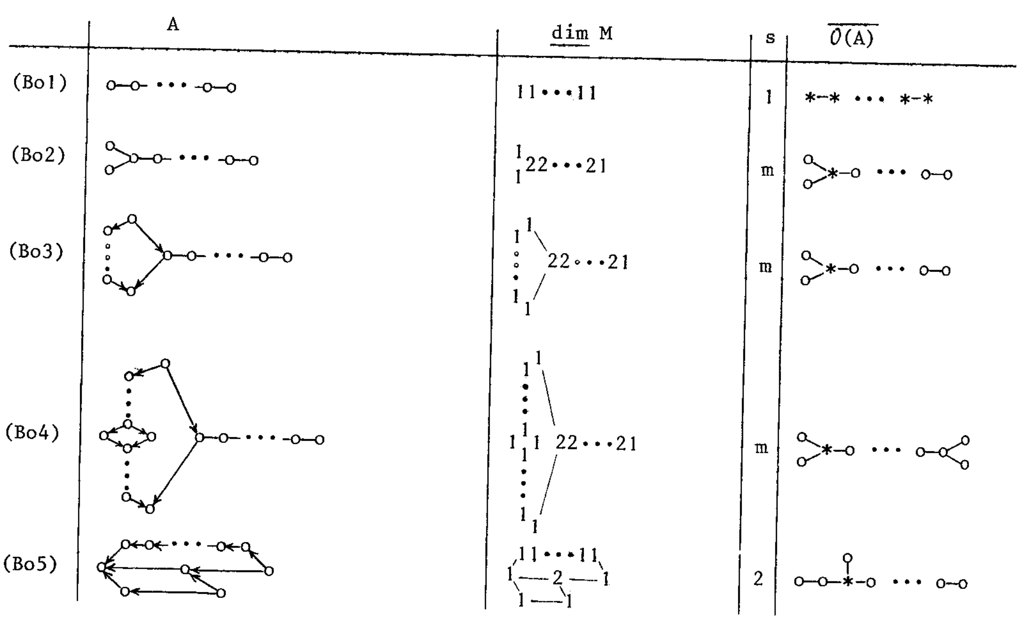
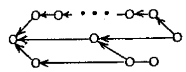
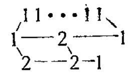
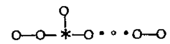
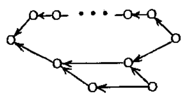
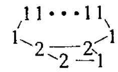
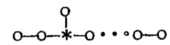
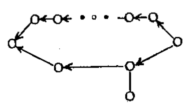
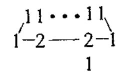
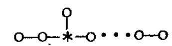
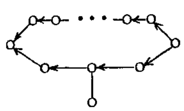
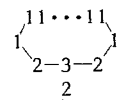
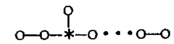
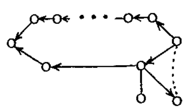
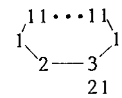
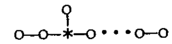
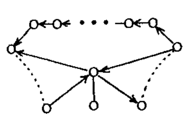
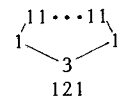
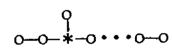
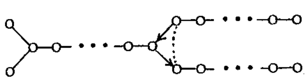
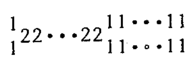
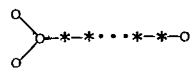
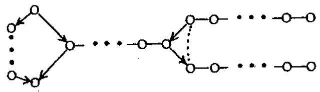
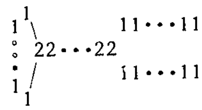
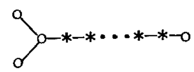
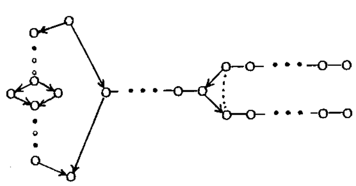
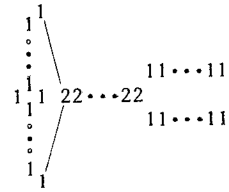
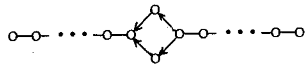
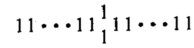
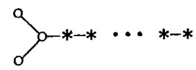
6.3 The large sincere directed algebras

