5. Extensions.

Given representations M, N of a quiver, we want to introduce a vector space $\operatorname{Ext}^1(M,N)$ which measures the possible extensions. Here, by an *extension* of R-modules (where R is a ring, for example the path algebra of a quiver) one means a short exact sequence

$$0 \to N \xrightarrow{f} Y \xrightarrow{g} M \to 0.$$

Note that such an exact sequence just means that $f\colon N \to Y$ is an injective homomorphism and g a cokernel of f, thus g is up to isomorphism uniquely determined by f, but the information given by f itself is (up to the isomorphism $f\colon N \to f(N)$) just the inclusion $f(N) \subseteq Y$. Let me repeat this as a slogan:

> SLOGAN: To consider extensions means nothing else then to study submodules of modules (to be precise: we do not mean the study of a submodule as a module in its own right, but the study of the **embedding** of the submodule into the given module).
>
> A typical question is the following: Given a submodule N of Y, is it a direct summand? Formulated in the language of "extensions", this is the question whether the sequence $0 \to N \to Y \to M \to 0$ splits or not.

The sections 5.1 - 5.3 deal with modules in general. Here, we start with a ring R, all modules are R-modules.

5.1. Split extension.

If N, M are modules and $\sigma\colon N \to M$, and $\rho\colon M \to N$ are maps with $\rho\sigma = 1_N$, then σ is said to be a *split monomorphism* (with *retraction* ρ), and ρ is said to be a *split epimorphism* (with *section* σ).

Lemma. *Let*

$$0 \to N \xrightarrow{f} Y \xrightarrow{g} M \to 0.$$

be an exact sequence. The following conditions are equivalent:
(1) *f is a split monomorphism.*
(1′) *g is a split epimorphism.*
(1″) *There is a submodule Y' such that $f(N) \oplus Y' = Y$.*

Proof should be well-known. For example, if (1) holds, thus there is $\rho\colon Y \to N$ with $\rho f = 1_N$. Let $Y' = \operatorname{Ker}(\rho)$. In order to see $Y = f(N) \oplus Y'$, take $y \in Y$ and write is as $f = f\rho(y) + (y - f\rho(y))$; here $f\rho(y) \in f(N)$ and $(y - f\rho(y)) \in Y'$. Also, If $y \in f(N) \cap Y'$, then $y = f(x)$ for some $x \in N$, thus $0 = \rho(y) = \rho f(x) = x$, and therefore $y = f(x) = 0$. Thus $(1''')$ holds. If $(1''')$ holds, then $g|Y'\colon Y' \to M$ is an isomorphism, thus take for σ the composition of $(g|Y')^{-1}$ with the inclusion map $Y' \to Y$; this yields $(1'')$.

Proposition. *Let R be a k-algebra, and let*

$$0 \to N \xrightarrow{f} Y \xrightarrow{g} M \to 0.$$

be an exact sequence, where N, M (thus also Y) are finite-dimensional k-modules. The following conditions are equivalent:
(1) *f is a split monomorphism.*
(2) *Y is isomorphic to $N \oplus M$.*
(3) $\dim_k \mathrm{End}(Y) = \dim_k \mathrm{End}(N \oplus M)$.
(3′) $\dim_k \mathrm{End}(Y) \geq \dim_k \mathrm{End}(N \oplus M)$.

Proof: Trivially, (1) $\implies$ (2) $\implies$ (3) $\implies$ (3′). Thus, let us assume (3′). We may assume that f is an inclusion map, thus $N \subseteq Y$ and that $M = Y/N$ with g the projection map. Let $\eta\colon \mathrm{End}(M) \to \mathrm{Hom}(N, M)$ be defined by $\eta(\phi) = g\phi f$ and let E be the kernel of η, thus

