2. Simple representations, thin representations.

2.1. Paths.

Given a representation M of the quiver Q, one defines the support quiver $Q(M)$ of M as follows: its vertices are the vertices x of Q with $M_x \neq 0$, the arrows of $Q(M)$ are the arrows α of Q such that $M_\alpha \neq 0$.

If Q is a quiver, a *path* $w = \alpha_1 \cdots \alpha_n$ in Q of length $n \geq 1$ is a sequence of arrows $\alpha_1, \dots, \alpha_n$ such that $t(\alpha_i) = h(\alpha_{i+1})$ for $1 \leq i \leq n-1$. One calls $h(w) = t(\alpha_1)$ the *head* of w and $t(w) = t(\alpha_n)$ the *tail* of w; we also say that this is *a path from* $t(w)$ *to* $h(w)$. Such a path w should be visualized as follows:

$$\underset{h(w)}{\circ} \xleftarrow{\ \alpha_1\ } \circ \leftarrow \quad \cdots \quad - \circ \xleftarrow{\ \alpha_n\ } \underset{t(w)}{\circ}$$

In addition, any vertex x of Q is considered as a *path of length* 0 with head x and tail x, and then denoted by e_x. Actually, in order to have available a common notation for all the paths, we write also $(h(\alpha_1)|\alpha_1, \cdots, \alpha_n|t(\alpha_n))$ for the paths of length at least 1 and $e_x = (x||x)$ for those of length 0.

If two paths $w = (x|\alpha_1, \dots, \alpha_n|y)$ and $w' = (x'|\alpha'_1, \dots, \alpha'_{n'}|y')$ are given, the *concatenation* ww' is defined provided $y = x'$ and then $ww' = (x|\alpha_1, \dots, \alpha_n, \alpha'_1, \dots, \alpha'_{n'}|y')$; otherwise (if $y \neq x'$), we write $ww' = 0$.

> Here, 0 is a new element called the *zero path*. We later will consider the so-called path algebra kQ of the quiver; it is the k-vector space with basis all the non-zero paths, and 0 will be the zero vector in kQ; using the product of paths which we just have introduced, kQ becomes an associative k-algebra.

A quiver Q is called *strongly connected* provided for every pair x, y of vertices of Q there is a path w of length at least 1 with $h(w) = x, t(w) = y$. (In the definition, the vertices x, y are **not** assumed to be different; in particular, a strongly connected quiver has for every vertex x a path w of length at least 1 with $h(w) = x = t(w)$.)

An *oriented cycle* is a path w of length at least 1 such that $h(w) = t(w)$. Such an oriented cycle $w = \alpha_1 \cdots \alpha_n$ is said to be *elementary* provided the vertices $t(\alpha_i)$ with $1 \leq i \leq n$ are pairwise different.

If $w = (x|\alpha_1, \dots, \alpha_n|y)$ is a path, then we write wM or $w(M)$ for the image $\alpha_1 \cdots \alpha_n(M_y)$ in M_x. In particular, if $e_x = (x||x)$ is a path of length 0, then $e_xM = M_x$.

> This explains why we draw arrows as pointing from right to left when we deal with paths: in this way we follow the convention of writing maps on the left of argument so that the composition of first a map f and second a map g has to be denoted as gf.

2.2. Simple representations.

A representation S of Q is said to be *simple* (or *irreducible* provided S is non-zero and any non-zero subrepresentation of S is equal to S. The representations $S(x)$ with $x \in Q_0$ are obviously simple.

The loop quiver. Let $\mathbb{L}$ be the loop quiver, it has just one vertex, say x and just one arrow, the loop $x \to x$. The representations of $\mathbb{L}$ are pairs (V, ϕ), where V is a vector space and $\phi\colon V \to V$ a linear map (a vector space endomorphism). If $(V, \phi), (V', \phi')$ are representations of $\mathbb{L}$, then an isomorphism $f\colon (V, \phi) \to (V', \phi')$ is an invertible linear map $f\colon V \to V'$ such that $\phi' = f\phi f^{-1}$. This shows that $(V, \phi), (V', \phi')$ are isomorphic if and only if the endomorphisms ϕ and ϕ' are similar.

