HOMOLOGICAL CONJECTURES IN REPRESENTATION THEORY OF FINITE-DIMENSIONAL ALGEBRAS

DIETER HAPPEL

Let A be a finite-dimensional k-algebra over an algebraically closed field k. We denote by $\operatorname{mod} A$ the category of finitely generated left A-modules. For an A-module ${}_AX$ we denote by $\operatorname{pd}_A X$ (resp. $\operatorname{id}_A X$) the projective (resp. injective) dimension of X. With $D = \operatorname{Hom}_k(-, k)$ we denote the standard duality with respect to the ground field. Then ${}_AD(A_A)$ is an injective cogenerator for $\operatorname{mod} A$. To formulate some of the homological conjectures we need some more notation. Let ${}_A\mathcal{I} \subset \operatorname{mod} A$ be the full subcategory containing the finitely generated injective A-modules. Let $K^b({}_A\mathcal{I})$ be the homotopy category of bounded complexes over ${}_A\mathcal{I}$. Let $D^b(A)$ be the derived category of bounded complexes over $\operatorname{mod} A$. We consider $K^b({}_A\mathcal{I})$ as a full subcategory of $D^b(A)$.

We define $K^b({}_A\mathcal{I})^\perp = \{X \in D^b(A) \mid \operatorname{Hom}(I, X) = 0 \text{ for all } I \in K^b({}_A\mathcal{I})\}$.

The following is the well-known hierarchy of some of the homological conjectures:

(1) Auslander Conjecture: Let ${}_AX$ be an A-module. There exists an integer n such that if $\operatorname{Ext}^i_A(X, Y) = 0$ for i sufficiently large, then $\operatorname{Ext}^i_A(X, Y) = 0$ for $i \geq n$.

(2) Finitistic Dimension Conjecture: $\operatorname{fd}(A) = \sup\{\operatorname{pd}_A X \mid \operatorname{pd}_A X < \infty\}$ is finite.

(3) Vanishing Conjecture: $K^b({}_A\mathcal{I})^\perp = 0$

(4) Nunke Condition: For an A-module ${}_AX$ there is $i \geq 0$ such that $\operatorname{Ext}^i_A({}_AD(A_A), {}_AX) \neq 0$.

(5) Generalized Nakayama Conjecture: For a simple module ${}_AS$ there is $i \geq 0$ such that $\operatorname{Ext}^i_A({}_AD(A_A), {}_AS) \neq 0$.

(6) Nakayama Conjecture: If in a minimal injective resolution of ${}_AA$

$$0 \to {}_AA \to I_0 \to I_1 \to \cdots$$

all I_j are projective, then A is a selfinjective algebra.

Note that there is a similar hierarchy of dual conjectures. But it is not clear that if A satisfies one of the conjectures (1) up to (5), then A satisfies the dual conjecture.

In the first section we briefly recall the well-known relationship between these conjectures. In the second section we report about some of the recent investigations on these conjectures. In the third section we will announce some reduction

1991 *Mathematics Subject Classification.* 16E10,18E30.

Key words and phrases. finitistic dimension, Nakayama conjecture, derived categories.

This article is an expanded version of a lecture given in October 1990 at the Université de Sherbrooke. I would like to thank I. Assem for the invitation and kind hospitality during this visit.

techniques using triangulated categories. For the proofs we refer to the references given.

1. Elementary remarks

Let us also introduce

$$\mathrm{fd}'(A) = \sup\{\mathrm{id}_A X \mid \mathrm{id}_A X < \infty\}$$

It is easy to construct examples such that $\mathrm{fd}(A) \neq \mathrm{fd}'(A)$. In fact consider the family of algebras A_n for $n \in \mathbb{N}$ given as the quiver algebra of

$$1 \xleftarrow{\alpha_n} 2 \xleftarrow{\alpha_{n-1}} 3 \xleftarrow{\alpha_{n-2}} \qquad \cdots \xleftarrow{\alpha_3} n-1 \xleftarrow{\alpha_2} n \circlearrowright \alpha_1$$

bound by $\alpha_1^2 = \alpha_1\alpha_2 = \cdots = \alpha_{n-1}\alpha_n = 0$. Then it is easy to see that $\mathrm{fd}(A) = n-1$ and $\mathrm{fd}'(A) = 0$.

Recall the following fact.

Lemma. *Let A be a finite-dimensional algebra. Then $\mathrm{fd}(A) = 0$ if and only if $\mathrm{Hom}_A(D(A_A), S) \neq 0$ for all simple A-modules S.*

Proof. In fact, let S be a simple A-module with $\mathrm{Hom}_A(D(A_A), S) = 0$, then $\mathrm{pd}_A\, \tau^- S = 1$, where τ^- denotes the Auslander-Reiten translation [AR2]. Conversely assume that $\mathrm{fd}(A) > 0$. Then there is a module Y with $\mathrm{pd}_A\, Y = 1$, hence $\mathrm{Hom}_A(D(A_A), \tau Y) = 0$. Thus there exists a simple S with $\mathrm{Hom}_A(D(A_A), S) = 0$. □

Let us recall the relationship of the conjectures mentioned above. It is shown in [H4] that (2) implies (3) and that (3) implies (4). The implication (4) to (5) is clear, and that (5) implies (6) was observed in [AR1]. The following relationship between (1) and (2) is due to Auslander [A2]. For the convenience of the reader we provide a proof. By A^{op} we denote the opposite algebra.