$$\dim_k \mathrm{End}(Y) \leq \dim_k E + \dim_k \mathrm{Hom}(N, M).$$

Note that

$$E = \{\phi \in \mathrm{End}(Y) \mid \phi(N) \subseteq N.\}.$$

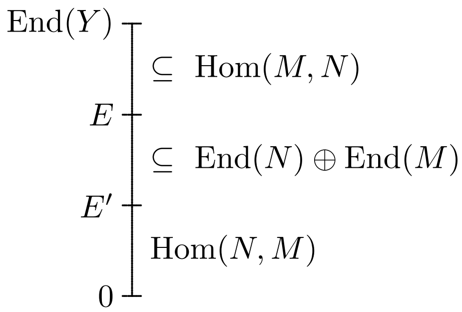
Define $\eta'\colon E \to \mathrm{End}(N) \oplus \mathrm{End}(M)$ by $\eta'(\phi) = (\phi|N, \overline{\phi})$, where $\overline{\phi}$ is the endomorphism of $M = Y/N$ induced by ϕ. Let E' be the kernel of η'. Then

$$E' = \{\phi \in \mathrm{End}(Y) \mid \phi(N) = 0, \phi(M) \subseteq N\},$$

thus E' is isomorphic as the vector space to $\mathrm{Hom}(M, N)$ (here, $\phi\colon M \to N$ corresponds to $f\psi g$). We see:

$$\begin{aligned} \dim_k E &\leq \dim_k E' + \dim_k \mathrm{End}(N) + \dim_k \mathrm{End}(M) \\ &= \dim_k \mathrm{Hom}(M, N) + \dim_k \mathrm{End}(N) + \dim_k \mathrm{End}(M). \end{aligned}$$

Here is a picture which shows the filtration of $\mathrm{End}(Y)$ we are dealing with, as well as the information on the corresponding factors which we have obtained:



Altogether we see that

$$\begin{aligned} \dim_k \mathrm{End}(Y) &\leq \dim_k E + \dim_k \mathrm{Hom}(N, M) \\ &\leq \dim_k \mathrm{Hom}(M, N) + \dim_k \mathrm{End}(N) + \dim_k \mathrm{End}(M) + \dim_k \mathrm{Hom}(N, M) \\ &= \dim_k \mathrm{End}(N \oplus M) \leq \dim_k \mathrm{End}(Y). \end{aligned}$$

This shows that all the inequality signs have to be equality signs, in particular, the map $\eta'\colon E \to \mathrm{End}(N) \oplus \mathrm{End}(M)$ has to be surjective, thus $(1_N, 0_N) = \eta'(\phi)$ for some $\phi \in E$.

But $\eta'(\phi) = (1_N, 0_N)$ means that $\phi|N = 1_N$ and $\phi(Y) \subseteq N$. Since $\phi(Y) \subseteq N$, we can write $\phi = f\rho$ for some $\rho\colon Y \to N$. Then $f = \phi f = f\rho f$, thus, since f is injective, $1_N = \rho f$. This shows that f is a split monomorphism.

> Of special interest seems to be the implication (2) $\implies$ (1). Whereas the converse implication is trivial, this one is not. In this context, it seems worthwhile to draw the attention to the weaker conditions that N is a direct summand of Y, or that M is a direct summand of Y. Such sequences $0 \to N \to N \oplus Y' \to M \to 0$ and $0 \to N \to M \oplus Y'' \to M \to 0$ are sometimes called *Riedtmann-Schofield sequences* and if such a sequence exists, one says that M is a *degeneration* of Y' (or that N is a degeneration of Y'', respectively), see for example Ringel: The ladder construction of Prüfer modules.

As mentioned above, given representations M, N of a quiver, we are going to introduce a vector space $\mathrm{Ext}^1(M, N)$ which measures the possible extensions. Actually, we are mainly interested to know whether $\mathrm{Ext}^1(M, N) = 0$ or not. This can be reformulated quite easily: The formulation $\mathrm{Ext}^1(M, N) = 0$ means just the following: Given a module Y with submodule N such that Y/N is isomorphic to M, then the embedding $N \to Y$ is *splits:* There is a submodule Y' of Y with $N \oplus Y' = Y$.