> Similarity of vector space endomorphisms (and of square matrices) is a basic concept in Linear Algebra. Recall that two $(n{\times}n)$-matrices Φ, Φ' with coefficients in the field k are said to be similar provided there is an invertible $(n \times n)$-matrix F such that $\Phi' = F\Phi F^{-1}$. In case the two linear maps $\phi\colon V \to V$ and $\phi'\colon V' \to V'$ are similar, the vector spaces V, V' are isomorphic, thus (in the finite-dimensional case) isomorphic to the vector space k^n for some natural number n. Such isomorphisms are obtained by choosing in V a basis $\mathcal{B}$ and in V' a basis $\mathcal{B}'$. With respect to these bases, we can write ϕ and ϕ' as $(n{\times}n)$-matrices Φ, Φ', respectively, and obviously ϕ, ϕ' are similar linear maps if and only if Φ, Φ' are similar matrices.
>
> Here we deal with a situation where the structure of the field k does play a role: In Linear Algebra, the classification problem for square matrices up to similarity is usually only discussed in case k is algebraically closed (say if $k = \mathbb{C}$ is the field of complex numbers). In that case, the classification is given by the Jordan normal form which we will recall below. For the moment, we are only interested in the simple representations of $\mathbb{L}$, they correspond to the "irreducible" square matrices.

Clearly, one-dimensional representations of any quiver have to be simple. A one-dimensional representation of $\mathbb{L}$ is up to isomorphism of the form (k, λ) where λ denotes the multiplication map $\lambda\cdot\colon k \to k$ (with $a \in k$ being sent to λa). Namely, if (V, ϕ) is a one-dimensional representation of $\mathbb{V}$, choose a non-zero vector $b \in V$. It generates V, since V is one-dimensional, thus we have $\phi(b) = \lambda b$ for some $\lambda \in k$. Note that we have $\phi(a) = \lambda a$ for all $a \in V$; namely write $a = \mu b$ with $\mu \in k$, then

$$\phi(a) = \phi(\mu b) = \mu\phi(b) = \mu\lambda b = \lambda(\mu b) = \lambda a.$$

This shows, that λ is an **invariant** of (V, ϕ) and that different λ's yield non-isomorphic representations.

In case k is an infinite field, we obtain in this way infinitely many isomorphism classes of simple representations of $\mathbb{L}$. In case k is algebraically closed we get in this way all isomorphism classes of simple representations, otherwise not.

> Recall that k is said to be *algebraically closed* provided any polynomial of degree at least 1 has a zero, or alternatively, provided every

endomorphism of a non-zero vector space over k has an eigenvector. It is the latter condition which we need here: If $\phi\colon V \to V$ is an endomorphism, and $v \in V$ is an eigenvector of ϕ, say with eigenvalue λ, then the subspace $\langle v\rangle$ of V generated by v yields a one-dimensional subrepresentation of (V,ϕ) (since $\phi(\langle v\rangle) \subseteq \langle v\rangle$), and this subrepresentation is isomorphic to (k,λ).

Also in case k is a finite field, there are infinitely many isomorphism classes of simple representations of $\mathbb{L}$, they can be constructed as follows: Take a finite field extension $k \subseteq K$. One knows (see any algebra course dealing with finite fields) that there is a so-called "primitive" element $t \in K$, namely an element which generates K as a k-algebra. Then (K,t) *is a simple representation of* $\mathbb{L}$, here t again denotes the multiplication map $t\colon K \to K$ which maps $a \in K$ to ta.