Proposition. *Let A be a finite-dimensional algebra. If the Auslander conjecture holds for the enveloping algebra $A^e = A \otimes_k A^{\mathrm{op}}$, then the finitistic dimension conjecture holds for A.*

Proof. This follows from the following identity for Hochschild-cohomolgy (see for example [CE], IX, 4.4). Let X, Y be A-modules. Then

$$H^i(A, \mathrm{Hom}_k(X, Y)) \simeq \mathrm{Ext}^i_A(X, Y)$$

where $H^i(A, {}_AM_A) = \mathrm{Ext}^i_{A^e}(A, M)$ for a bimodule M. Let X be an A-module with $\mathrm{pd}_A\, X < \infty$ and Y an arbitrary A-module. The identities above show that then $\mathrm{Ext}^i_{A^e}(A, \mathrm{Hom}_k(X, Y)) = 0$ for i sufficiently large. By assumption there is n such that if $\mathrm{Ext}^i_{A^e}(A, \mathrm{Hom}_k(X, Y)) = 0$ for i sufficiently large then we know that $\mathrm{Ext}^i_{A^e}(A, \mathrm{Hom}_k(X, Y)) = 0$ for $i \geq n$. We infer that $\mathrm{pd}_A\, X \leq n$. Thus $\mathrm{fd}(A) \leq n$. □

The following is a most probably incomplete list of references for the early treatment of these and related questions [A1], [AB1], [AB2], [AR1], [B1], [B2], [J1], [J2], [Na], [Nu], [Mu], [Se], [Sm], [T1], [T2].

There exist also conjectures dealing with infinite-dimensional modules, see [B1], [ZH4]. We refer to [ZH3] and [ZH4] for some recent interesting developments.

2. Special classes of algebras

In recent years some of the conjectures above have been verified for particular classes of finite-dimensional algebras.

2.1 Monomial algebras. The results given here are obtained in [GKK] and [IZ].

Let $\overrightarrow{\Delta}$ be a finite quiver and let $k\overrightarrow{\Delta}$ be the path algebra of $\overrightarrow{\Delta}$ over k. Let I be an ideal in $k\overrightarrow{\Delta}$ generated by paths of length at least two and containing all paths of length s for some integer s. Then $A = k\overrightarrow{\Delta}/I$ is called a *monomial algebra*. It has been shown in [GHZ] that projective resolutions of monomial algebras are rather well-behaved. This can also be seen by the following remarkable property of monomial algebras (compare [ZH2] or [ZH4]). If w is a path in $\overrightarrow{\Delta}$ we denote by $\bar{w}$ the residue class in A. Clearly there are only finitely many w such that $\bar{w} \neq 0$. We denote by $M(w) = A\bar{w}$ the A-module generated by $\bar{w}$.

Proposition. *Let A be a monomial algebra and $f\colon P \to Q$ a map between projective A-modules. Then* $\operatorname{Ker} f = \bigoplus_w M(w)$.

The following was shown in [GKK] and [IZ].

Theorem. *Let A be a monomial algebra. Then* $\operatorname{fd}(A) < \infty$.

In 2.3 we will give some remarks about the proof in [IZ]. For further investigations on homological properties of monomial algebras we refer to [ZH2].

2.2 Algebras with vanishing conditions. We follow the articles of [GZH] (see also [FS] and [ZH1]) and [DH].

For a finite-dimensional algebra A we denote by J the Jacobson radical of A.

Theorem. *Let A be a finite-dimensional algebra with $J^3 = 0$ then* $\operatorname{fd}(A) < \infty$.

The main idea of the proof in [GZH] is that the projective resolutions of modules of Loewy-length two can be controlled by a linear map which can be shown to be nilpotent.

Methods of linear algebra are also used in the next result from [DH].

Theorem. *Let A be a finite-dimensional algebra such that there is an integer s with $J^{2s+1} = 0$ and A/J^s representation-finite. Then the generalized Nakayama conjecture holds for A.*

We point out that variations of the proof can be used to verify the generalized Nakayama conjecture for other classes of algebras satisfying suitable conditions. For details we refer to [DH].

2.3 Finiteness conditions. It is trivial that all these conjectures hold for a representation-finite (i.e. there are only finitely many indecomposable modules up to isomorphism) algebra. There are some concepts which generalize this.

Let ${}_AX$ be an A-module and let

$$0 \to X \to I_0 \xrightarrow{\mu_o} I_1 \xrightarrow{\mu_1} I_2 \cdots$$

be a minimal injective resolution. Set $\Omega^{-i}X = \operatorname{Ker}\mu_i$ for $0 \leq i < \infty$. A module ${}_AX$ is called *cosyzygy-finite* or *ultimately closed* (compare [J1]) if $\{\Omega^{-i}X, i \geq 0\} \subseteq \operatorname{add} Y$ for an A-module Y, where $\operatorname{add} Y$ is the additive category of direct sums of direct summands of Y. We refer to [CF] for some related notions. The following result is due to [IZ].

Proposition. *Let A be a finite-dimensional algebra such that A/J is cosyzygy-finite, then* $\mathrm{fd}(A) < \infty$.

Proof. Let S be a simple A-module. Since S is cosyzygy-finite there is an integer r_S such that $\Omega^{-r_S} S \in \mathrm{add}(\bigoplus_{i<r_S} \Omega^{-i} S)$. Choose r_S minimal with this property and let $r = \max\{r_S \mid S \,\text{simple}\}$. Note that for a simple module S which satisfies $\mathrm{id}_A S = s < \infty$ we have that $r > s$. If $\mathrm{fd}(A) = \infty$ there is an A-module X with $\infty > \mathrm{pd}_A X = t+1 > r$. So there is a simple S with $\mathrm{Ext}_A^{t+1}(X, S) \neq 0$. But then $\mathrm{id}_A S = \infty$ by the remark above.

