cdots \otimes \mathrm{v}_{i_r}$ (each $i_j \in \{1, \dots, n\}$) onto the element:

$$\begin{cases} vB, & \text{if } i_k = i_{k+1} \\ B', & \text{if } i_k < i_{k+1} \\ B' + (v - v^{-1})B, & \text{if } i_k > i_{k+1}. \end{cases}$$

where $B' := \mathrm{v}_{i_1} \otimes \cdots \otimes \mathrm{v}_{i_{k-1}} \otimes \mathrm{v}_{i_{k+1}} \otimes \mathrm{v}_{i_k} \otimes \mathrm{v}_{i_{k+2}} \otimes \cdots \otimes \mathrm{v}_{i_r}$ is the basis element obtained by acting on B by the transposition $(k, k+1) \in \mathfrak{S}_r$. If $v = 1$ the action of $\mathscr{H}$ is the same as the place-permutation action of $K\mathfrak{S}_r$ considered in Section 1.

Definition. The q-Schur algebra $S_{K,v}(n, r)$ is the algebra $\mathrm{End}_{\mathscr{H}}((K^n)^{\otimes r})$.

When $v = 1$ the q-Schur algebra $S_{K,v}(n, r)$ coincides with the classical Schur algebra $S_K(n, r)$.

2.2. All the results stated in Section 1 have q-analogues in the present context. To get the analogue of 1.2 one needs a suitable q-analogue of $A_K(n)$. One approach is to define $A_{K,v}(n)$ as the free K-algebra on generators c_{ij} $(1 \leqslant i, j \leqslant n)$ subject to the relations

$$\begin{aligned} c_{ik}c_{jk} &= vc_{jk}c_{ik} \quad \text{if } i<j \\ c_{ki}c_{kj} &= vc_{kj}c_{ki} \quad \text{if } i<j \\ c_{ij}c_{kl} &= c_{kl}c_{ij} \quad \text{if } i<k \text{ and } j>l \\ c_{ij}c_{kl} &= c_{kl}c_{ij} + (v-v^{-1})c_{il}c_{kj} \quad \text{if } i<k \text{ and } j<l. \end{aligned}$$

Notice that $A_{K,v}(n) = A_K(n)$ in case $v = 1$. One gets a bialgebra structure on $A_{K,v}(n)$ by defining comultiplication Δ and counit ε by the same formulas as in the classical case. Letting $A_{K,v}(n,r)$ be the subspace spanned by monomials of degree r, we have

$$A_{k,v}(n) = \bigoplus\nolimits_{r=0}^{\infty} A_{K,v}(n,r)$$

where as before $A_{K,v}(n,r)$ is a subcoalgebra.

Proposition. *For any commutative ring K, and invertible $v \in K$, we have an isomorphism $S_{K,v}(n,r) \simeq A_{K,v}(n,r)^*$ as K-algebras.*

2.3. There is an analogue of the Schur functor, relating $S_{K,v}(n,r)$-modules to $\mathscr{H}$-modules. By analogy with the classical case, it may be defined by $\mathrm{Hom}_S((K^n)^{\otimes r}, -)$ where now $S = S_{K,v}(n,r)$. As in the classical case, when $n \geqslant r$ this is carried by an idempotent in S.

2.4. The q-Schur algebras also have a connection with representations of the finite general linear groups. Dipper and James showed that when K is taken to be a field of characteristic not dividing q, then the algebras $S_{K,v}(n,r)$ control the representations of $\mathrm{GL}_n(\mathbb{F}_q)$ over K.

2.5. In order to obtain a base change property it is necessary to replace $\mathbb{Z}$ by the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in the intedeterminate v. Let K be a commutative ring, and choose an invertible $v \in K$. Regard K as an $\mathcal{A}$-algebra via the map obtained by sending $v \in \mathcal{A} \to v \in K$.

(1) *(base change)* $S_{K,v}(n,r) \simeq K \otimes_{\mathcal{A}} S_{\mathcal{A},v}(n,r)$.

