3. Homomorphisms.

3.1. Definition, some properties.

If M, M' are representations of the quiver Q, a *homomorphism* $f\colon M \to M'$ is of the form $f = (f_x)_x$ with linear maps $f_x\colon M_x \to M'_x$ for all $x \in Q_0$ such that the following diagrams for every arrow $\alpha\colon x \to y$ commute

$$\begin{tikzcd} M_x \arrow[r, "f_x"] \arrow[d, "M_\alpha"'] & M'_x \arrow[d, "M'_\alpha"] \\ M_y \arrow[r, "f_y"] & M'_y \end{tikzcd}$$

To repeat: one has such a diagram for every arrow α of the quiver; the vertical data on the left are part of M, those on the right are part of M', the horizontal maps are those which combine to form f.

Of course, given a representation M, there is always the identity homomorphism $1_M\colon M \to M$ with $(1_M)_x$ the identity map of M_x. Also, for any pair M, M' of representations, there is the zero homomorphism $0\colon M \to M'$ (with $0_x\colon M_x \to M'_x$ being the zero map).

Examples. Consider the three representations

$$(0 \to k), \quad (k \to 0), \quad (1_k\colon k \to k)$$

of the quiver Q of type $\mathbb{A}_2$, and let us determine whether there are non-zero homomorphisms $M \to M'$ or not. Of course, If $M = (0 \to k)$ and $M' = (k \to 0)$, there cannot be a non-zero homomorphism $f\colon M \to M'$, since $f = (f_1, f_2)$ and for $f_1\colon M_1 \to M'_1$ and for $f_2\colon M_2 \to M'_2$ there only exist the zero maps. Now let $M = (0 \to k)$ and $M' = (1\colon k \to k)$, and look for pairs $f = (f_1, f_2)$ with $f_1\colon M_1 \to M'_1$ and $f_2\colon M_2 \to M'_2$. For f_1 the only possibility is the zero map, whereas for $f_2\colon k \to k$ we may try to take any scalar multiplication, say take the multiplication by $c \in k$ (as a map $k \to k$). But of course, we have to check whether the following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & k \arrow[d, "1"'] \\ k \arrow[r, "c"] & k \end{tikzcd}$$

it always is, thus there are non-zero homomorphisms $(0 \to k) \longrightarrow (1\colon k \to k)$. (Note that when drawing this square, as well as the following ones, we follow the convention mentioned above: the vertical maps are those of the form M_α, M'_α, whereas the horizontal ones are those of the form f_1 and f_2.) On the other hand, if we are looking for homomorphisms $(1\colon k \to k) \longrightarrow (0 \to k)$, we have to deal with the diagram

$$\begin{array}{ccc} k & \longrightarrow & 0 \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow \\ k & \xrightarrow{c} & k \end{array}$$

and here it turns out that the diagram commutes only in case $c = 0$, thus there is no non-zero homomorphism $(1\colon k \to k) \longrightarrow (0 \to k)$.

In a similar way, one deals with homomorphisms between $(k \to 0)$ and $(1\colon k \to k)$. The only homomorphism $(k \to 0) \longrightarrow (1\colon k \to k)$ is the zero homomorphism, since the following diagram on the left commutes only for $c = 0$.

$$\begin{array}{ccc} k & \xrightarrow{c} & k \\ \big\downarrow & & \big\downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & k \end{array} \qquad\qquad \begin{array}{ccc} k & \xrightarrow{c} & k \\ {\scriptstyle 1}\big\downarrow & & \big\downarrow \\ k & \longrightarrow & 0 \end{array}$$

On the other hand, the above diagram on the right commutes for all c, thus any $c \in k$ defines a homomorphism $(1\colon k \to k) \longrightarrow (k \to 0)$.

Summarizing these considerations, we see that we can order the indecomposable representations of Q

$$(0 \to k), \quad (1_k\colon k \to k), \quad (k \to 0)$$

so that non-invertible homomorphisms go from left to right.