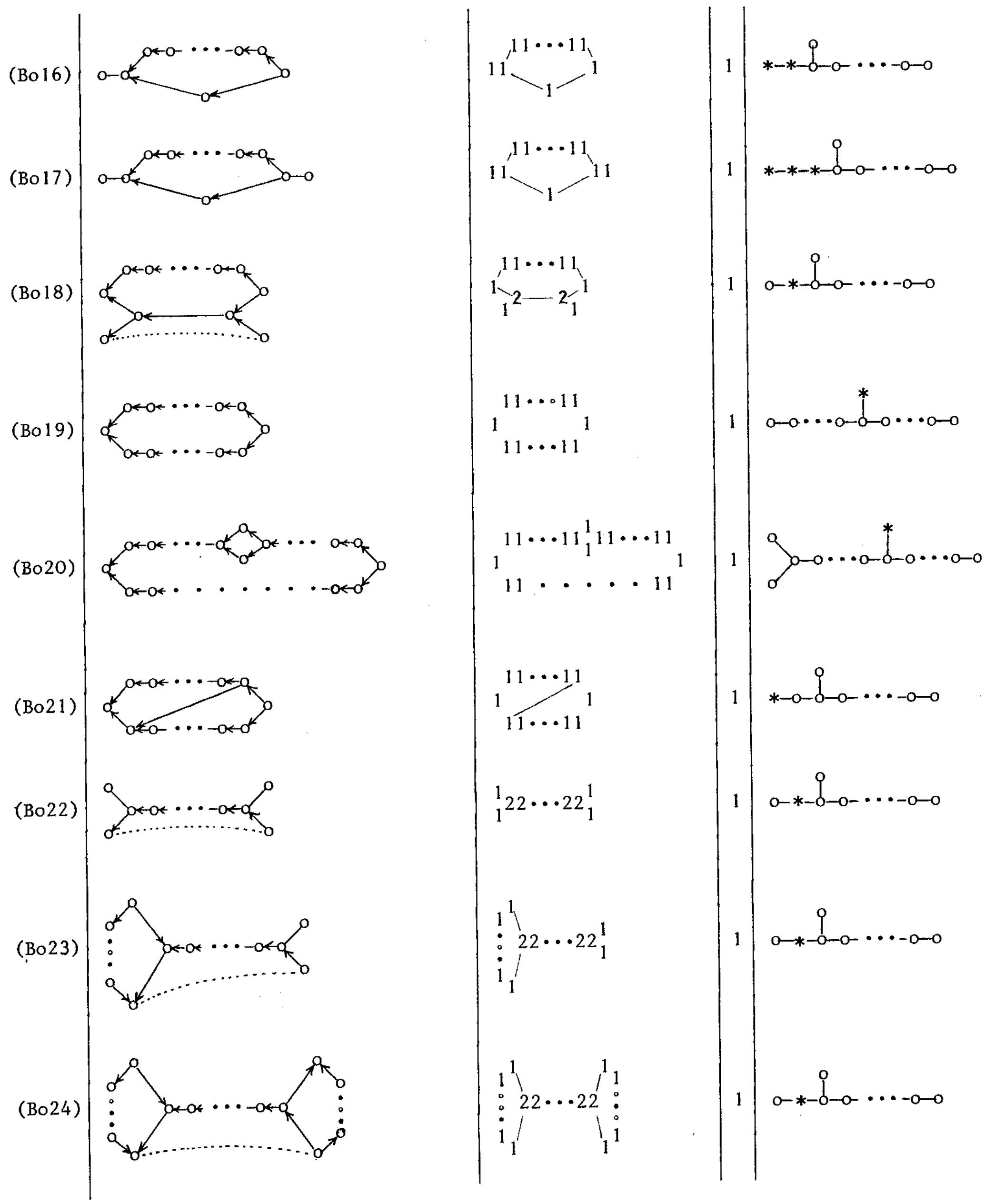
Given a finite-dimensional algebra A, let us denote by n(A) the number of isomorphism classes of simple A-modules.