> It was Kaplansky who stressed that it sometimes may be sufficient to work with the condition $\mathrm{Ext}^1(M, N) = 0$ without introducing the groups Ext^1 .

If M is a module with $\mathrm{Ext}^1(M, M) = 0$, then M is said *to have no self-extensions.* An indecomposable module without self-extensions is called an *exceptional* module. The kQ-modules which we are interested in, are mainly the exceptional kQ-modules.

> Warning. This terminology is in some sense irritating. For example, for a Dynkin quiver, all the indecomposables are exceptional, thus to be exceptional is nothing special! One of the reasons for the naming comes from commutative ring theory, where it is very unusual to deal with a module without self-extensions. This has influenced people dealing with vector bundles, since they usually coming from commutative ring theory. Thus, the first appearance of the word "exceptional" in the sense as mentioned here, was in the realm of vector bundles, here we should mention the school of Rudakov. It was shifted to quiver representation by Crawley-Boevey, since it turned out that there were several parallel results.

5.2. Equivalence classes of short exact sequences.

Definition: Let $\epsilon = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ and $\epsilon' = (0 \to X \xrightarrow{f'} Y' \xrightarrow{g'} Z \to 0)$ be exact sequences (with identical first and last modules). These extensions are called *equivalent* provided there is a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow h & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z & \longrightarrow & 0. \end{array}$$

Note that, *if such a diagram exists, then the map h is necessarily an isomorphism.*

Proof: First, let us show that h is injective. Thus, take $y \in Y$ with $h(y) = 0$. Then $g(y) = g'h(y) = 0$, thus there is $x \in X$ with $f(x) = y$. Then $f'(x) = hf(x) = h(y) = 0$, and, since f' is injective, $x = 0$, thus $y = f(x) = 0$.

Second, in order to see that h is surjective, start with $y' \in Y'$. There is $y \in Y$ with $g(y) = g'(y')$, since g is surjective. Now

$$g'(y' - h(y)) = g'(y) - g'h(y) = g'(y) - g(y) = 0,$$

thus $y' - h(y) = f'(x)$ for some $x' \in X'$. Then $y' = h(y) + f'(x) = h(y) + hf(x) = h(y + f(x))$ shows that y is in the image of h.

As a consequence, it is obvious that the relation for short exact sequences to be equivalent is really an equivalence relation: if ϵ, ϵ' (in this order) are equivalent, say using the map h, then also ϵ', ϵ are equivalent, use h^{-1}; if in addition also ϵ', ϵ'' are equivalent, say using the map h', then ϵ, ϵ'' are equivalent: use $h'h$.

The set of equivalence classes of exact sequences $\epsilon = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ with X, Z being fixed will be denoted by $\mathrm{Ext}^1(Z, X)$.

> We have introduced here $\mathrm{Ext}^1(Z, X)$ just as a set (or as a set with a distinguished element, namely the equivalence class of split exact sequences). Usually, one defines on $\mathrm{Ext}^1(Z, X)$ an addition, the so-called Baer addition, so that $\mathrm{Ext}^1(Z, X)$ becomes an abelian group. In case one deals with a k-algebra R, the set $\mathrm{Ext}^1(Z, X)$ should be endowed even with the structure of a k-space. We avoid this at the moment, but later we will identify the set Ext^1 (in the case where R is the path algebra of a quiver) with a k-space, and this k-space structure of $\mathrm{Ext}^1(Z, X)$ is the usual one.
>
> When we speak about the **set** of equivalence classes, we have to worry whether there may be set-theoretical difficulties. Fortunately, in the usual categories we are working with, say the category of modules over a ring R, the class of modules which are isomorphic to a fixed one may not be a set (but just a class), however the class of isomorphism classes of modules with fixed cardinality is a set, as is the class of equivalence classes of exact sequences of the form $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ with X, Z both being fixed.