Now let K be a field. The algebra $S_{K,v}(n,r)$ is quasihereditary; equivalently, the category of $S_{K,v}(n,r)$-modules is a highest weight category.

2.6. To get an analogue of 1.7 we must first define the quantized enveloping algebra $\mathbf{U} = \mathbf{U}$ corresponding to U. Let $X^\vee$ be the free $\mathbb{Z}$-module with basis $h_1, \dots, h_n$ and let $\varepsilon_1, \dots, \varepsilon_n \in X := X^{\vee *}$ be the corresponding dual basis: ε_i is given by $\varepsilon_i(h_j) := \delta_{i,j}$ for $j = 1, \dots, n$. For $i = 1, \dots, n-1$ let $\alpha_i \in X$ be defined by $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$.

Definition. Let $\mathbf{U}$ be the associative $\mathbb{Q}(v)$-algebra with 1 generated by the elements E_i, F_i $(i = 1, \dots, n-1)$ and v^h $(h \in X^\vee)$ with the defining relations

(Q1) $v^0 = 1$, $v^h v^{h'} = v^{h+h'}$;

(Q2) $v^h E_i v^{-h} = v^{\alpha_i(h)} E_i$, $v^h F_i v^{-h} = v^{-\alpha_i(h)} F_i$;

(Q3) $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{v - v^{-1}}$, where $K_i := v^{h_i - h_{i+1}}$;

(Q4) $E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ if $|i - j| = 1$; $E_i E_j = E_j E_i$ otherwise;

(Q5) $F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ if $|i - j| = 1$; $F_i F_j = F_j F_i$ otherwise.

$\mathbf{U}$ is a Hopf algebra with comultiplication Δ given by $\Delta(v^h) = v^h \otimes v^h$, $\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i$, $\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i$; the counit ε and antipode S are defined by the formulas $\varepsilon(v^h) = 1$, $\varepsilon(E_i) = \varepsilon(F_i) = 0$, $S(v^h) = v^{-h}$, $S(E_i) = -E_i K_i^{-1}$, $S(F_i) = -K_i F_i$.

Let $\{\mathrm{v}_1, \dots, \mathrm{v}_n\}$ be the standard basis of $\mathbb{Q}(v)^n$. Make $\mathbb{Q}(v)^n$ into a $\mathbf{U}$-module via $v^h \mathrm{v}_j = v^{\varepsilon_j(h)} \mathrm{v}_j$, $E_i \mathrm{v}_j = \delta_{j,i+1} \mathrm{v}_i$, $F_i \mathrm{v}_j = \delta_{j,i} \mathrm{v}_{i+1}$.

Let $[m] \in \mathcal{A}$ be defined by $[m] := \sum_{i=0}^{m-1} v^{2i-m+1}$. Set $[m]! := [m][m-1] \cdots [1]$ and set $E_i^{(m)} := \frac{E_i^m}{[m]!}$, $F_i^{(m)} := \frac{F_i^m}{[m]!}$ (quantized divided powers). Let $\mathbf{U}_\mathcal{A}$ be the $\mathcal{A}$-subalgebra of $\mathbf{U}$ generated by the v^h and the divided powers $E_i^{(m)}$ and $F_i^{(m)}$ for $m \geq 0$. Then $\mathbf{U}_\mathcal{A}$ is again a Hopf algebra and we have

$$\Delta(E_i^{(m)}) = \sum_{k=0}^{m} v^{k(m-k)} E_i^{(m-k)} \otimes K_i^{k-m} E_i^{(k)},$$
$$\Delta(F_i^{(m)}) = \sum_{k=0}^{m} v^{-k(m-k)} F_i^{(m-k)} K_i^{k} \otimes F_i^{(k)}.$$

