Aufgabenzettel 3: English version

3.1. *Let p be a prime, and t, n natural numbers with $1 \leq n \leq p^t$. If s is maximal such that p^s divides n, then*

$$\binom{p^t}{n}_p = p^{t-s}.$$

Hints for the proof: First, we show: Let x, x', y be positive real numbers with $\frac{x+x'}{y} \in \mathbb{N}$. *If $\frac{x}{y} \in \mathbb{N}$, then*

$$\lfloor \frac{x+x'}{y} \rfloor = \lfloor \frac{x}{y} \rfloor + \lfloor \frac{x'}{y} \rfloor.$$

If $\frac{x}{y}$ is not a natural number, then

$$\lfloor \frac{x+x'}{y} \rfloor = \lfloor \frac{x}{y} \rfloor + \lfloor \frac{x'}{y} \rfloor + 1.$$

Show that this implies: $\lfloor \frac{p^t}{p^i} \rfloor = \lfloor \frac{n}{p^i} \rfloor + \lfloor \frac{p^t - n}{p^i} \rfloor$ if and only if p^i divides n. And show that this implies the assertion.

3.2. Let p be a prime with $\sqrt{2n} < p \leq 2n$. Take t maximal with $p \leq \frac{2n}{t}$. Show:
(a) If t is even, then $\binom{2n}{n}_p = 1$,
(b) If t is odd, then $\binom{2n}{n}_p = p$.
(This generalizes the assertions (1), (2) of 1.3.; as an example, write down the prime divisors $p > \sqrt{2n}$ of $\binom{400}{200}$.)

3.3. (Erdös). Use the following considerations in order to show:

$$\pi(n) \geq \frac{1}{2 \ln 2} \ln n.$$

(a) Show that there are at most $\sqrt{n}$ square numbers m with $m \leq n$ is
(b) Show that there are at most $2^{\pi(n)}$ squarefree numbers m with $m \leq n$.
(c) Show that the unique factorization property implies that

$$n \leq \sqrt{n}\, 2^{\pi(n)},$$

and that this implies the assertion.

(d) Campare this inequality with the inequality $\pi(n) \geq \frac{\ln 2}{4} \cdot \frac{n}{\ln n}$ which has been shown in the lecture.

3.4. Fermat numbers. These are the numbers of the form $F_n = 2^{2^n} + 1$ with $n \in \mathbb{N}_0$. (a) Let $G_n = F_n - 2 = 2^{2^n} - 1$. Show that $F_n G_n = G_{n+1}$ for $n \in \mathbb{N}_0$.

(b) Show that this implies $\prod_{t=0}^{n} F_t = G_{n+1}$.

(c) Use this in order to show that the Fermat numbers have pairwise no proper common divisor.

(d) Show that this implies that there are infinitely many prime numbers.

Präsenz-Aufgaben.

1. The sieve of Eratosthenes (276-194 v.u.Z.). Use exercise 1.4 in order to write down all prime numbers $p \leq 200$ as follows: First, determine the primes $p_1 < \cdots < p_6$ with $p_i \leq \sqrt{200}$. Now take a list of all numbers n with $1 \leq n \leq 200$. Delete first the number 1, then all multiples of the numbers p_i, for $1 \leq i \leq 6$ (thus, all even numbers, all numbers divisible by 3, all numbers divsible by 5, and so on). Claim: The remaining numbers are precisely the prime numbers p with $\sqrt{200} < p \leq 200$. Warum?

2. Cyclic groups.

(a) Determine all elements $\overline{z}$ in the group $G = (\mathbb{Z}/22, +)$ with $G = \langle \overline{z} \rangle$.

(b) Exhibit an element $\overline{z}$ in the group $G = ((\mathbb{Z}/23)^*, \cdot)$ with $G = \langle \overline{z} \rangle$.

(c) How does one find additional elements $\overline{z}$ in $G = ((\mathbb{Z}/23)^*, \cdot)$ such that $G = \langle \overline{z} \rangle$?

(d) Replace $p = 23$ in (b) and (c) by the prime numbers $p = 17$ and $p = 11$.