If M, M', M'' are representations of the quiver Q, and $f\colon M \to M'$, $g\colon M' \to M''$ are homomorphisms, then the definition $(gf)_x = g_x f_x$ yields a homomorphism $gf = ((gf)_x)_x\colon M \to M''$, the *composition* of these homomorphisms. Note that the composition is both associative and bilinear.

Let M, N be representations of the quiver Q. Let $\mathrm{Hom}(M, N)$ be the set of homomorphisms $f\colon M \to N$. This set $\mathrm{Hom}(M, N)$ *is a k-space* with respect to the following addition and scalar multiplication: Let $f = (f_x)_x$ and $f' = (f'_x)_x$ be homomorphisms $M \to N$ and $c \in k$, we define $f + f'$, $cf\colon M \to N$ by $(f + f')_x = f_x + f'_x$ and $(cf)_x = cf_x$. (Here, one has to check that $f + f'$ as well as cf are again homomorphisms; also one has to check that with this definition of addition and scalar multiplication, the vector space axioms are satisfied.) In particular, the zero homomorphism $M \to N$ is the zero element of the vector space $\mathrm{Hom}(M, N)$. It should be stressed that for finite-dimensional representations M, N, also $\mathrm{Hom}(M, N)$ is a finite-dimensional k-space.

If M, M', N, N' are representations of the quiver Q, and $f\colon M \to M'$ is a homomorphism, then the composition yields a k-linear map

$$\mathrm{Hom}(f, N)\colon\ \mathrm{Hom}(M', N) \to \mathrm{Hom}(M, N),$$

it is defined by $\mathrm{Hom}(f,N)(h) = hf$ for $h \in \mathrm{Hom}(M',N)$. Similarly, if $g\colon N \to N'$ is a homomorphism, then the composition yields a k-linear map

$$\mathrm{Hom}(M,g)\colon \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N'),$$

it is defined by $\mathrm{Hom}(f,N)(h) = gh$ for $h \in \mathrm{Hom}(M,N)$.

Let $f\colon M \to M'$ be a homomorphism. We say that f is a *monomorphism,* or an *epimorphism,* or an *isomorphism,* provided all the maps f_x are injective, or surjective, or bijective, respectively. Note that *if* $f\colon M \to M'$ *is an isomorphism, then* $f^{-1}\colon M' \to M$ defined by $(f^{-1})_x = (f_x)^{-1}$ *is again a homomorphism,* and of course also an isomorphism. (Proof: Let $\alpha\colon x \to y$ be an arrow of Q. It follows from $M'_\alpha f_x = f_y M_\alpha$ that $M_\alpha (f_x)^{-1} = (f_y)^{-1} M'_\alpha$. This is what is needed in order that $((f_x)^{-1})_x$ is a homomorphism.) If an isomorphism $f\colon M \to M'$ exists, then M, M' are said to be *isomorphic.*

Note that the composition of two monomorphisms, epimorphisms, isomorphisms is again a monomorphism, epimorphism, isomorphism, respectively. The following is quite easy to check: *A homomorphism* $f\colon M \to M'$ *is an isomorphism if and only if there is a homomorphism* $g\colon M' \to M$ *such that* $gf = 1_M$ *and* $fg = 1_{M'}$.

Let us also record the following observations: If $f\colon M \to M'$ is a homomorphism such that all the maps f_x for $x \in Q_0$ are inclusion maps, then M' is a subrepresentation of M and f is called the corresponding *inclusion map*. Of course, an inclusion map is a monomorphism. Also recall that given a subrepresentation M' of a representation M, then we form the factor representation $M'' = M/M'$ and there are the canonical projection maps $q_x\colon M_x \to (M/M')_x = M''_x$, they combine to a homomorphism $q\colon M \to M'' = M/M'$. Of course, q is an epimorphism.

If $f\colon M \to M'$ is a homomorphism of representations of Q, then its *kernel* $\mathrm{Ker}(f)$ is the subrepresentation of M with $(\mathrm{Ker}(f))_x = \mathrm{Ker}(f_x)$, and the *image* $\mathrm{Im}(f)$ is the subrepresentation of M' with $(\mathrm{Im}(f))_x = \mathrm{Im}(f_x)$. Finally, define $M'/\mathrm{Im}(f)$ to be the *cokernel* of f. Also note: Given a monomorphism $u\colon M \to M'$, then M is isomorphic to the image of u. If $q\colon M \to M'$ is an epimorphism, then M' is isomorphic to $M/\mathrm{Ker}(q)$.