These formulas are used to verify that the $\mathcal{A}$-lattice $\mathcal{A}^n$ in $\mathbb{Q}(v)^n$ is invariant under the action of $\mathbf{U}_\mathcal{A}$. Now, make the transition from $\mathcal{A}$ to the commutative ring K: Let $v \in K$ be invertible and consider K as an $\mathcal{A}$-algebra via the algebra map $v \in \mathcal{A} \to v \in K$. Let $\mathbf{U}_K := K \otimes_\mathcal{A} \mathbf{U}_\mathcal{A}$. Then $\mathbf{U}_K$ inherits a Hopf algebra structure from $\mathbf{U}_\mathcal{A}$ and $K^n \simeq K \otimes_\mathcal{A} \mathcal{A}^n$ is a $\mathbf{U}_K$-module. The comultiplication is used to make the tensor power $(K^n)^{\otimes r}$ into a $\mathbf{U}_K$-module as well. At this point one may check that the action of $\mathbf{U}_K$ on

$(K^n)^{\otimes r}$ commutes with the action of $\mathscr{H}$ defined earlier. Here is the promised analogue of Proposition 1.7, proved in [Jim86] in case v is not a root of unity, and in [DPS98] in general.

Proposition. *Let K be a commutative ring. Schur–Weyl duality holds between the actions of $\mathbf{U}_K$ and $\mathscr{H}$; that is, the subalgebra of $\mathrm{End}_K((K^n)^{\otimes r})$ generated by the image of each action equals the centralizer algebra for the other.*

In particular, the image of the representation $\rho : \mathbf{U}_K \to \mathrm{End}_K((K^n)^{\otimes r})$ is the q-Schur algebra $S_{K,v}(n, r)$.

2.7. There is also a q-analogue of 1.8, proved in [DG02].

Proposition. *The kernel of $\mathbf{U} \to S_{\mathbb{Q}(v),v}(n, r)$ is generated by the elements $v^{H_1+\cdots+H_n}v^{-r} - 1$ and $(v^{H_i} - 1)(v^{H_1} - v)\cdots(v^{H_i} - v^r)-$, for $1 \leqslant i \leqslant n$.*

Thus it follows that $S_{\mathbb{Q}(v),v}(n, r)$ may be defined as the associative algebra with 1 on the same generators as $\mathbf{U}$, subject to the relations (Q1)–(Q5) defining $\mathbf{U}$, along with the additional relations

(Q6) $v^{H_1}v^{H_2}\cdots v^{H_n} = v^r$;

(Q7) $(v^{H_i} - 1)(v^{H_1} - v)\cdots(v^{H_i} - v^r) = 1$.

The integral form $S_{\mathcal{A},v}(n, r)$ may then be obtained as the image of $\mathbf{U}_{\mathcal{A}}$ under the quotient map $\mathbf{U} \to S_{\mathbb{Q}(v)}(n, r)$. In other words, it is the $\mathcal{A}$-subalgebra of $S_{\mathbb{Q}(v),v}(n, r)$ generated by the quantized divided powers $E_i^{(m)}$, $F_i^{(m)}$ along with the v^h. One then obtains all q-Schur algebras $S_{K,v}(n, r)$ via change of base ring: $S_{K,v}(n, r) \simeq K \otimes_{\mathcal{A}} S_{\mathcal{A},v}(n, r)$ for any commutative ring K, where as before K is regarded as an $\mathcal{A}$-algebra via the algebra map $v \in \mathcal{A} \to v \in K$, for a choice of invertible $v \in K$.

3 Generalized Schur algebras

Now we consider a second generalization of the theory from Section 1. Instead of quantization, this direction is based on the idea of truncation of a module category, which approaches the study of representations of reductive algebraic groups through finite dimensional algebras. The basic results are due to Donkin [Don86]. Another good reference for this is [Jan03, Part II, Chapter A].