Altogether we see: *For any field k, there are infinitely many isomorphism classes of simple representations of* $\mathbb{L}$. Let us add: *If (V,ϕ) is a simple representation of* $\mathbb{L}$*, then ϕ is bijective unless (V,ϕ) is isomorphic to* $(k,0) = S(*)$, where $*$ is the unique vertex of $\mathbb{L}$. Namely, if ϕ is not bijective, then it has an eigenvector with eigenvalue 0, thus $(k,0)$ is a subrepresentation of (V,ϕ).

If k is algebraically closed, the classification of the similarity classes of linear endomorphisms of vector spaces or, equivalently, of square matrices with coefficients in k, is given by the Jordan normal form: any square matrix is similar to the direct sum of Jordan blocks, a Jordan block is of the form

$$\begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}$$

this is called the Jordan block $J(\lambda,n)$ of size n with eigenvalue λ. Two Jordan normal forms are similar provided they consist (up to possible permutations of the block) of the same Jordan blocks. The diagonal sum of square matrices $A_1,\dots,A_m$ is the matrix

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{bmatrix}.$$

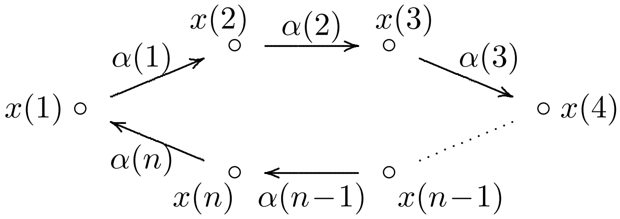
Let us reformulate these considerations in terms of the representation theory of quivers. The diagonal sum of square matrices corresponds to the direct sum of representations of $\mathbb{L}$. The Jordan blocks $J(\lambda,n)$ yield indecomposable representations of the loop quiver $\mathbb{L}$, namely

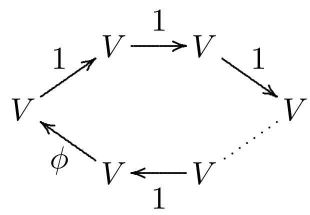
$(k^n, J(\lambda, n))$; different Jordan blocks yield non-isomorphic representations, and, if k is algebraically closed, any indecomposable representation of $\mathbb{L}$ is isomorphic to some $(k^n, J(\lambda, n))$. In case we deal with an arbitrary (not necessarily algebraically closed) field, then, as we have mentioned, there are additional simple representations S of $\mathbb{L}$; and there are the corresponding indecomposable representations with a "filtration" of arbitrary length with all factors isomorphic to a fixed simple representation S.

Elementary oriented cycles. Recall that an elementary oriented cycle is a path

$$w = \alpha(1) \cdots \alpha(n)$$

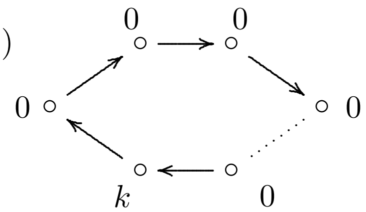
of length at least 1 such that $h(w) = t(w)$ and such that the vertices $x(i) = t(\alpha(i))$ with $1 \leq i \leq n$ are pairwise different. Given such an elementary oriented cycle w in the quiver Q, and a representation (V, ϕ) of the loop quiver $\mathbb{L}$, we define a representation $M = M(w, V, \phi)$ of Q as follows: Let $M_{x(i)} = V$ for $1 \leq i \leq n$ and $M_y = 0$ for the remaining vertices; let $M_{\alpha(i)}$ be the identity map for $1 \leq i < n$, let $M_{\alpha(n)} = \phi$, and $M_\beta = 0$ for the remaining arrows β. Thus, the essential part of M looks as follows:



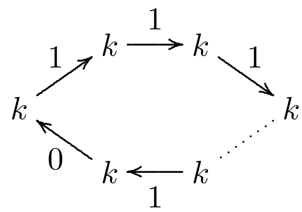


Note that if $(V, \phi) = (k, 0) = S(*)$, and $n \geq 2$, then $M(w, V, \phi) = M(w, k, 0)$ has the simple module $S(x_n)$ as submodule:

$S(x(n))$



$M(w, k, 0)$



Thus, $M(x, k, 0)$ is not simple (for $n \geq 2$). Let us now consider the representations $M(w, V, \phi)$, where (V, ϕ) is a simple representation of V which is not isomorphic to $(k, 0) = S(*)$.