If M, M', N, N' are representations of Q, then there are canonical identifications:

$$\begin{aligned} \mathrm{Hom}(M, N \oplus N') &= \mathrm{Hom}(M,N) \oplus \mathrm{Hom}(M,N'), \\ \mathrm{Hom}(M \oplus M', N) &= \mathrm{Hom}(M,N) \oplus \mathrm{Hom}(M',N). \end{aligned}$$

3.2. Endomorphism rings.

Let Q be a quiver and M a representation of Q. A homomorphism $f\colon M \to M$ is called an *endomorphism* of M. In case f is invertible, one calls it an *automorphism.* The set $\mathrm{End}(M)$ of all endomorphisms of M is a ring, even a k-algebra, it is called the *endomorphism ring* of M. (Since $\mathrm{End}(M) = \mathrm{Hom}(M,M)$, it is a k-space, the composition of endomorphisms yields an associative multiplication which is bilinear, thus satisfies the distributivity laws. The identity map $1 = 1_M$ is the unit element of the ring $\mathrm{End}(M)$. If $c \in k$, the scalar multiple $c \cdot 1$ is the scalar multiplication on M (sending $a \in M$ to ca); these scalar multiples $c \cdot 1$ commute with all endomorphisms, thus the map $k \to \mathrm{End}(M)$ which sends c to $c \cdot 1$ is a ring homomorphism from k into the center of $\mathrm{End}(M)$, in this way, $\mathrm{End}(M)$ is a k-algebra.) In case $M \neq 0$, the ring homomorphism $k \to \mathrm{End}(M)$ defined by $c \mapsto c \cdot 1$ is injective; we may consider this as an embedding of k into $\mathrm{End}(M)$. Of course, if M is a finite-dimensional representation, then $\mathrm{End}(M)$ is a finite-dimensional k-algebra.

Let us stress that the endomorphism ring $\mathrm{End}(M)$ of a representation M is usually non-commutative (as one knows already from the case of the quiver $\mathbb{A}_1$; the representations of this quiver are just vector spaces, and the endomorphism ring of a vector space V is commutative only in case the dimension of V is at most 1).

Of special interest are the idempotents in $\mathrm{End}(M)$. Recall that an element e of a ring is called an *idempotent* provided $e^2 = e$; the elements 0 and 1 of $\mathrm{End}(M)$ are always idempotents, and it is interesting to know whether there are additional idempotents.

Lemma. *Given any representation M of a quiver Q, there is a bijection between the set of idempotents in $\mathrm{End}(M)$ and the direct decompositions $M = M' \oplus M''$, where the idempotent e corresponds to the direct decomposition $M = \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$, and conversely, the direct decomposition $M = M' \oplus M''$ corresponds to the canonical projection of M onto M' (with kernel M'').*

The canonical projection of $M = M' \oplus M''$ onto M' is given by the map which sends $a' + a''$ (where $a' \in M', a'' \in M''$) onto a'.

Proof: Many things have to be verified.

Let us start with e an idempotent. We know already that both $\mathrm{Im}(e)$ and $\mathrm{Ker}(e)$ are subrepresentations of M, thus we only have to verify we obtain in this way a direct decomposition. First, $\mathrm{Im}(e) \cap \mathrm{Ker}(e) = 0$; namely, if $a' \in \mathrm{Im}(e)$, then a' is of the form $a' = e(a)$ for some $a \in M$; if a' also belongs to $\mathrm{Ker}(e)$, then $0 = e(a') = e(ea) = (e^2)a = ea = a'$. Second, $\mathrm{Im}(e) + \mathrm{Ker}(e) = M$; namely, if $a \in M$, then $a = e(a) + (1-e)(a)$ and $e(a)$ belongs to $\mathrm{Im}(e)$, whereas $(1-e)(a)$ obviously belongs to $\mathrm{Ker}(e)$.