3.1. Let G be an arbitrary reductive algebraic group (scheme) over a field K, and fix a maximal torus $T \subset G$. Let $X(T)$ be the abelian group of characters on T, as usual, and recall that every (rational) G-module V may be written as a direct sum of its weight spaces: $V = \bigoplus_{\lambda \in X(T)} V_\lambda$ where $V_\lambda = \{v \in V : tv = \lambda(t)v, \text{ all } t \in T\}$. As usual, fix a Borel subgroup B with $T \subset B \subset G$ and let B correspond to the negative roots. This choice determines the sets of positive roots, simple roots, and dominant weights $X(T)^+$. The latter is defined by

$$X(T)^+ = \{\lambda \in X(T) : \langle \lambda, \alpha^\vee \rangle \geqslant 0, \text{ all simple roots } \alpha\}.$$

The choice of B also determines the standard dominance order on $X(T)$, where one declares that $\lambda \leqslant \mu$ if and only if $\mu - \lambda$ can be written as a positive integral linear combination of simple roots.

Let K_λ be the one dimensional B-module with T acting as the character λ and the unipotent radical of B acting trivially. For each $\lambda \in X(T)^+$ one has the G-module $\nabla(\lambda) = \operatorname{ind}_B^G K_\lambda$. Let $\Delta(\lambda)$ be the contravariant dual of $\nabla(\lambda)$; $\Delta(\lambda$ is the Weyl module of highest weight λ. Then $\nabla(\lambda)$ has a simple socle which we denote by $L(\lambda)$. The set of isomorphism classes of simple G-modules is given by $\{L(\lambda) : \lambda \in X(T)^+\}$.

3.2. Fix a set $\pi \subset X(T)^+$ of dominant weights. For a given G-module M let $\mathscr{O}_\pi M$ be the largest submodule of M such that every composition factor of M has highest weight belonging to π. Let $\mathscr{C}(\pi)$ be the full subcategory of the category of G-modules consisting of G-modules M satisfying the condition Clearly $M \to \mathscr{O}_\pi M$ defines a functor $\mathscr{O}_\pi : \{G\text{-modules}\} \to \mathscr{C}(\pi)$. This functor is left exact, and takes injective G-modules to injective objects of $\mathscr{C}(\pi)$. Another property is that the right derived functors $R^i\mathscr{O}_\pi$ commute with direct limits.

Definition. The set $\pi \subset X(T)_+$ is called *saturated* if $\lambda \leqslant \mu$ for $\lambda \in X(T)_+$, $\mu \in \pi$ implies $\lambda \in \pi$.

Saturated subsets of $X(T)_+$ exist in abundance. For example, given any $\mu \in X(T)_+$, the set of all $\lambda \in X(T)_+$ satisfying $\lambda \leqslant \mu$ is saturated. In general, a saturated subset of $X(T)_+$ is a union of such subsets.

3.3. Recall that an ascending chain of submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots$ of a G-module M is called a *good filtration* of M if $M = \bigcup M_i$ and each nontrivial subquotient M_i/M_{i-1} is isomorphic to some $\nabla(\lambda_i)$ with $\lambda_i \in X(T)_+$. One denotes by $\mathscr{F}(\nabla)$ the full subcategory of $\{G\text{-modules}\}$ the objects of which are G-modules with a good filtration. For an object M of $\mathscr{F}(\nabla)$ let $(M : \nabla(\lambda))$ be the number of filtration factors isomorphic with $\nabla(\lambda)$, for $\lambda \in X(T)_+$.

Suppose π is a saturated subset of $X(T)_+$. If M is an object of $\mathscr{F}(\nabla)$ such that all $(M : \nabla(\lambda))$ are finite, one can show that $\mathscr{O}_\pi M$ again belongs to $\mathscr{F}(\nabla)$ and

(1) $$(\mathscr{O}_\pi M : \nabla(\lambda)) = \begin{cases} (M : \nabla(\lambda)) & \text{if } \lambda \in \pi, \\ 0 & \text{otherwise.} \end{cases}$$

Next one shows that if $M \in \mathscr{F}(\nabla)$ then $R^i\mathscr{O}_\pi M = 0$ for all $i > 0$. From this it follows by an induction that if M belongs to $\mathscr{C}(\pi)$ then $R^i\mathscr{O}_\pi M = 0$ for all $i > 0$. With these facts in hand one can now show by an argument with Grothendieck's spectral sequence that for modules V, V' in $\mathscr{C}(\pi)$ we have

(2) $\mathrm{Ext}^i_G(V, V') \simeq \mathrm{Ext}^i_{\mathscr{C}(\pi)}(V, V')$, for all $i \geqslant 0$.