(1) Theorem (Bongartz). Let A be a sincere, directed algebra with $n(A) > 13$. Then A or A^{op} is an algebra presented in the table below.

In the table, we first give the quiver and its zero relations; in addition, one has to take all possible commutativity relations. The unoriented edges occurring in the list may be oriented arbitrarily [however, it is not allowed to replace a branch arbitrarily: the only possible changes of branches involving zero relations are already taken into account: the changes from (2) to (12), from (3) to (13) and from (4) to (14)]. We note that the algebras exhibited in the table have a unique sincere maximal module M, and we present in the second column its dimension vector. The third column gives the numbers of sincere positive roots, thus the number of isomorphism classes of sincere indecomposable modules. Here, m denotes the length of the largest branch. [Starting from the maximal root $z = \underline{\dim}\, M$, it is very easy to derive the remaining sincere positive roots using reflections σ_i]. The final column gives the orbit graph of the corresponding algebra. By a star, we indicate the possible positions of the τ-orbit of the sincere maximal module (it depends on the orientation of the unoriented edges). Observe that the orbit graph gives a quadratic form $\mathbb{Z}$-equivalent to the quadratic form χ_A. (For, since A is sincere and directed, there is a slice $\mathcal{S}$ in A-mod, say with slice module S, and the underlying graph of the quiver of $\mathcal{S}$ coincides with $\overline{\mathcal{O}(A)}$. According to 4.1.7, the quadratic form of A is $\mathbb{Z}$-equivalent to the quadratic form of End S.)

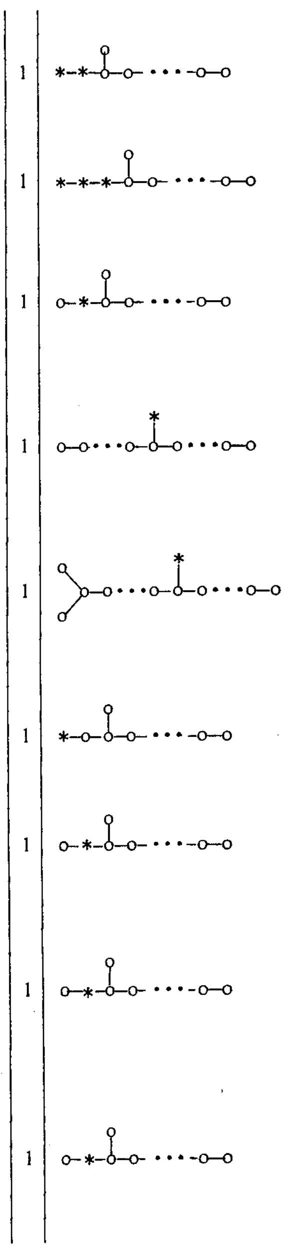


(Bo6)			4	
(Bo7)			4	
(Bo8)			3	
(Bo9)			7	
(Bo10)			4	
(Bo11)			3	
(Bo12)			1	
(Bo13)			1	
(Bo14)			1	
(Bo15)			1	



(2) Corollary. Let A be a sincere directed algebra. Then there is a codimension 1 subspace U in $K_o(A)$ such that the restriction of χ_A to U is positive definite.