We have seen that if the sequences $\epsilon = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ and $\epsilon' = (0 \to X \xrightarrow{f'} Y' \xrightarrow{g'} Z \to 0)$ are equivalent, then Y, Y' are isomorphic, but not every isomorphism $h\colon Y \to Y'$ will not provide a commutative diagram as required for the equivalence — the easiest example to have in mind is the following (here we assume that R is a k-algebra): assume that $h\colon Y \to Y'$ is an isomorphism which provides a a commutative diagram as required, and let $c \neq 0$ be an element of k, then also $ch\colon Y \to Y'$ is an isomorphism, but it will **not** provide such a commutative diagram unless $X = 0 = Z$.

5.3. Construction of extensions using projective modules.

Proposition. *Assume that there is given a surjective map $p\colon P \to Z$ with P projective, let ΩZ be the kernel of p, thus we deal with the exact sequence*

$$\epsilon : \qquad 0 \to \Omega Z \xrightarrow{u} P \to Z \to 0.$$

Then any short exact sequence ending in Z is induced from the sequence ϵ.

This means that given a short exact sequence $0 \to X \to Y \to Z \to 0$, there is a map $\phi\colon \Omega Z \to X$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega Z & \xrightarrow{u} & P & \longrightarrow & Z & \longrightarrow & 0 \\ & & \phi\big\downarrow & & \big\downarrow\phi' & & \big\| & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array} .$$

Note that up to equivalence of short exact sequences, the upper sequence and the map ϕ together determine the lower exact sequence uniquely: Namely for such a commutative diagram, the left square

$$\begin{array}{ccc} \Omega Z & \xrightarrow{u} & P \\ \phi\big\downarrow & & \big\downarrow\phi' \\ X & \longrightarrow & Y, \end{array}$$

is a pushout diagram, thus up to isomorphism Y is of the form

$$Y = P \oplus X/\{(-u(a), \phi(a) \mid a \in \Omega Z\}.$$

The proposition asserts that there is a surjective map

$$\delta\colon \operatorname{Hom}(\Omega Z, X) \to \operatorname{Ext}^1(Z, X),$$

and the kernel of this map are the morphisms $\Omega Z \to X$ which factor through $u\colon \Omega Z \to P$.

These assertions are usually formulated in terms of the long exact sequence which one obtains when we apply the functor $\mathrm{Hom}(-, X)$ to the exact sequence

$$0 \to \Omega Z \xrightarrow{u} P \to Z \to 0 \qquad (\epsilon).$$

Namely, we obtain the exact sequence

$$0 \to \mathrm{Hom}(Z, X) \to \mathrm{Hom}(P, X) \xrightarrow{\mathrm{Hom}(u,X)} \mathrm{Hom}(\Omega Z, X) \xrightarrow{\delta} \mathrm{Ext}^1(Z, X) \to 0$$

The map δ is called the *connecting homomorphism,* it attaches to $\phi\colon \Omega Z \to X$ the exact sequence induced from ϵ by ϕ.

Let us return to quivers and their representations.

5.4. Realization of extensions of quiver representations by quiver data.

Let Q be a quiver. Given representations M, N of Q, let

$$D(M, N) = \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)}).$$

This is the set whose elements we will use in order to construct short exact sequences starting with N and ending in M. For any $e = (e_\alpha)_\alpha$ in $D(M, N)$, we may consider the representation $W(M, N, e)$ as follows:

$$W(M, N, e)_x = N_x \oplus M_x, \quad W(M, N, e)_\alpha = \begin{bmatrix} N_\alpha & e_\alpha \\ 0 & M_\alpha \end{bmatrix}$$

Note that N is a submodule of $Y = W(M, N, e)$ and the corresponding factor module Y/N can be identified with M; thus, there is the following short exact sequence

$$\epsilon(M, N, e) = \left(0 \to N \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} W(M, N, e) \xrightarrow{[0\ 1]} M \to 0 \right).$$

Lemma. *If Y is a representation of Q with a subrepresentation N and $M = Y/N$, with inclusion map $u\colon N \to P$ and projection map $p\colon Y \to M$, then there is a commutative diagram*