Now $\mathrm{Ext}_A^{t+1}(X, S) \simeq \mathrm{Ext}_A^1(X, \Omega^{-t} S)$ shows that there is an indecomposable direct summand Y_t of $\Omega^{-t} S$ such that $\mathrm{Ext}_A^1(X, Y_t) \neq 0$. By the choice r there is an integer $m < r$ such that Y_t is a direct summand of $\Omega^{-m} S$. But then Y_t is a direct summand of $\Omega^{-s} S$ for infinitely many s. Thus $0 \neq \mathrm{Ext}_A^1(X, \Omega^{-s} S) \simeq \mathrm{Ext}_A^s(X, S)$ for infinitely many s, in contrast to $\mathrm{pd}_A X < \infty$. □

The proof of the result in 2.1 is then obtained by showing that for a monomial algebra A the simple modules are cosyzygy-finite (see 2.1).

We point out that a similar proof shows that $\mathrm{fd}(A) < \infty$ if ${}_A A$ is cosyzygy-finite.

It is easy to construct examples where the simples do not have this property. In fact, let $A = k[x, y]/(x^2, y^2)$. Then the unique simple A-module S is not cosyzygy-finite.

A quite different approach was taken in [AR3] (see also [AR4]) while using the concept of contravariantly finite subcategories as introduced in [AS1] and [AS2]. We recall the relevant notions. Let $\mathcal{D}$ be a full subcategory of $\mathrm{mod}\, A$. It is always assumed to be closed under direct sums, direct summands and isomorphisms. The subcategory $\mathcal{D}$ is called *contravariantly finite* in $\mathrm{mod}\, A$ if every $X \in \mathrm{mod}\, A$ has a *right $\mathcal{D}$-approximation*, i.e. there is a morphism $F_X \to X$ with $F_X \in \mathcal{D}$ such that the induced morphism $\mathrm{Hom}_A(D, F_X) \to \mathrm{Hom}_A(D, X)$ is surjective for all $D \in \mathcal{D}$.

If the A-module X admits a right $\mathcal{D}$-approximation, then X clearly admits a *minimal* right $\mathcal{D}$-approximation (i.e. a right approximation of minimal length).

The subcategory $\mathcal{D}$ is called *resolving* if $\mathcal{D}$ is closed under extensions, kernels of surjective maps and contains ${}_A A$. Note that for a contravariantly finite subcategory which is resolving every right approximation is surjective [AR3].

Theorem. *For a contravariantly finite resolving subcategory $\mathcal{D}$ the objects in $\mathcal{D}$ consist of the summands of modules which have a filtration with composition factors the minimal right-approximations of the simple modules.*

An easy but important observation is that the subcategory

$$\mathcal{P}(A) = \{{}_A X \mid \mathrm{pd}_A X < \infty\}$$

is a resolving subcategory.

Corollary. *If $\mathcal{P}(A)$ is contravariantly finite, then* $\mathrm{fd}(A) < \infty$.

There are examples of algebras A such that $\mathcal{P}(A)$ is not contravariantly finite [ITS].

A different sort of finiteness condition can be obtained as follows. Consider the following subcategory $\mathcal{S}$ of $\mathrm{mod}\, A$.

$$\mathcal{S} = \{{}_A X \mid \mathrm{pd}_A X < \infty, \text{ each proper submodule } U \text{ satisfies } \mathrm{pd}_A U = \infty\}$$

Proposition. *If the length of the indecomposable modules in $\mathcal{S}$ is bounded, then* $\mathrm{fd}(A) < \infty$.

Proof. We sketch the proof using some ideas from [Sc]. Clearly it is enough to show that the projective dimension of modules in $\mathcal{S}$ is bounded. For a given integer r we consider the algebraic variety $\mathcal{M}_r(A)$ of A-modules of length r. Consider the subset $\mathcal{P}_t \subseteq \mathcal{M}_r(A)$ formed by those modules X which satisfy $\operatorname{pd}_A X \leq t$. As in [Sc] it follows that $\mathcal{P}_t$ is an open subset in the Zariski-topology. So we obtain an ascending chain of open subsets $\mathcal{P}_0 \subseteq \mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots$ which has to become stationary, since $\mathcal{M}_r(A)$ is finite-dimensional. So there exists t_r with $\operatorname{pd}_A X \leq t_r$ for all ${}_AX \in \mathcal{M}_r(A)$.

By assumption there is an integer m such that the length of indecomposable modules in $\mathcal{S}$ is bounded by m. Hence $\operatorname{fd}(A) \leq \max\{r_t \mid 1 \leq t \leq m\}$, which shows the assertion. □

2.4 Bounds for the finitistic dimension. An interesting question is to obtain good bounds for the finitistic dimension. In the cases discussed above this problem was solved. Let us list the corresponding results.

Let A be a monomial algebra. Consider the following set $\mathcal{W}$ of paths in $\overrightarrow{\Delta}$.

$$\mathcal{W} = \{w \mid \bar{w} \neq 0,\, \operatorname{pd}_A M(w) < \infty\}$$

Let $r = \max\{\operatorname{pd}_A M(w) \mid w \in \mathcal{W}\}$ if $\mathcal{W}$ is non-empty and $r = -1$ if $\mathcal{W}$ is empty.

The result in 2.1 clearly has the following application.

Corollary. *Let A be a monomial algebra. Then* $\operatorname{fd}(A) \leq r + 2$.

We refer to [ZH4] for a slightly different definition of $\mathcal{W}$ to obtain a sharper bound.

In [GZH] the following bound for $\operatorname{fd}(A)$ was found for an algebra A with $J^3 = 0$. For this let $S_1, \ldots, S_n$ be a complete set of simple A-modules. We may assume that there is m such that $\operatorname{pd}_A S_i = \infty$ for $1 \leq i \leq m$ and that $\operatorname{pd}_A S_i < \infty$ for $m < i \leq n$. Let $d = \max\{\operatorname{pd}_A S_i \mid m < i \leq n\}$.