Next, start with a direct decomposition $M = M' \oplus M''$ and let e be the canonical projection onto M'. Here one has to verify that this is indeed a homomorphism (it is the composition of the projection $M \to M/M''$, the identification $M/M'' \to M'$ and the embedding $M' \to M$). In addition, we need to know that $e^2 = e$, but this is obvious from the definition.

If we start with the idempotent e, and consider the direct decomposition $M = \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$, then it turns out that the canonical projection of M onto $\mathrm{Im}(e)$ is precisely e. Namely, consider an element $a = a' + a''$ with $a' \in \mathrm{Im}(e)$ and $a'' \in \mathrm{Ker}(e)$. Let $a' = e(b)$ for some $b \in M$. If we apply e to $a = a' + a''$, we obtain $e(a) = e(a') + e(a'') = e(e(b)) = e(b) = a'$ (using that $e^2 = e$ and that $e(a'') = 0$).

Conversely, if we start with the direct decomposition $M = M' \oplus M''$ and consider the canonical projection e of M onto M', then clearly M' is the image of e, whereas M'' is contained in the kernel of e. It only remains to observe that M'' has to be the kernel of e: if an element $a \in M$ belongs to the kernel of a, then write $a = a' + a''$ with $a' \in M', a'' \in M''$;

as we know, $a' \in \mathrm{Im}(e), a'' \in \mathrm{Ker}(e)$. Since both a, a'' belong to $\mathrm{Ker}(e)$, also $a' = a - a''$ belongs to $\mathrm{Ker}(e)$. Thus $a' \in \mathrm{Im}(e) \cap \mathrm{Ker}(e) = 0$ (this we have shown for any idempotent e), therefore $a = a'' \in M''$.

Corollary. *Let M be a non-zero representation. Then M is indecomposable if and only if the only idempotents in* $\mathrm{End}(M)$ *are* 0 *and* 1.

As an example, let us calculate the endomorphism ring in one example. We deal with the 3-subspace quiver with vertices labeled $0, 1, 2, 3$ as shown below on the left, and we consider the representation M shown on the right.

$$
\begin{tikzcd}
\overset{1}{\circ} \arrow[dr] & \overset{2}{\circ} \arrow[d] & \overset{3}{\circ} \arrow[dl] \\
& \underset{0}{\circ} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
k0 \arrow[dr] & 0k \arrow[d] & \Delta \arrow[dl] \\
& k^2 &
\end{tikzcd}
$$

with $\Delta = \{(c, c) \mid c \in k\}$, or better $\Delta = \{\left[\begin{smallmatrix} c \\ c \end{smallmatrix}\right] \mid c \in k\}$.

Let $f = (f_0, f_1, f_2, f_3)$ be an endomorphism, thus $f_0 \colon k^2 \to k^2$ is given by a (2×2)-matrix F with coefficients in k. The commutativity of the diagram

$$
\begin{tikzcd}
k0 \arrow[r, "f_1"] \arrow[d, "u"'] & k0 \arrow[d, "u"] \\
k^2 \arrow[r, "f_0"] & k^2
\end{tikzcd}
$$

(here, u denotes the inclusion map) implies that F is an upper triangular matrix, Similarly, the commutativity of the diagram

$$
\begin{tikzcd}
0k \arrow[r, "f_2"] \arrow[d, "u"'] & 0k \arrow[d, "u"] \\
k^2 \arrow[r, "f_0"] & k^2
\end{tikzcd}
$$

shows that F is a lower triangular matrix. Thus F is a diagonal matrix, say $F = \left[\begin{smallmatrix} d_1 & \\ & d_2 \end{smallmatrix}\right]$. But there is a third commutativity condition:

$$
\begin{tikzcd}
\Delta \arrow[r, "f_3"] \arrow[d, "u"'] & \Delta \arrow[d, "u"] \\
k^2 \arrow[r, "f_0"] & k^2
\end{tikzcd}
$$

it asserts that F maps Δ into Δ. But $F \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} d_1 & \\ & d_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}\right]$. It follows that $d_1 = d_2$, thus F is a scalar matrix, say the multiplication by $c \in k$. But then also f_1, f_2, f_3 (being restrictions of f_0) are the multiplication by c, therefore $\mathrm{End}(M) = k$. In particular, we see that M *is indecomposable.*