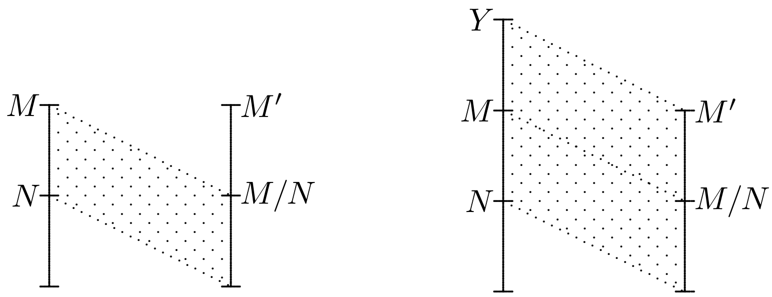
$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} & W(M, N, e) & \xrightarrow{[0\ 1]} & M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow{u} & Y & \xrightarrow{p} & M & \longrightarrow & 0 \end{array}$$

Proof: For any vertex x, choose a submodule C_x such that $N_x \oplus C_x = Y_x$. Using the map p, we actually may identify C_x with M_x, thus we assume $N_x \oplus M_x = Y_x$. Let $\alpha\colon x \to y$ be an arrow. Note that $Y_\alpha(M_x) \subseteq N_y$, thus we may consider the restriction of Y_α to M_x and denote it by $e_\alpha\colon M_x \to N_y$. Using these identifications, we see that the identity map

$$\phi_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : W(M,N,e)_x = N_x \oplus M_x \longrightarrow N_x \oplus M_x = Y_x$$

yields an isomorphism (even an identification) $\phi = (\phi_x)_x\colon W(M,N,e) \to Y$. Of course, this is the required isomorphism which we need in the Lemma.

We say that an abelian category such as a module category is *hereditary* provided the following condition is satisfied: If M is a module with a submodule N and there is an embedding $M/N \to M'$ for some module M', then there exists a module Y with submodule M, such that there is an isomorphism $Y/N \to M'$ which is the identity on M/N (by assumption, M/N is both a submodule of Y/N as well as of M').



Let us stress, for those familiar with Ext^2 or at least with Ext^1, that this definition of heredity coincides with the usual one, namely with the condition that globally $\mathrm{Ext}^2 = 0$, or, equivalently, that for any monomorphism u and any object N, the induced map $\mathrm{Ext}^1(N,u)$ is surjective. What we did, is that we have reformulated the surjectivity assertion, by saying that for any exact sequence ϵ, which we can assume to be of the form $0 \to N \to M \to M/N \to 0$ (with an inclusion map $N \to M$), and for any embedding $u\colon M/N \to M'$, there exists an exact sequence $\epsilon' = (0 \to N \to Y \to M' \to 0)$ which induces ϵ, thus $\epsilon = \mathrm{Ext}^1(N,u)(\epsilon')$.

Theorem. *The category* $\mathrm{Rep}(Q,k)$ *is hereditary.*

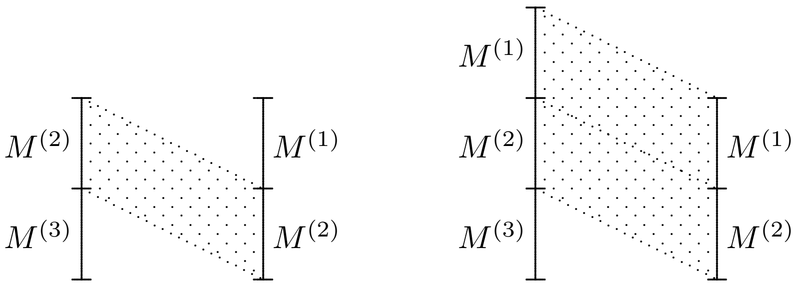
Proof: We assume that we have given three representations, say $M^{(1)}, M^{(2)}, M^{(3)}$, and extensions

$$W(M^{(1)}, M^{(2)}, e) \qquad \text{and} \qquad W(M^{(2)}, M^{(3)}, e').$$

Then the required representations are those of the form

$$\left(M_x^{(1)} \oplus M_x^{(2)} \oplus M_x^{(3)}, \begin{bmatrix} M^{(1)} & e_x & * \\ & M^{(2)} & e'_x \\ & & M^{(3)} \end{bmatrix} \right),$$

where $*$ is arbitrary.