Claim: *The construction $(V, \phi) \mapsto M(w, V, \phi)$ furnishes an injection from the set of isomorphism classes of the simple representations of $\mathbb{L}$ different from $S(*)$ into the set of isomorphism classes of the simple representations of Q.*

Proof: Let (V, ϕ) be a simple representation of $\mathbb{L}$ which is not isomorphic to $S(*)$ and let $M = M(w, V, \phi)$. Let M' be a non-zero submodule of M. We claim that $M_{x(1)} \neq 0$. In general, if M' is a submodule of M and $\alpha\colon x \to y$ is an arrow, such that M_α is injective, then with $M'_x \neq 0$ also $M'_y \neq 0$, since $0 \neq M_\alpha(M'_x) \subseteq M'_y$. Thus, in our case where all the maps $M_{\alpha(i)}$ are invertible, we see that $M'_{x(i)} \neq 0$ implies that also $M'_{x(i+1)} \neq 0$ (as usual we let $x(n+1) = x(1)$).

Let $U = M'_{x(1)} (\neq 0)$. Since the maps $M_{\alpha(i)}$ are the identity map for $1 \le i < n$, we see inductively that $U = M_{\alpha(i)}(U) \subseteq M_{x(i+1)}$, thus $U \subseteq M'_{x(n)}$. Finally, $\phi(U) = M_{\alpha(n)}(U) \subseteq M_{x(1)} = U$ shows that $(U, \phi|U)$ is a non-zero submodule of (V, ϕ). Since (V, ϕ) is simple, we must have $U = V$ and therefore $M'_{x(i)} = M_{x(i)}$ for all i, thus $M' = M$. This shows that $M(w, V, \phi)$ is a simple representation of Q.

Now assume that (V, ϕ) and (V', ϕ') are simple representations of $\mathbb{L}$ and that there is an isomorphism $f = f_{x(i)} \colon M = M(w, V, \phi) \to M(w, V', \phi') = M'$. There are the commutative squares

$$\begin{tikzcd} M_{x(i)} \arrow[r, "f_{x(i)}"] \arrow[d, "M_{\alpha(i)}"'] & M'_{x(i)} \arrow[d, "M'_{\alpha(i)}"] \\ M_{x(i+1)} \arrow[r, "f_{x(i+1)}"] & M'_{x(i+1)} \end{tikzcd}$$

which are of the form

$$\text{for } 1 \le i < n \quad \begin{tikzcd} V \arrow[r, "f_{x(i)}"] \arrow[d, "1"'] & V' \arrow[d, "1"] \\ V \arrow[r, "f_{x(i+1)}"] & V' \end{tikzcd} \qquad \text{and for } i = n \quad \begin{tikzcd} V \arrow[r, "f_{x(n)}"] \arrow[d, "\phi"'] & V' \arrow[d, "\phi'"] \\ V \arrow[r, "f_{x(1)}"] & V' \end{tikzcd}$$

It follows first that $f_{x(i)} = \psi$ for some fixed linear map ψ and all then $1 \le i \le n$, and that $\phi'\psi = \psi\phi$. Since ψ is invertible, we see that ϕ, ϕ' are similar linear maps, thus (V, ϕ) and (V', ϕ') are isomorphic representations of $\mathbb{L}$.