Proposition. *Let A be a finite-dimensional algebra with vanishing radical cube. Then* $\operatorname{fd}(A) \leq 2m + d + 1$.

In the situation of 2.3 the theorem gives the following bound for $\operatorname{fd}(A)$ if $\mathcal{P}(A)$ is contravariantly finite. Again let $S_1, \ldots, S_n$ be a complete set of simple A-modules. Let $F_i \in \mathcal{P}(A)$ be the minimal right $\mathcal{P}(A)$-approximation of S_i.

Corollary. *If $\mathcal{P}(A)$ is a contravariantly finite subcategory. Then* $\operatorname{fd}(A) \leq \max\{\operatorname{pd}_A F_i \mid 1 \leq i \leq n\}$.

3. Reduction techniques

The following is a summary of results in [H3], [H4] and [H6].

For the convenience of the reader we recall some of the terminology for complexes which we have to use.

Let $\mathfrak{a}$ be an arbitrary additive subcategory of $\operatorname{mod} A$.

A *complex* $X^\bullet = (X^i, d_X^i)_{i\in\mathbb{Z}}$ over $\mathfrak{a}$ is a collection of objects X^i from $\mathfrak{a}$ and morphisms $d^i = d_X^i \colon X^i \to X^{i+1}$ such that $d^i d^{i+1} = 0$. A complex $X^\bullet = (X^i, d_X^i)$ is *bounded below* if $X^i = 0$ for all but finitely many $i < 0$. It is called *bounded above* if $X^i = 0$ for all but finitely many $i > 0$. It is *bounded* if it is bounded below and bounded above. It is said to have *bounded cohomology* if $H^i(X^\bullet) = 0$ for all but finitely many $i \in \mathbb{Z}$, where by definition $H^i(X^\bullet) = \operatorname{Ker} d_X^i / \operatorname{Im} d_X^{i-1}$. Denote by $C(\mathfrak{a})$ the category of complexes over $\mathfrak{a}$, by $C^{-,b}(\mathfrak{a})$ (resp.$C^{+,b}(\mathfrak{a})$, resp. $C^b(\mathfrak{a})$) the full subcategories of complexes bounded above with bounded cohomology (resp. bounded below with bounded cohomology, resp. bounded above and below).

If $X^\bullet = (X^i, d_X^i)_{i\in\mathbb{Z}}$ and $Y^\bullet = (Y^i, d_Y^i)_{i\in\mathbb{Z}}$ are two complexes, a morphism $f^\bullet\colon X^\bullet \to Y^\bullet$ is a sequence of morphisms $f^i\colon X^i \to Y^i$ of $\mathfrak{a}$ such that

$$d_X^i f^{i+1} = f^i d_Y^i$$

for all $i \in \mathbb{Z}$. The *translation functor* is defined by

$$(X^\bullet[1])^i = X^{i+1} \quad , \quad (d_{X[1]})^i = -(d_X)^{i+1} \ .$$

The mapping cone $C_{f^\bullet}$ of a morphism $f^\bullet\colon X^\bullet \to Y^\bullet$ is the complex

$$C_{f^\bullet} = ((X^\bullet[1])^i \oplus Y^i, d_{C_f}^i)$$

with 'differential'

$$d_{C_f}^i = \begin{pmatrix} -d_X^{i+1} & f^{i+1} \\ 0 & d_Y^i \end{pmatrix} \ .$$

We denote by $K^{-,b}(\mathfrak{a}), K^{+,b}(\mathfrak{a})$ and $K^b(\mathfrak{a})$ the homotopy categories of the categories of complexes introduced above. Note that all these categories are triangulated categories in the sense of [V].

Recall that two morphisms $f^\bullet, g^\bullet\colon X^\bullet \to Y^\bullet$ are called *homotopic*, if there exist morphisms $h^i\colon X^i \to Y^{i-1}$ such that $f^i - g^i = d_X^i h^{i+1} + h^i d_Y^{i-1}$ for all $i \in \mathbb{Z}$.

We have denoted by ${}_A\mathcal{P}$ (resp. ${}_A\mathcal{I}$) the full subcategory of $\operatorname{mod} A$ formed by the projective (resp. injective) A-modules. Then we identify the derived category $D^b(A)$ of bounded complexes over $\operatorname{mod} A$ with $K^{-,b}({}_A\mathcal{P})$ or with $K^{+,b}({}_A\mathcal{I})$. In case A has finite global dimension this yields the identification of $D^b(A)$ with $K^b({}_A\mathcal{P})$ or with $K^b({}_A\mathcal{I})$, since the natural embedding of $K^b({}_A\mathcal{P})$ into $K^{-,b}({}_A\mathcal{P})$ is an equivalence in this case. We identify the derived category $D^-(A)$ of complexes bounded above over $\operatorname{mod} A$ with $K^-({}_A\mathcal{P})$ and we identify the derived category $D^+(A)$ of complexes bounded below over $\operatorname{mod} A$ with $K^+({}_A\mathcal{I})$. For a more detailed analysis of the derived category we refer to [G],[Gr] and [V].

3.1 Auslander-Reiten triangles. In [H1] we introduced the notion of an Auslander-Reiten triangle in a triangulated category. We first recall the relevant definitions.

Let $\mathcal{C}$ be a triangulated category such that $\operatorname{Hom}_{\mathcal{C}}(X,Y)$ is a finite-dimensional k-vector space for all $X, Y \in \mathcal{C}$ and assume that the endomorphism ring of an indecomposable object is local. This assumption ensures that $\mathcal{C}$ is a Krull-Schmidt category. We denote by $X[1]$ the value of the translation functor on the object X of $\mathcal{C}$.