Proposition. *Let M, M' be representations of the quiver Q such that the support quivers $Q(M)$ and $Q(M')$ are disjoint. Then*

$$\mathrm{End}(M \oplus M') = \mathrm{End}(M) \times \mathrm{End}(M').$$

Given two rings R, R' we denote by $R \times R'$ the product; it is defined on the set $R \times R'$ using component wise addition and multiplication.

Proof: In general, we have

$$\mathrm{End}(M \oplus M') = \begin{bmatrix} \mathrm{End}(M) & \mathrm{Hom}(M', M) \\ \mathrm{Hom}(M, M') & \mathrm{End}(M') \end{bmatrix}.$$

Since $Q(M) \cap Q(M') = \emptyset$, we have $\mathrm{Hom}(M', M) = 0 = \mathrm{Hom}(M, M')$.

3.3. Recollection of general results.

Here we should insert some general results from ring and module theory.

The rings which we will consider here are (associative, and not necessarily commutative) rings with 1. Note that we allow that a ring consists just of one element, this is the zero-ring (and there it holds that $0 = 1$). The zero ring arises naturally as the endomorphism ring of the zero representation of a quiver (and similarly as the endomorphism ring of the zero modules in module theory).

Recall that a ring R is said to be *local,* provided it is not the zero ring and has a unique maximal left ideal I. This maximal left ideal is necessarily a two-sided ideal, and contains every left ideal, it is called the *radical* of R. Also, R/I is a division ring. Note that a ring R is local if and only if the set of non-invertible elements is closed under addition, thus if R is local, also the opposite ring is local (thus R is local if and only if R has a unique maximal right ideal).

The only idempotents of a local ring are 0 and 1 (but there are many non-local rings which have only these two idempotents, for example the ring $\mathbb{Z}$ of the integers).

Fitting Lemma. *An endomorphism of a finite-dimensional indecomposable module is either bijective or nilpotent.*

Corollary. *A finite-dimensional algebra which has only* $0, 1$ *as idempotents, is a local ring with nilpotent radical.*

Proof: Just consider the algebra as a module over itself.

Corollary. *Let Q be a quiver and M a finite-dimensional indecomposable representation of Q. Then* $\mathrm{End}(M)$ *is a local ring with nilpotent radical.*

The locality of endomorphism rings of indecomposable objects has strong consequences, the most important one is the uniqueness of direct decompositions, as formulated in the theorem of Krull-Remak-Schmidt. Since this is usually formulated for module categories, we will discuss this result when we have identified the category of representations of a finite quiver Q with the category of finite-dimensional modules over the path algebra kQ.

3.4. Homomorphisms between thin indecomposable representations.

Proposition. *Let M, M' be thin indecomposable representations of a tree quiver Q. Then* $\operatorname{Hom}(M, M')$ *is at most one-dimensional.*

Proof. Let $f = (f_x)_x \colon M \to M'$ be a homomorphism. Clearly $f_x \neq 0$ implies that x belongs to $Q(M) \cap Q(M')$ (this holds true for general quivers). We assume that Q is a tree, and that M, M' are indecomposable representations. Thus $Q(M), Q(M')$ are again trees and if $Q(M) \cap Q(M') \neq \emptyset$, then $Q(M) \cap Q(M')$ is a tree. If there is an arrow $\alpha \colon x \to y$ in $Q(M) \cap Q(M')$, then we see that $f_x = f_y$:

$$\begin{array}{ccc} M_x & \xrightarrow{f_x} & M'_x \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ M_y & \xrightarrow{f_y} & M'_y \end{array} \qquad\qquad \begin{array}{ccc} k & \xrightarrow{f_x} & k \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ k & \xrightarrow{f_y} & k \end{array}$$

Note that the proof shows: *If M, M' are thin indecomposable representations of a tree quiver Q and $f \colon M \to M'$ is a non-zero homomorphism, then $Q(M) \cap Q(M')$ is a connected subquiver of Q and the image of f is the thin representation with support quiver $Q(M) \cap Q(M')$.*

In general, it is not difficult to decide whether $\operatorname{Hom}(M, M')$ is zero or 1-dimensional, Let us write down the rule in a special case:

The case of a linearly ordered $\mathbb{A}_n$-quiver.