Thus, we see: *If Q has an orientec cycle, then there are infinitely many isomorphism classes of simple representations.*

Lemma. *The support quiver of a simple representation which is not of the form $S(x)$ for any vertex x is strongly connected.*

Proof. Let S be a simple representation which is not isomorphic to a representation of the form $S(x)$. If the support quiver $Q(S)$ of S has only one vertex, say x, there must be a loop $x \to x$, since otherwise $Q(S)$ is the quiver $\mathbb{A}_1$ and the representations of this quiver are just the direct sums of copies of $S(x)$. On the other hand, the quiver with one vertex and at least one loop is strongly connected.

Now assume that the support quiver of S has at least two vertices. Let $x \neq y$ be vertices of $Q(S)$ and assume that there is no path from x to y. For any vertex z, let $W(z, x)$ be the set of paths from x to z and let N_z be the sum of the subspaces $w(S_x)$. where $w \in W(z, x)$. Clearly, N is a subrepresentation of S. We have $N_x = S_x$, thus $N \neq 0$. And we have $N_y = 0$, whereas $S_y \neq 0$, thus $N \neq S$. This shows that N is a non-zero proper subrepresentation of S, thus S is not simple. This contradiction shows that there is a path from x to y, for any pair $x \neq y$ of vertices. But then there is also a proper path from x to itself, namely the concatenation of a path from x to some vertex $y \neq x$ with a path from y to x.

Theorem. *Let Q be a finite quiver. The following conditions are equivalent:*
(i) *There is an oriented cycle in Q.*
(ii) *There are infinitely many isomorphism classes of simple representations.*
(iii) *There is at least one simple representation which is not isomorphic to a representation of the form $S(x)$ with $x \in Q_0$.*

Proof: (i) $\implies$ (ii): Let w be an elementary oriented cycle and consider the representations M with support quiver being given by (the vertices and arrows of) w.

(ii) $\implies$ (iii): The number of representations of the form $S(x)$ is $|Q_0|$, thus finite.

(iii) $\implies$ (i). Let S be simple, not isomorphic to $S(x)$. It has been shown in the Lemma that the support quiver $Q(S)$ of S is strongly simply connected, thus $Q(S)$, and therefore Q, has an oriented cylce.

2.3. Factor representations and filtrations.

Let Q be a quiver and M a representation of Q. If M' is a subrepresentations of M, thus for every vertex x of Q, there is given a subspace M'_x of M_x, and if $\alpha\colon x \to y$ is an arrow, then $M_\alpha(M'_x) \subseteq M'_y$. We can form the factor spaces $M''_x = M_x/M'_x$, and since for the arrow $\alpha\colon x \to y$, we have $M_\alpha(M'_x) \subseteq M'_y$, one knows that M_α induces a linear map $M''_x = M_x/M'_x \to M_y/M'_y = M''_y$ which we denote by M''_α (the elements of M''_x are residue classes of the form $v + M'_x$ with $v \in M_x$ and M''_α is defined by $M''_\alpha(v + M'_x) = M_\alpha(v) + M'_y$; it is basic knowledge in Linear Algebra (and also easy to check) that this definition of M''_α is well-defined and yields again a linear map. We see that we obtain a representation $M'' = (M''_x, M''_\alpha)_{x,\alpha}$ which is called a *factor representation* (or a *factor module*) of M.

Sometimes we will consider filtrations of a representation M. A *filtration* of M is given by a chain of submodules of M, say

$$0 = M^{(0)} \subseteq M^{(1)} \subseteq M^{(2)} \subseteq \cdots \subseteq M^{(m)} = M,$$

and one calls the factor representations $M^{(i)}/M^{(i-1)}$ with $1 \leq i \leq m$ the *factors* of the filtration.

2.4. Nilpotent representations.

We say that a representation M is *nilpotent* provided it has a filtration with all factors being of the form $S(x)$ (with x vertices of the quiver).

Proposition. *Let Q be a quiver and M a representation of Q of dimension m. Then the following conditions are equivalent:*
(a) *M is nilpotent.*

(b) $w(M) = 0$ *for every path* w *of length* m,

(c) *There is a natural number* t *such that* $w(M) = 0$ *for any path of length* t.