A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ in $\mathcal{C}$ is called an *Auslander-Reiten triangle* if the following conditions are satisfied:

(AR1) X, Z are indecomposable,
(AR2) $w \neq 0$,
(AR3) If $f\colon W \to Z$ is not a retraction, then there exists $f'\colon W \to Y$ such that $f'v = f$.

We will say that $\mathcal{C}$ has Auslander-Reiten triangles if for all indecomposable objects $Z \in \mathcal{C}$ there exists a triangle satisfying the conditions above.

Recall that a *translation quiver* $\overrightarrow{\Gamma} = (\Gamma_0, \Gamma_1, \tau)$ is given by a (locally finite) quiver (Γ_0, Γ_1) (Γ_0 denotes the vertex set, Γ_1 denotes the set of arrows) together with an injective map $\tau\colon \Gamma_0' \to \Gamma_0$ defined on a subset $\Gamma_0' \subseteq \Gamma_0$ such that for any $z \in \Gamma_0'$, and any $y \in \Gamma_0$ the number of arrows from y to z is equal to the number of arrows from τz to y. The map τ is called the *translation*. If $\Gamma_0' = \Gamma_0$ and τ is a bijection we say that $\overrightarrow{\Gamma}$ is a stable translation quiver.

If the triangulated category $\mathcal{C}$ has Auslander-Reiten triangles then $\overrightarrow{\Gamma}(\mathcal{C})$ has the structure of a translation quiver (see [H2]).

We refer to [H1] and [H2] for some properties of Auslander-Reiten triangles.

Theorem. *Let A be a finite-dimensional k-algebra. Then the following are equivalent.*

(i) $\operatorname{pd}_A D(A_A) < \infty$
(ii) $K^b({}_A\mathcal{P})$ *has Auslander-Reiten triangles.*

We recall that the finitistic dimension conjecture implies the following for a finite-dimensional algebra A (see [AR3] or [H4]).

$$(*) \quad \operatorname{id}_A A < \infty \quad \text{if} \quad \operatorname{pd}_A D(A_A) < \infty.$$

The theorem and its dual now imply.

Corollary. *Let A be a finite dimensional algebra such that* $\operatorname{pd}_A D(A_A) < \infty$. *If the translation τ on $\overrightarrow{\Gamma}(K^b({}_A\mathcal{P}))$ is surjective, then* $\operatorname{id}_A A < \infty$.

We refer to 3.2 for problems related to $(*)$.

3.2 Tilting invariance. A module $T \in \operatorname{mod} A$ is called a (generalized) *tilting module* if the following conditions are satisfied:

(i) $\operatorname{pd}_A T < \infty$
(ii) $\operatorname{Ext}^i_A(T,T) = 0$ for all $i > 0$
(iii) There is a long exact sequence $0 \to {}_AA \to T_0 \to \cdots \to T_m \to 0$ with $T_j \in \operatorname{add} T$.

A module ${}_AT$ satisfying the properties (i) and (ii) is called a *partial tilting* module.

We refer to [H1],[H2] and [Mi] for an outline of tilting theory in this case and to [Ri1], [Ri2] and [Ri3] for the general notion of derived equivalence.

If T is a tilting module and $B = \operatorname{End}_A T$. Then it is known that $\operatorname{gl.dim} A < \infty$ if and only if $\operatorname{gl.dim} B < \infty$. The next result is a generalization of this.

Theorem. *Let A be a finite-dimensional algebra and T a tilting module. Let* $B = \operatorname{End}_A T$. *Then* $\operatorname{fd}(A) < \infty$ *if and only if* $\operatorname{fd}(B) < \infty$.

We now come back to the property $(*)$ in 3.1.

If ${}_AM$ is an A-module we may decompose ${}_AM = \oplus_{i=1}^s M_i^{n_i}$ with M_i indecomposable, $M_i \not\simeq M_j$ for $i \neq j$ and $n_i > 0$. In this case we denote the number s of non-isomorphic indecomposable direct summands of M by $\delta(M)$.

It is easy to see that if ${}_AT$ is a tilting module, then $\delta(T) = \operatorname{rk} K_0(A) = n$, where $K_0(A)$ is the Grothendieck group of A.

It is not known that a partial tilting module which satisfies $\delta(T) = \operatorname{rk} K_0(A) = n$ is a tilting module, unless $\operatorname{pd}_A T \leq 1$ by a result in [Bo]. It is shown in [RS] that this holds if A is representation-finite.

Note that $(*)$ is a special case of this problem.

3.3 Recollement. Let $\mathcal{C}, \mathcal{C}'$ and $\mathcal{C}''$ be triangulated categories. Following [BBD] (see also [GV]) a *recollement* of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ is given by

$$\begin{tikzcd}[column sep=large] \mathcal{C}' \arrow[r, "i_*", "i_!"'] & \mathcal{C} \arrow[l, shift right=7, "i^!"'] \arrow[l, shift left=7, "i^*"] \arrow[r, "j^!", "j^*"'] & \mathcal{C}'' \arrow[l, shift right=7, "j_*"'] \arrow[l, shift left=7, "j_!"] \end{tikzcd}$$

such that

(RI) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs of exact functors and that $i_* = i_!,\ j^! = j^*$

(RII) $j^* i_* = 0$

(RIII) $i^* i_* \simeq id$, $id \simeq i^! i_!$, $j^* j_* \simeq id$ and $id \simeq j^! j_!$

(RIV) For $X \in \mathcal{C}$ there are triangles

$$j_! j^! X \to X \to i_* i^* X \to j_! j^! X[1]$$

$$i_! i^! X \to X \to j_* j^* X \to i_! i^! X[1].$$

(The morphisms in (RIII) and (RIV) are the adjunction morphisms.)