Remark. Any Serre subcategory of a hereditary category is hereditary, thus the category of nilpotent representations is also hereditary.

Reformulation. *If $e\colon X \to I$ is an epimorphism, $m\colon I \to Y$ is a monomorphism, then there is a monomorphism $m'\colon X \to J$ and an epimorphism $e'\colon J \to Y$ such that the sequence*

$$0 \to X \xrightarrow{\begin{bmatrix} m' \\ e \end{bmatrix}} I \oplus J \xrightarrow{\begin{bmatrix} -e' & m \end{bmatrix}} Y \to 0$$

is exact.

Proof: Choose J with submodule X such that $J/m(I) = Y$, let $m'\colon X \to J$ the inclusion map and $e'\colon J \to Y$ the projection.

Lemma. *If X, Y are indecomposable objects of finite length in a hereditary category, and $\operatorname{Ext}^1(Y,X) = 0$, then any non-zero morphism $X \to Y$ is a monomorphism or an epimorphism.*

Proof. Let $f\colon X \to Y$ be a non-zero morphism which is neither a monomorphism nor an epimorphism, let I be the image of f. Then there is an exact sequence

$$0 \to X \to I \oplus J \to Y \to 0.$$

But this sequence cannot split, since otherwise $X \oplus Y$ is isomorphic to $I \oplus Y$, but $I \neq 0$ and an indecomposable direct summand I' of I has length smaller than the length of X or Y, thus cannot be isomorphic to X of Y, contrary to the Krull-Remak-Schmidt theorem.

Corollary. *If M is an exceptional object of finite length in a hereditary category, then $\operatorname{End}(M)$ is a division ring.*

Proof: If $f\colon M \to M$ is non-invertible, then it is neither a monomorphism nor an epimorphism. The previous result shows that $f = 0$.

5.5. Modules without self-extensions.

We are interested in the exceptional modules or, more generally, in modules without self-extensions.

Proposition 1. *Let M, N be representations of the quiver Q. Let $\alpha\colon x \to y$ be an arrow of the quiver. Assume that M_α has a non-trivial kernel, and that N_α has a non-trivial cokernel. Then* $\operatorname{Ext}^1(M,N) \neq 0$.

Proof. Write $M_x = \operatorname{Ker}(M_\alpha) \oplus C$ for some subspace C, and take a non-zero element $b \in M_y$ which does not belong to $\operatorname{Im}(N_\alpha)$. Let $e_\alpha\colon M_x \to N_y$ be defined as follows: it shall be zero on C and it shall map $\operatorname{Ker}(M_\alpha)$ surjectively onto $\langle b\rangle$ (such a linear map exists, sine we assume that $\operatorname{Ker}(M_\alpha)$ is non-zero. For the remaining arrows β of the quiver, let $e_\beta = 0$. We consider $W = W(M,N,e)$, in particular

$$W_\alpha = \begin{bmatrix} M_\alpha & e_\alpha \\ & N_\alpha \end{bmatrix} : M_x \oplus N_x \longrightarrow M_y \oplus N_y.$$

Clearly, the image of W_α is $\operatorname{Im}(M_\alpha) \oplus (\operatorname{Im} N_\alpha) + \langle b\rangle$, and the latter plus sign concerns also a direct sum inside N_y. Therefore W_α has rank equal to $\operatorname{rank} M_\alpha + \operatorname{rank} N_\alpha + 1$. But this shows that $W = W(M,N,e)$ cannot be isomorphic to $M \oplus N$, thus $\operatorname{Ext}^1(M,N) \neq 0$.