Proof. (a) $\implies$ (b). We use induction on m. If $m = 1$, then $M = S(x)$ for some vertex x and then all maps M_α are zero-maps. Now if w is a path of length 1, then $w = \alpha$ for some arrow α and $w(M) = \alpha(M_{t(\alpha)}) = 0$.

Now assume that the implication has been shown for some $m \geq 1$, let M be of dimension $m + 1$. Since M is nilpotent, it has a filtration such that all factors are (1-dimensional and) of the form $S(x)$ for vertices x. This shows that there is a submodule M' of M of dimension m which is nilpotent and such that M/M' is isomorphic to some $S(x)$. By induction, we know that $w(M') = 0$ for all paths of length m. Let α be any arrow. We claim that $\alpha(M) \subseteq M'$. Let $\alpha\colon y \to z$, thus $\alpha(M) = M_\alpha(M_y)$. Now $M''_\alpha\colon M_y/M'_y \to M_z/M'_z$ is induced by M_α and is the zero map, since $M'' = S(x)$, thus $M_\alpha(M_y) \subseteq M'_z$.

If w is a path of length $m+1$, it is of the form $w = w'\alpha$, where w' is a path of length m and α is an arrow (thus a path of length 1). It follows that

$$w(M) = (w'\alpha)(M) = w'(\alpha(M)) \subseteq w(M') = 0.$$

This concludes the proof.

For (b) $\implies$ (c), nothing has to be shown.

(c) $\implies$ (a). Let M be non-zero and assume that there is a natural number t such that $w(M) = 0$ for any path of length t. For any vertex x and any natural number i, let $I(x, i)$ be the sum of the images $w(M)$, where w is a path of length i with head z, thus $I(x, i) \subseteq M_x$. We show the following: If $I(x, i) = M_x$ for all x and some $i \geq 1$, then also $I(x, i+1) = M_x$ for all x. Namely, let $\alpha_j\colon x(j) \to y$ be the arrows with head y. Then $M_{x(j)} = I(x(j), i)$ and therefore

$$M_y = I(y, 1) = \sum_j \alpha_j(M_{x(j)}) = \sum \alpha_j(I(x(j), i) = \sum_w w(M)$$

where the last sum is indexed by all paths of length $j+1$ with head y (observe that any path of length $j+1$ with head y is of the form $\alpha_j w'$ with w' a path of length j with head $x(j)$.

Since by assumption, we have $I(x, t) = 0$ for all x, it follows that there has to be a vertex y such that there has to exist some z such that $I(z, 1)$ is a proper subspace of M_z. Define M' as follows: let M'_z be a maximal subspace of M_z which includes $I(z, 1)$, and let $M'_x = M_x$ for $x \neq z$. Since $I(z, 1) \subseteq M'_z$, it follows that M' is a subrepresentation of M and also that M/M' is isomorphic to $S(z)$.

Of course, with M also M' satisfies the condition (c), thus M' is nilpotent, and therefore also M is nilpotent.

> In order to show the implication (c) $\implies$ (a), one also may start to construct a subrepresentation M' of M which is isomorphic to some $S(x)$ as follows: Let t be minimal such that $w(M) = 0$ for all paths of length t. If $t = 0$, then M itself is a direct sum of representations of the form $S(x)$. If $t \geq 0$, there is a path w with $w(M) \neq 0$. Any non-zero element in $w(M)$ generates a subrepresentation which is

of the form $S(h(w))$. Now one may look at M/M'. By induction, M/M' is nilpotent, thus it remains to be seen that a representation M with a subrepresentation M' such that both M' and M/M' are nilpotent, is nilpotent.