We refer to [BBD] for properties of recollements and to [Kö] for necessary and sufficient conditions that $D^-(A)$ has a recollement relative to $D^-(A')$ and $D^-(A'')$ for some finite-dimensional algebras A, A', A''.

In particular we mention the following result from [Kö]. If $D^-(A)$ has a recollement relative to $D^-(A')$ and $D^-(A'')$ for some finite-dimensional algebras A, A', A'' and one of the algebras A, A', A'' has finite global dimension then $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$.

Theorem. *Let A be a finite-dimensional algebra and assume that $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$ for some finite-dimensional algebras A', A''. Then* $\mathrm{fd}(A) < \infty$ *if and only if* $\mathrm{fd}(A') < \infty$ *and* $\mathrm{fd}(A'') < \infty$.

We point out that the theorem above generalizes a theorem in [W].

We give two examples in which this theorem may be applied. We stress that there exist proofs of these results avoiding the use of triangulated categories.

Let A', A'' be finite-dimensional algebras and let ${}_{A'}M_{A''}$ be a bimodule. Consider the triangular matrix algebra A of the form

$$A = \begin{pmatrix} A' & M \\ 0 & A'' \end{pmatrix}$$

with multiplication

$$\begin{pmatrix} a' & m \\ 0 & a'' \end{pmatrix} \begin{pmatrix} b' & m' \\ 0 & b'' \end{pmatrix} = \begin{pmatrix} a'b' & a'm' + mb'' \\ 0 & a''b'' \end{pmatrix}$$

where $a', b' \in A', m, m' \in M$ and $a'', b'' \in A''$.

Then $D^-(A)$ has a recollement relative to $D^-(A')$ and $D^-(A'')$.

Assume that A' or A'' has finite global dimension. Then $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$. In particular, $\mathrm{fd}(A) < \infty$ if $\mathrm{fd}(A') < \infty$ and $\mathrm{fd}(A'') < \infty$.

In the next example we will use the concept of perpendicular categories as introduced in [GL], see also [H5].

Let $X \in \operatorname{mod} A$ with $\operatorname{pd}_A X \leq 1$. We define the right *perpendicular category* $X^\perp$ to be the full subcategory of mod$A$ whose objects Z satisfy

$$\operatorname{Hom}_A(X, Z) = 0 = \operatorname{Ext}^1_A(X, Z).$$

It is straightforward to see that $X^\perp$ is an abelian category, which is closed under extensions and that the inclusion functor $X^\perp \hookrightarrow \operatorname{mod} A$ is exact. The next result states some useful properties of $X^\perp$ under additional assumptions. For the proof we refer to [GL] or [H5].

Theorem. *Let $X \in \operatorname{mod} A$ such that $\operatorname{pd}_A X \leq 1$ and $\operatorname{Ext}^1_A(X, X) = 0$, then there exists ${}_AQ \in X^\perp$ such that $X^\perp \simeq \operatorname{mod} A_0$, with $A_0 = \operatorname{End}_A Q$. If X is indecomposable, then* $\operatorname{rk} K_0(A_0) = \operatorname{rk} K_0(A) - 1$.

Now assume that A admits a simple A-module S with $\operatorname{pd}_A S = 1$. Then $D^-(A)$ has a recollement relative to $D^-(A')$ and $D^-(A'')$, where $A' = \operatorname{End}_A Q$ for a projective generator ${}_AQ$ of $S^\perp$ and $A'' = k$. Since $\operatorname{gl.dim} k = 0$ we infer that $D^b(A)$ has a recollement relative to $D^b(A')$ and $D^b(A'')$. In particular, $\operatorname{fd}(A) < \infty$ if $\operatorname{fd}(A') < \infty$.

3.4 Grothendieck groups. The generalized Nakayama conjecture is related to a problem about Grothendieck groups of triangulated categories. First we need a reformulation of the generalized Nakayama conjecture which is due to [AR1].

(5'): Let ${}_AM$ be a generator for $\operatorname{mod} A$ with $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$, then ${}_AM$ is projective.

The following is shown in [AR1]. The generalized Nakayama conjecture holds for all finite-dimensional algebras if and only if the conjecture (5') holds for all finite-dimensional algebras.

Let us indicate one direction. We assume that (5') holds for all finite-dimensional algebras with $\operatorname{rk} K_0(A) = n - 1$ and we claim that the generalized Nakayama conjecture holds for all finite-dimensional algebras with $\operatorname{rk} K_0(A) = n$. In fact, let A be an algebra with $\operatorname{rk} K_0(A) = n$ and let

$$\cdots \to P_2 \to P_1 \to P_0 \to D(A_A) \to 0$$

be a minimal projective resolution of $D(A_A)$. Assume that there is a simple A-module S with $\operatorname{Ext}^i_A(D(A_A), S) = 0$ for all i. Let $P(S)$ be the projective cover of S. Then $P(S)$ is not a direct summand of P_i for all i. Let ${}_AA = P \oplus P(S)^r$ such that $P(S)$ is not a summand of P. Let $B = \operatorname{End}_A P$. Then it follows from [Ri1] that we have a full exact embedding of triangulated categories $D^-(B) \to D^-(A)$. By the choice of P we infer that $K^b({}_A\mathcal{I})$ is contained in $D^-(B)$. Using the obvious identifications we may consider $D(A_A)$ as B-module. But $\delta(D(A_A)) = n$ and $\operatorname{Ext}^i_B(D(A_A), D(A_A)) = 0$ for all $i > 0$ yields a contradiction to (5').