We consider a linearly ordered quiver Q of type $\mathbb{A}_n$, say with the following vertices:

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n{-}1 \longleftarrow n$$

For every pair of integers i, j with $1 \le i \le j \le n$, we define a representation $[i, j]$ with $[i, j]_x = k$ if $i \le x \le j$ and $[i, j]_x = 0$ otherwise, and such that $[i, j]_\alpha$ is the identity map whenever possible. We know that we obtain in this way all the indecomposable representations of Q, one from each isomorphism class.

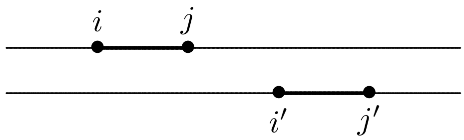
Proposition.

$$\operatorname{Hom}([i, j], [i', j']) = \begin{cases} k & \textit{if} \quad i \le i' \le j \le j' \\ 0 & \qquad \textit{otherwise} \end{cases},$$

and the image of any non-zero homomorphism $[i,j] \to [i',j']$ is just $[i',j]$.
If $i = i' \le j \le j'$, then any non-zero homomorphism $[i,j] \to [i',j']$ is a monomorphism.
If $i \le i' \le j = j'$, then any non-zero homomorphism $[i,j] \to [i',j']$ is an epimorphism.

Proof: Note that the considerations to be done will be the same as those in the special case $\mathbb{A}_2$ discussed in 3.1. Let $f\colon [i,j] \to [i',j']$.

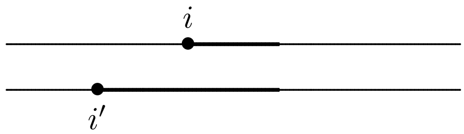
We distinguish several cases. First, assume that $j < i'$.



In this case $Q([i,j]) \cap Q([i',j']) = \emptyset$. Similarly, if $j' < i$, then $Q([i,j]) \cap Q([i',j']) = \emptyset$. In both cases we have $\mathrm{Hom}([i,j],[i',j']) = 0$.

From now on, we assume that $i' \le j$ and $i \le j'$.

Let $i' < i$, thus we deal with

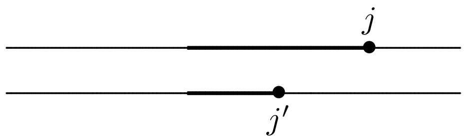


and both $i \le j, j'$. In particular, $i \in Q([i,j]) \cap Q([i',j'])$. Let us consider the arrow $i{-}1 \leftarrow i$:

$$\begin{array}{ccc} [i,j]_i & \xrightarrow{f_i} & [i',j']_i \\ \downarrow & & \downarrow \\ [i,j]_{i-1} & \longrightarrow & [i',j']_{i-1} \end{array} \qquad \begin{array}{ccc} k & \xrightarrow{f_i} & k \\ \downarrow & & \downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & k \end{array}$$

we see that we must have $f_i = 0$. But if $f_x = 0$ for some x in the intersection of the support quivers, then $f = 0$.

Let $j' < j$, thus we deal with

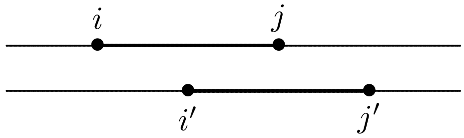


and both $i, i' \le j'$. In particular, $j' \in Q([i,j]) \cap Q([i',j'])$. Let us consider the arrow $j' \leftarrow j'{+}1$:

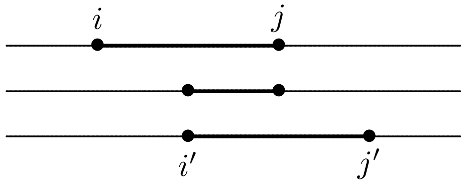
$$\begin{array}{ccc} [i,j]_{j'+1} & \longrightarrow & [i',j']_{j'+1} \\ \downarrow & & \downarrow \\ [i,j]_{j'} & \xrightarrow[f_{j'}]{} & [i',j']_{j'} \end{array} \qquad \begin{array}{ccc} k & \longrightarrow & 0 \\ {\scriptstyle 1}\downarrow & & \downarrow \\ k & \xrightarrow[f_{j'}]{} & k \end{array}$$

we see that we must have $f_{j'} = 0$, and again it follows that $f = 0$. Thus, in all cases discussed so far, $\mathrm{Hom}([i,j],[i',j']) = 0$.

It remains to consider the case $i' \leq i \leq j' \leq j$.



In this case we claim that there is a non-zero map $f\colon [i,j] \to [i',j']$, or even better that there is an epimorphism $g\colon [i,j] \to [i',j]$ and a monomorphism $h\colon [i',j] \to [i',j']$



so that we can take for f the composition $f = hg$. In order to define such maps f, g, h, just take $f_x = g_x = h_x = 1_k$ for $i' \leq x \leq j$ and zero otherwise (of course, one has to check that the diagrams in question commute). It follows that in this last case, $\mathrm{Hom}([i,j],[i',j'])$ is non-zero (and one-dimensional).

Example: $n = 4$. We consider the quiver Q of type $\mathbb{A}_4$

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4$$

with the indecomposable representations $[i,j]$ with $1 \leq i \leq j \leq 4$. The following picture arranges these representations in a triangle.

$$\begin{tikzcd}
{[1,1]} \arrow[dr] \arrow[rr, dotted, no head] & & {[2,2]} \arrow[dr] \arrow[rr, dotted, no head] & & {[3,3]} \arrow[dr] \arrow[rr, dotted, no head] & & {[4,4]} \\
& {[1,2]} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & {[2,3]} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & {[3,4]} \arrow[ur] & \\
& & {[1,3]} \arrow[ur] \arrow[dr] \arrow[rr, dotted, no head] & & {[2,4]} \arrow[ur] & & \\
& & & {[1,4]} \arrow[ur] & & &
\end{tikzcd}$$

The arrows $[i,j] \to [i,j+1]$ (drawn in southeast direction) indicate the existence of a so-called "irreducible" monomorphism, the arrows $[i,j] \to [i+1,j]$ (drawn in northeast direction) indicate the existence of a so-called "irreducible" epimorphism. Note that all the paths in southeast direction indicate the existence of corresponding monomorphisms, the paths in northeast direction indicate the existence of corresponding epimorphisms. As we know, any non-zero homomorphism $[i,j] \to [i',j']$ has as image an indecomposable representation, namely $[i',j]$, and there is a corresponding concatenation of a northeast path followed by a southeast path (for example, any non-zero homomorphism $[1,3] \to [3,4]$ has a factorization $[1,3] \to [2,3] \to [3,3] \to [3,4]$).

Thus, we deal with a visualization of the category $\mathcal{A}$ of indecomposable representations of Q, it is called the "Auslander-Reiten quiver" of the category of representations of Q. We also should mention the meaning of the dotted lines: they indicate "relations": In the upper line, they indicate that the composition of maps corresponding to a southeast arrow and the next northeast arrow is zero. The lower dotted lines mark the commutativity of the corresponding squares. Actually, this Auslander-Reiten quiver (with its vertices, arrows and dotted lines) provides a presentation of the category $\mathcal{A}$. by generators and relation.

Such an Auslander-Reiten quiver is not just a quiver, but a so-called translation quiver; the translation is indicated by the dotted lines. Any translation quiver may be considered as a 2-dimensional simplicial complex. In our case, the triangles can be seen quite well, all are bounded by a northeast arrow, a southeast arrow and a dotted line. In our example, these (small) triangles fit together to form a large triangle.

Again we see that we can order the indecomposable representations in such a way, that non-invertible homomorphisms go in one direction (here from left to right.