The following is easy to verify: Let M be a nilpotent representation, say with a filtrations

$$0 = M^{(0)} \subseteq M^{(1)} \subseteq M^{(2)} \subseteq \cdots \subseteq M^{(m)} = M,$$

such that the factors $M^{(i)}/M^{(i-1)}$ with $1 \le i \le m$ are of the form $S(x)$. Then *m is the dimension of M and for any vertex x, the number of factors $M^{(i)}/M^{(i-1)}$ isomorphic to $S(x)$ is equal to the k-dimension of M_x.*

These numbers are called the Jordan-Hölder multiplicities. We will discuss them later in detail.

2.5. Thin representations.

Recall that a representation $M = (M_x, M_\alpha)_{x,\alpha}$ is thin provided all the vector spaces M_x are of dimension at most 1. Also we recall that the support quiver $Q(M)$ is given by the vertices x with $M_x \neq 0$ and the arrows α with $M_\alpha \neq 0$.

Let us start with two quite trivial general assertions.

Lemma.
(a) *If M is indecomposable, $Q(M)$ is connected.*
(b) *If M, M' are isomorphic representations, then $Q(M) = Q(M')$.*

Proof: (a) Assume that we can write $Q(M)$ as the disjoint union of two (non-empty) quivers Q', Q''. Let $M'_x = M_x$ if x is a vertex of Q' and $M'_x = 0$ otherwise. Similarly, let $M''_x = M_x$ if x is a vertex of Q'' and $M''_x = 0$ otherwise. Then one easily sees that both M', M'' are non-zero subrepresentations of M and that $M = M' \oplus M''$, thus M is decomposable.

(b) If $f = (f_x)_x \colon M \to M'$ is an isomorphism, then $f_x \colon M_x \to M'_x$ is an isomorphism for all vertices x, thus $M_x \neq 0$ if and only if $M'_x \neq 0$. Also, if $\alpha \colon x \to y$ is an arrow, then we have the following commutative diagram

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & M'_x \\ {\scriptstyle M_\alpha}\downarrow & & \downarrow{\scriptstyle M'_\alpha} \\ M_y & \xrightarrow{f_y} & M'_y \end{array}$$

thus $M'_\alpha = f_x^{-1} M_\alpha f_y$ shows that $M_\alpha \neq 0$ if and only if $M'_\alpha \neq 0$.

Proposition 1. *Let M be a thin representation of Q. Then M is indecomposable if and only if the support quiver of M is connected.*

Proof: One direction is true in general, as the lemma above shows. Now let T be a tree, M a thin representation and $Q(M)$ connected. Now since $Q(M)$ is a connected subquiver of a tree quiver, it is again a tree quiver, thus, without loss of generality, we can assume that $Q = Q(M)$. If Q has only one vertex x, then $M = S(x)$. Thus we can assume that Q has at least two vertices. Since Q is a tree it is obtained from a quiver Q' by attaching an arm of the form $\mathbb{A}_2$ at a vertex x, say adding an arrow α with $\{h(\alpha), t(\alpha)\} = \{x, \omega\}$. By induction, the restriction M' of M to Q' is indecomposable. Since M_α is non-zero, it follows that with M' also M is indecomposable.

Proposition 2. *Let Q be a tree and M, M' indecomposable thin representations of Q. Then M, M' are isomorphic if and only if M, M' have the same support quiver.*

Proof: One direction is true in general. We have to consider the reverse implication, thus assume that M, N are indecomposable thin representations of a tree quiver Q such that $Q(M) = Q(N)$. As in the previous proof, we can assume that $Q = Q(M) = Q(N)$ and that Q has at least 2 vertices and is obtained from a quiver Q' by attaching an arm of the form $\mathbb{A}_2$ at a vertex x, say adding an arrow α with $\{h(\alpha), t(\alpha)\} = \{x, \omega\}$. Now by induction, the restrictions M' of M and N' of N to Q' are isomorphic, say there is an isomorphism $f = (f_x)_x \colon M' \to N'$. We need to define $f_\omega \colon M_\omega \to N_\omega$ so that we obtain an isomorphism $M \to N$. But this just means that we have to choose f_ω so that one of the following diagrams commutes, the left one in case $\alpha \colon x \to \omega$, the right one in case $\alpha \colon \omega \to x$.