We recall now the definition of the Grothendieck group of a triangulated category [Gr]. For this let $\mathcal{C}$ be a triangulated category. Let $\mathcal{F}$ be the free abelian group on the isomorphism classes of objects in $\mathcal{C}$. The isomorphism class of an object $X \in \mathcal{C}$ is denoted by $[X]$. Let $\mathcal{F}'$ be the subgroup of $\mathcal{F}$ generated by $[X] + [Z] - [Y]$ for all triangles $X \to Y \to Z \to X[1]$ in $\mathcal{C}$. Then by definition the *Grothendieck group* of $\mathcal{C}$ is $K_0(\mathcal{C}) = \mathcal{F}/\mathcal{F}'$.

Let $F : \mathcal{C}' \to \mathcal{C}$ be an exact functor of triangulated categories. Then there is an induced map $K_0(F) : K_0(\mathcal{C}') \to K_0(\mathcal{C})$.

For example consider the embedding of $K^b({}_A\mathcal{P})$ into $D^b(A)$. Then the induced map on the level of Grothendieck groups turns out to be the Cartan map (see [B3] for a definition). In particular we see that $K_0(F)$ need not to be injective, if F is an embedding. We also state the following.

Proposition. *Let $\mu\colon D^b(A) \to D^-(A)$ be the canonical embedding. Then $K_0(\mu) = 0$.*

It was shown in [Gr] that $K_0(D^b(A)) \simeq K_0(A)$, which is isomorphic to $\mathbb{Z}^n$, with $n = \delta({}_A A)$.

Following [V] we call a full triangulated subcategory $\mathcal{C}'$ of a triangulated subcategory $\mathcal{C}$ an *épaisse subcategory*, if $\mathcal{C}'$ is closed under direct summands. We consider the following condition:

(5"): Let $\mathcal{C}$ be an épaisse subcategory of $D^b(A)$ such that $K_0(A)$ is finitely generated. Then $\operatorname{rk} K_0(\mathcal{C}) \leq n$.

Remark. If (5") holds then the generalized Nakayama conjecture holds.

Proof. By the mentioned result of [AR1] it is enough to verify condition (5'). Let ${}_A M$ be a generator which satisfies $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$. Then it is easy to see that we obtain a full embedding $K^b(\operatorname{add} M) \to D^b(A)$ (compare [H1]). So we may consider $K^b(\operatorname{add} M)$ as an épaisse subcategory of $D^b(A)$. A straightforward calculation shows that $K_0(K^b(\operatorname{add} M)) \simeq \mathbb{Z}^{\delta(M)}$. So $\delta(M) \leq n$ by (5"). Since M is a generator we know that ${}_A A$ is a direct summand of M, hence M is projective. ☐

Note that the proof actually shows that for an arbitrary A-module M which satisfies $\operatorname{Ext}^i_A(M, M) = 0$ for all $i > 0$ the number $\delta(M) \leq n$ in case (5") holds.

It is easy to construct counterexamples to the condition (5") if we leave out the assumptions that $\mathcal{C}$ is épaisse or that $K_0(\mathcal{C})$ is finitely generated. For instance let A be a finite-dimensional tame hereditary algebra and let $\mathcal{C} = D^b(\mathcal{R})$ be the derived category of the abelian subcategory $\mathcal{R} \subset \operatorname{mod} A$ of regular A-modules. Clearly $\mathcal{C}$ is an épaisse subcategory of $D^b(A)$, but $K_0(\mathcal{C})$ is not finitely generated. We thank H. Lenzing for this example, which led to a reformulation of a more optimistic version of (5").

References

[A1] Auslander, M., On dimensions of modules and algebras III, Nagoya Math.J., 1955, vol. 9, 65–77.

[A2] Auslander, M., private communication, 1989.

[AB1] Auslander, M.; Buchsbaum, D., Homological dimension in noetherian rings, Proc. Nat. Acad. Sci. U.S.A., 1956, vol. 42, 36–38.

[AB2] Auslander, M.; Buchsbaum, D., Homological dimension in noetherian rings II, Trans. Amer. Math. Soc., 1957 vol. 85, 390–405.

[AR1] Auslander, M.; Reiten, I., On a generalized version of the Nakayama conjecture, Proc. Amer. Math. Soc., 1975 vol. 52, 69–74.

[AR2] Auslander, M; Reiten, I., Representation theory of Artin algebras III, Comm. Algebra, 1975 vol. 3, 239–294.

[AR3] Auslander, M; Reiten, I., Applications of contravariantly finite subcategories, Adv. Math., 1991 vol. 86, 111–152.

[AR4] Auslander, M; Reiten, I., Homologically finite subcategories, preprint

[AS1] Auslander, M; Smalø, S., Preprojective modules over Artin algebras, J. Algebra, 1980 vol. 66, 61–122.

[AS2] Auslander, M; Smalø, S., Almost split sequences in subcategories, J. Algebra vol. 69, 1981, 426–454.

[B1] Bass, H., Finitistic dimension and a homological generalization of semiprimary rings, Trans. Amer. Math. Soc., 1960 vol. 95, 466–488.

[B2] Bass, H., Injective dimension in noetherian rings, Trans. Amer. Math. Soc., 1962, vol. 102, 18–29.

[B3] Bass, H., Algebraic K-Theory, Benjamin, New York 1968.

[BBD] Beilinson, A.A.; Bernstein, J.; Deligne, P., Faisceaux pervers, Astérique, 1982, vol. 100.

[Bo] Bongartz, K., Tilted algebras, Springer Lecture Notes, 1981, vol. 903, 26–38.

[CE] Cartan, H.; Eilenberg, S., Homological Algebra, Princeton University Press, 1956.

[CF] Colby, R.R.; Fuller, K.R., A note on the Nakayama conjecture, Tsukuba J. Math., 1990, vol. 11, 343–352.

