SOME OPEN PROBLEMS FOR MODULES OF PROJECTIVE DIMENSION TWO AND ALGEBRAS OF GLOBAL DIMENSION TWO

DIETER HAPPEL

Let Λ be a finite dimensional algebra over some algebraically closed field k. Let $\operatorname{mod}\Lambda$ be the category of finite dimensional left $\Lambda-$modules.

In the first part of the talk we will consider selforthogonal and exceptional modules. Recall that a $\Lambda-$module X is said to be **selforthogonal** provided that $\operatorname{Ext}^i_\Lambda(X,X) = 0$ for all $i > 0$ and is called **exceptional** if it is selforthogonal and $\operatorname{proj.dim}_\Lambda X < \infty$. Note that we do not assume X to be indecomposable. Also X is said to be a **partial tilting module** if X is a direct summand of a tilting module T, i.e T is exceptional and there is an exact sequence

$$0 \to {}_\Lambda\Lambda \to T^0 \to T^1 \to \cdots \to T^d \to 0,$$

where $T^i \in \operatorname{add} T$ for all i.

It can be shown that d can be chosen to coincide with $\operatorname{proj.dim}_\Lambda X$

It is well known that an exceptional $\Lambda-$module X with $\operatorname{proj.dim}_\Lambda X \leq 1$ is a partial tilting module [Bo]. It has been shown in [RS] that this is no longer true for $\operatorname{proj.dim}_\Lambda X \geq 2$. We will recall the example and mention some results under which additional assumptions, an exceptional $\Lambda-$module X is a partial tilting module.

A famous open problem which fits into the context described above, states that the injective cogenerator $D(\Lambda_\Lambda)$ of $\operatorname{mod}\Lambda$ is a (partial) tilting module provided $\operatorname{proj.dim}_\Lambda D(\Lambda_\Lambda) < \infty$. This is open even for $\operatorname{proj.dim}_\Lambda D(\Lambda_\Lambda) = 2$. We will show how this problem fits into the context of the hierarchy of the classical homological conjectures. (For a survey see [Ha].)

It is well known that there exists a $\Lambda-$module X with $\operatorname{proj.dim}_\Lambda X = 1$ if and only if there is a simple $\Lambda-$module S such $\operatorname{Hom}_\Lambda(D(\Lambda_\Lambda), S) = 0$. It seems that there are no analogous statements known for the existence of modules of projective dimension $d \geq 2$. We will give an example that the naive extension will fail.

In the second part we will address some problems concerning restrictions for algebras of finite global dimension. For this let $\vec{\Delta}$ be a finite quiver. Recall that a two-sided ideal I of the path algebra $k\vec{\Delta}$ is said to be **admissible** if $\Lambda = k\vec{\Delta}/I$ is finite dimensional and the quiver of Λ coincides with $\vec{\Delta}$.

It seems that the following problems deserve some attention. For which quivers $\vec{\Delta}$ do exist two sided ideals I such that $\operatorname{gl.dim} k\vec{\Delta}/I < \infty$? It is known that $\vec{\Delta}$ should not contain a loop [I] or [L]. For a given quiver $\vec{\Delta}$ and a natural number d consider

$$\mathcal{A}(\vec{\Delta}, d) = \{k\vec{\Delta}/I \,|\, \text{gl.dim}\, k\vec{\Delta}/I = d\}.$$

and

$$\mathcal{A}(\vec{\Delta}) = \bigcup_{d=1}^{\infty} \mathcal{A}(\vec{\Delta}, d).$$

So for example $\mathcal{A}(\vec{\Delta}, 1)$ is empty if $\vec{\Delta}$ contains an oriented cycle and $\mathcal{A}(\vec{\Delta})$ is empty if $\vec{\Delta}$ contains a loop.

It is known that the global dimension of a finite dimensional algebra is not bounded by a function depending on the number of simple $\Lambda-$modules [G] and also not bounded by a function on the Loewey length of Λ [KK]. However one may consider the following related problem. For this denote by

$$a(\vec{\Delta}, d) = \sup\{\dim_k \Lambda \,|\, \Lambda \in \mathcal{A}(\vec{\Delta}, d)\}$$

and by

$$a(\vec{\Delta}) = \sup\{\dim_k \Lambda \,|\, \Lambda \in \mathcal{A}(\vec{\Delta})\}.$$

Then one may wonder whether or not these are actually natural numbers. Trivially if $\vec{\Delta}$ has no oriented cycle then $a(\vec{\Delta}) = \dim_k k\vec{\Delta}$ and $\mathcal{A}(\vec{\Delta}, d)$ is non-empty for only finitely many d.

References

[Bo] Bongartz, Klaus, Tilted algebras. Representations of algebras (Puebla, 1980), pp. 26–38, Lecture Notes in Math., 903, Springer, Berlin-New York, 1981.

[G] Green, Edward L., Remarks on projective resolutions. Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp. 259–279, Lecture Notes in Math., 832, Springer, Berlin, 1980.

[Ha] Happel, Dieter, Homological conjectures in the representation theory of finite dimensional algebras, Sherbrooke Preprint Series 1990.

[I] Igusa, Kiyoshi, Notes on the no loops conjecture. J. Pure Appl. Algebra 69 (1990), no. 2, 161–176.

[KK] Kirkman, Ellen; Kuzmanovich, James, Algebras with large homological dimensions. Proc. Amer. Math. Soc. 109 (1990), no. 4, 903–906.

[L] Lenzing, Helmut, Nilpotente Elemente in Ringen von endlicher globaler Dimension. Math. Z. 108 1969 313–324.

[RS] Rickard, Jeremy; Schofield, Aidan, Cocovers and tilting modules. Math. Proc. Cambridge Philos. Soc. 106 (1989), no. 1, 1–5.