Proof. We consider the quadratic form given by $\chi(\overline{O(A)})$. If $n(A) > 13$, then we delete a suitable branching vertex of $\overline{O(A)}$. On the other hand, given a tree with at most four endpoints (see 6.1.1) and at most 13 vertices, we always may delete



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we always may delete

one vertex in order to obtain a disjoint union of Dynkin graphs.

A direct consequence of the classification and Ovsienko's theorem is also the following

(3) Corollary (Bongartz). Let A be a directed algebra, and M an indecomposable A-module. Then $|M| \leq 2 \cdot n(A) + 48$.

Proof of the corollary: Obviously, we may assume that A is sincere. We have $\underline{\dim}\, M \leq z$, where z is a maximal positive root for χ_A. If $n(A) \geq 14$, then A or A^{op} belongs to the list, and z is tabulated, thus we see that $|M| \leq 2n(A)-3$ (this bound is optimal for (Bo 2)). Now assume $n(A) \leq 13$. Let ω be an exceptional vertex for z, thus $z_\omega \leq 2$. According to theorem 1 of Ovsienko, $z_a \leq 6$ for all vertices a. Thus $|M| \leq \sum_a z_a \leq 2n(A) + \sum_{a \neq \omega} (z_a - 2) \leq 2n(A) + 12 \cdot 4$. Of course, the term 48 is not optimal, since never $z_a = 6$ for all $a \neq \omega$.)

(4) Outline of the proof of (1). It is easy to see that the only sincere directed algebras A with $\overline{\mathcal{O}(A)} = \mathbb{A}_n$ are the algebras (Bo1). It remains to consider the cases where $\mathcal{O}(A)$ has three or four endpoints. The possibilites for A in case $\mathcal{O}(A)$ has three endpoints are discussed in (6), (7), and (8). Assertion (10) gives a further restriction for the orbit quivers of sincere directed algebras having four endpoints. The possibilities for A in case $\mathcal{O}(A)$ has four endpoints, are discussed in (11). This then finishes the proof of (1). We need a preliminary result dealing with a maximal sincere module M whose τ-orbit cuts off a branch of $\mathcal{O}(A)$; this will be given in (5).

We introduce the following notation: Given an indecomposable A-module M, let [[M]] denote its τ-orbit in $\Gamma(A)$. In case M belongs to a preprojective component, for example, if A is directed, then [[M]] will be considered as an element of the orbit quiver $\mathcal{O}(A)$ of A. Also, we recall from 2.5, that given a one-point extension $A = A_o[R]$, we may consider the A-modules as triples $X = (X_o, X_\omega, \gamma_X)$, where X_o is an A_o-module, X_ω a vectorspace, and $\gamma_X : X_\omega \longrightarrow \mathrm{Hom}(R, X_o)$ a linear map. In particular, given an A_o-module Y, let $\bar{Y} = (Y, \mathrm{Hom}(R,Y), 1_{\mathrm{Hom}(R,Y)})$.

(5) Let A_o be a directed algebra, let $R_1 \to R_2 \to \dots \to R_m$ be a sectional path, with R_i, $1 \leq i \leq m$, indecomposable, $[[R_1]]$ **endpoint**, and such that, for $1 \leq i < m$, $[[R_i]]$ cuts off a branch of $\mathcal{O}(A_o)$. Assume that $\tau^{-m+s}R_s$ is injective, for some $1 \leq s \leq m$. Let $A = A_o[R_1]$. Then there is at most one indecomposable sincere A-module, namely $\bar{R}_m$. Always, $[[\bar{R}_m]]$ cuts off a branch of $\mathcal{O}(A)$ of length m.

Proof. We may assume that m is minimal such that some $\tau^{-m+s}R_s$ with $1 \leq s \leq m$, is injective. Let $R_{ij} = \tau^{-i+1}R_{j-i+1}$, $1 \leq i \leq j \leq m$. Let $t = m-s+1$, thus R_{tm} is injective. Thus, in $\Gamma(A_o)$, there is the following part, which obviously is a wing for R_{1m}.