[DH] Dräxler, P.; Happel, D., A proof of the generalized Nakayama conjecture for for algebras with $J^{2s+1} = 0$ and A/J^s representation finite, J. of Pure and Applied Alg. to appear.

[FS] Fuller, K.R; Saorín, M., On the finitistic dimension conjecture and the theorem of E. Green and B. Zimmermann-Huisgen , preprint.

[G] Grivel, P.P., Catégories dérivées et foncteur dérivés, in Algebraic D-modules, Academic Press, New York 1987, 1–108.

[GHZ] Green, E.; Happel, D.; Zacharia, D., Projective resolutions over artin algebras with zero relations, Illinois Journal of Mathematics, 1985 vol. 29, 180–190.

[GKK] Green, E.; Kirkmann, E.; Kuzmanovich, J., Finitistic dimension of finite dimensional monomial algebras, J. Algebra, 1991 vol. 136, 37–51.

[GL] Geigle, W.; Lenzing, H., Perpendicular categories with applications to representations and sheaves, preprint.

[Gr] Grothendieck, A., Groupes des classes des catégories abeliennes et triangulée, SGA 5, Springer Lecture Notes 589, Heidelberg 1977, 351–371.

[GV] Grothendieck, A.; Verdier, J.L., Topos, Exposé IV in Springer Lecture Notes, 1972, vol. 269, 299–519.

[GZH] Green, E; Zimmermann Huisgen, B., Finitistic dimension of artinian rings with vanishing radical cube, Math. Zeitschrift , 1991, vol. 206, 505–526.

[H1] Happel, D., On the derived category of a finite-dimensional algebra, Comment. Math. Helv., 1987, vol. 62, 339–389.

[H2] Happel, D., Triangulated categories in the representation theory of finite-dimensional algebras, Cambridge University Press, 1988, vol. 119.

[H3] Happel, D., Auslander-Reiten triangles in derived categories of finite-dimensional algebras, Proc. AMS, to appear.

[H4] Happel, D., On Gorenstein algebras, in Representation Theory of Finite Groups and Finite-Dimensional Algrebras, Birkhäuser Verlag, Basel, 1991, 389–404.

[H5] Happel, D., Partial tilting modules and recollement, Proceedings Malcev Conference, to appear.

[H6] Happel, D., Reduction techniques for homological conjectures, in preparation.

[ITS] Igusa, K.; Todorov, G.; Smalø, S., Finite projectivity and contravariant finiteness, Proc. Amer. Math. Soc., 1991.

[IZ] Igusa, K.; Zacharia, D., Syzygy pairs in a monomial algebra, Proc. Amer. Math. Soc., 1990, vol. 108, 601–604.

[J1] Jans, J. P., Some genralizations of finite projective dimension, Illinois J. Math., 1961, vol. 5, 334–344.

[J2] Jans, J. P., Duality in noetherian rings, Proc. Amer. Math. Soc., 1961, vol. 12, 829–835.

[Kö] König, S., Tilting complexes,perpendicular categories and recollement of derived module categories of rings , J. of Pure and Applied Algebra, 1991, vol. 73, 211–232.

[Na] Nakayama, T., On algebras with complete homology, Abh. Math. Sem. Univ. Hamburg, 1958 vol. 22, 300–307.

[Nu] Nunke, R.J., Modules of extensions over Dedekind rings, Illinois J. Math., 1959, vol. 3, 222–242.

[Mi] Miyashita, Y., Tilting modules of finite projective dimension, Math. Zeitschrift, 1986, vol. 193, 113–146.

[Mu] Müller, B.J., The classification of algebras by dominant dimension, Can. J. Math., 1968, vol. 20, 398–409.

[Ri1] Rickard, J., Morita theory for derived categories, J. London Math. Soc., 1989, vol. 39, 436–456.

[Ri2] Rickard, J., Derived categories and stable equivalence, J. Pure Appl. Algebra, 1989, vol. 61, 303–317.

[Ri3] Rickard, J., preprint, Derived equivalences as derived functors

[RS] Rickard, J.; Schofield, A., Cocovers and tilting modules, Math. Proc. Camb. Phil. Soc., 1989, vol. 106, 1–5.

[Sc] Schofield, A., Bounding the global dimension in terms of the dimension, Bull. London Math. Soc., 1985, vol. 17, 393–394.

[Se] Serre, J.-P., Sur la dimension homologique des anneaux et des modules noethériens, Proc. Intl. Symp. on Algebraic Number Theory, Tokyo, 1955, 175–189.

[Sm] Small, L., A change of rings theorem, Proc. Amer. Math. Soc., 1968, vol. 19, 662–666.

[T1] Tachikawa, H., On dominant dimensions of QF-3 algebras, Trans. Amer. Math. Soc., 1964, vol. 112, 249–266.

[T2] Tachikawa, H., Quasi-Frobenius rings and generalizations, Springer Lecture Notes, Heidelberg, 1973, vol. 351.

[V] Verdier, J.L. ,Catégories dérivées, état 0, Springer Lecture Notes, 1977, vol. 569, 262–311.

[W] Wiedemann, A., On stratifications of derived module categories, preprint.

[ZH1] Zimmermann Huisgen, B., Bounds on finitistic and global dimension for finite dimensional algebras with vanishing radical cube, preprint.

[ZH2] Zimmermann Huisgen, B., Predicting syzygies over finite dimensional monomial relation algebras, Manuscripta Mathematica, 1991, vol. 70, 157–182.

[ZH3] Zimmermann Huisgen, B., Field-dependent homological behaviour of finite dimensional algebras, preprint.

[ZH4] Zimmermann Huisgen, B., Homological domino effects and the first finitistic dimension conjecture, preprint.

Fakultät für Mathematik, Universität Bielefeld, Postfach 8640, 4800 Bielefeld 1, FRG