3.4. Let π be a finite saturated subset of $X(T)_+$. Set $A_G(\pi) = \mathscr{O}_\pi K[G]$. One easily checks that $A_G(\pi)$ is a finite dimensional subcoalgebra of $K[G]$.

Definition. The *generalized Schur algebra* associated to G, π is the finite dimensional algebra $S_G(\pi) = A_G(\pi)^*$.

Note that by taking $G = \mathrm{GL}_n$ (regarded now as a group scheme over K) and taking π to be the set of dominant weights of the G-module $(K^n)^{\otimes r}$, we have $S_G(\pi) = S_K(n, r)$, the classical Schur algebra from Section 1. Some basic properties of $S_G(\pi)$ are:

(1) The category $\mathscr{C}(\pi)$ is equivalent to the category of $S_G(\pi)$-modules.
(2) $\dim S_G(\pi) = \sum_{\lambda\in\pi} (\dim \nabla(\lambda))^2$.
(3) $S_G(\pi)$ is quasihereditary.

3.5. There is an alternative construction of $S_G(\pi)$ as a quotient of the algebra $U(G)$ of distributions on G. Recall that $U(G)$ is by definition a certain subalgebra of $K[G]^*$. The quotient map $U(G) \to S_G(\pi)$ sends an element of $U(G)$ to its restriction to $A_G(\pi)$; the kernel of this quotient map is

$$I_G(\pi) = \{F \in U(G) : F(V) = 0, \text{ all } V \in \mathscr{C}(\pi)\}$$

and we have $S_G(\pi) \simeq U(G)/I_G(\pi)$.

3.6. Suppose now that our G is a scheme over a commutative ring K, and continue to assume that π is finite and saturated. The construction of the category $\mathscr{C}(\pi)$ can be extended to this situation, but we need a different definition. In this case, we say that a G-module M belongs to π if the set of dominant weights of M is contained in π. Then $\mathscr{C}(\pi)$ is the full subcategory of $\{G\text{-modules}\}$ whose objects belong to π.

Return now to the assumption that K is a field. In order to avoid double subscripts, write $I_G(\pi)_{\mathbb{Q}}$, $S_G(\pi)_{\mathbb{Q}}$ respectively for the $I_G(\pi)$, $S_G(\pi)$ in case $G = G_{\mathbb{Q}}$, and set

$$I_G(\pi)_{\mathbb{Z}} = U(G_{\mathbb{Z}}) \cap I_G(\pi)_{\mathbb{Q}}.$$

One can show that an element lies in $I_G(\pi)_{\mathbb{Z}}$ if and only if it annihilates all $\Delta(\lambda)_{\mathbb{Q}}$ with $\lambda \in \pi$. This is true if and only if it annihilates all $\Delta(\lambda)_{\mathbb{Z}}$ with $\lambda \in \pi$. From this one can show that

(1) $I_G(\pi)_{\mathbb{Z}}$ is the set of all elements of $U(G_{\mathbb{Z}})$ annihilating all $G_{\mathbb{Z}}$-modules that are free of finite rank over $\mathbb{Z}$ and belong to $\mathscr{C}(\pi)$.

Definition. Set $S_G(\pi)_{\mathbb{Z}} = U(G_{\mathbb{Z}})/I_G(\pi)_{\mathbb{Z}}$.

One can recover the original $S_G(\pi)_{\mathbb{Q}}$ from its integral form $S_G(\pi)_{\mathbb{Z}}$ because: $S_G(\pi)_{\mathbb{Q}} \simeq \mathbb{Q} \otimes_{\mathbb{Z}} S_G(\pi)_{\mathbb{Z}}$. This extends to arbitrary fields:

(2) $S_G(\pi) \simeq K \otimes_{\mathbb{Z}} S_G(\pi)_{\mathbb{Z}}$.

This base change property means that again the theory can be constructed from an integral form in characteristic zero.

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