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & N_x \\ {\scriptstyle M_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle N_\alpha} \\ M_\omega & \xrightarrow{f_\omega} & N_\omega \end{array} \qquad \begin{array}{ccc} M_x & \xrightarrow{f_x} & N_x \\ {\scriptstyle M_\alpha}\big\uparrow & & \big\uparrow{\scriptstyle N_\alpha} \\ M_\omega & \xrightarrow{f_\omega} & N_\omega \end{array}$$

thus, in case $\alpha \colon x \to \omega$, take $f_\omega = N_\alpha f_x M_\alpha^{-1}$, in case $\alpha \colon \omega \to x$, take $f_\omega = N_\alpha^{-1} f_x M_\alpha$.

As an immediate consequence of Propositions 1 and 2 we see: *If Q is a tree and M is an indecomposable thin representation with support quiver Q, then M is isomorphic to the representation M' with $M'_x = k$ for all vertices x and M'_α the identity map, for all arrows α;* we may call M' the *normal form*).

Proposition 3. *Let Q be a tree and M an indecomposable thin representation of Q. If Y is a representation with a subrepresentation X such that both X and Y/X are isomorphic to M, then X is a direct summand of Y* (thus, there is a subrepresentation Z of Y with $Y = X \oplus Z$ and Z is necessarily also isomorphic to M.)

> We later may reformulate this assertion by saying that for a tree quiver Q any indecomposable thin representation is "exceptional".

Proof: Again, we assume that $Q = Q(M)$ and use induction. In case Q has only one vertex $*$, the representation Y has to be the direct sum of two copies of $S(*)$.

Now consider the case that Q has at least 2 vertices. Before we continue, let us note that for α an arrow in $Q = Q(M)$, the map M_α is bijective, thus X_α and $(Y/X)_\alpha$

are bijective and therefore also Y_α is bijective. We assume that Q is obtained from a quiver Q' by attaching an arm of the form $\mathbb{A}_2$ at a vertex x, say adding an arrow α with $\{h(\alpha), t(\alpha)\} = \{x, \omega\}$. We consider the restrictions X' of X and Y' of Y to Q'. By induction, there is a subrepresentation Z' of Y' such that $Y' = X' \oplus Z'$. If $\alpha\colon x \to \omega$, let $Z_\omega = Y_\alpha(Z_x)$. If $\alpha\colon \omega \to x$, let $Z_\omega = Y_\alpha^{-1}(Z_x)$. Then $Y = X \oplus Z$.

Elementary cycle. We consider now a quiver of type $\widetilde{\mathbb{A}}_n$, such a quiver has $n+1$ vertices, say labeled $0, 1, \dots, n$ and $n+1$ arrows $\alpha(0), \dots, \alpha(n)$ with $\{t(\alpha(i)), h(\alpha(i)\} = \{i, i+1\}$ (modulo $n+1$). *The isomorphism classes of thin indecomposable representations with support quiver Q are indexed by the non-zero element of the base field k.* Thus, if k is an infinite field, there are infinitely many isomorphism classes of indecomposable thin representations. If k is a finite field, say with q elements, then the number of isomorphism classes of indecomposable thin representations with support quiver Q is equal to $q-1$.

Proof: If M is an indecomposable thin representation with support quiver Q, it is isomorphic to a representation M' with $M'_x = k$ for all $x \in Q_0$ and $M'_{\alpha(i)}$ the identity map for all the arrows $\alpha(i)$ with $1 \le i \le n$. Namely, let Q' be obtained from Q by deleting one arrow, say $\alpha(0)$, but keeping all the vertices. Then Q' is a tree, and we may write $M'|Q'$ in normal form. Now $M'_{\alpha(0)}$ is an arbitary non-zero element of k and this element is uniquely determined by the isomorphism class of M.