Corollary. *Let M be a representation of a quiver Q without self-extensions. Then, for any arrow α, the map M_α has maximal rank.*

(We recall that a vector space map $V \to V'$ is said to have maximal rank, provided its rank is as large as possible, namely $\min\{\dim_k V, \dim_k V'\}$, or, equivalently, provided the map is a monomorphism or an epimorphism.)

Proposition 2. *Let M be a module without self-extensions. Let w be a path with $wM = 0$. Let α be an arrow with $t(\alpha = h(w)$ and $\alpha w M = 0$. Then $M_{h(\alpha)} = 0$.*

Proof: Write $M_x = wM \oplus C$ for some subspace C, and choose some non-zero element $b \in M_y$. Let $e_\alpha\colon M_x \to N_y$ be defined as follows: it shall be zero on C and it shall map wM surjectively onto $\langle b\rangle$. For the remaining arrows β of the quiver, let $e_\beta = 0$. We consider $W = W(M,M,e)$ and the extension

$$0 \to M \to W(M,M,e) \to M \to 0.$$

This sequence does not split. Now assume that $\alpha w M = 0$, then also $\alpha w(M \oplus M) = 0$. However, by construction, $\alpha w W(M,M,e) = \langle b\rangle \neq 0$ and therefore $W(M,M,e)$ is not isomorphic to $M \oplus M$. But this implies $\operatorname{Ext}^1(M,M) \neq 0$, contrary to the assumption. This contradiction shows that we must have $\alpha w M \neq 0$.

Corollary. *Let M be a nilpotent module without self-extensions. Then the support quiver $Q(M)$ does not have cyclic paths.*

Proof: Assume that there is a cyclic path v in the support quiver. Since M is nilpotent, there is some t with $v^t M = 0$. Let $v^t = \alpha_s \cdots \alpha_1$ Choose $m \geq 0$ maximal with $wM \neq 0$, where $w = \alpha_m \cdots \alpha_1$. Then $m < s$ and $\alpha w M = 0$ for $\alpha = \alpha_{m+1}$. According to the Proposition, $M_{h(\alpha)} = 0$. But then α does not belong to the support quiver of Q, a contradiction.

It is not too difficult to show that the support quiver of no module without self-extensions has cyclic paths.

5.6. The standard guide.

Given representations M, N of the quiver Q and an element $e \in D(M,N)$, we have constructed a representation $W(M,N,e)$, or better even, an extension

$$\epsilon(M,N,e) = \left(0 \to N \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} W(M,N,e) \xrightarrow{[0\ 1]} M \to 0\right)$$

and we know that we obtain in this way all extensions. One may ask when are two such extensions equivalent.

> For example, if we start with the quiver of tape $\mathbb{A}_2$ and consider $M = N$ the two-dimensional indecomposable representation, then obviously all the extensions $\epsilon(M,N,e)$ are equivalent.

If M, N are representations, we consider the following linear map which we call the *standard guide* Ξ_{MN} for M and N:

$$\Xi_{MN} \colon \bigoplus_x \operatorname{Hom}_k(M_x, N_x) \longrightarrow \bigoplus_a \operatorname{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)}) = D(M,N),$$

defined by

$$(\Xi_{MN}(f))_\alpha = N_\alpha f_{t(\alpha)} - f_{h(\alpha)} M_\alpha.$$

where $f = (f_x)_x$ with k-linear maps $f_x \colon M_x \to N_x$. First, let us note:

Proposition. *The kernel of* Ξ_{MN} *is* $\operatorname{Hom}(M,N)$.

Now let us look at the cokernel. Note that if $e = (e_\alpha)_\alpha$ is an element of the target $D(M,N)$ of Ξ_{MN}, then there is defined the representation $W(M,N,e)$ and the extension $\epsilon(M,N,e)$.

Theorem. *The map* $e \mapsto \epsilon(e)$ *yields a bijection*

$$\operatorname{Cok}(\Xi_{MN}) \longrightarrow \operatorname{Ext}^1(M,N).$$

Under this bijection, the zero element $e = 0$ *is sent to the split exact sequence.*

Proof: The last sentence is trivial. Thus, let us prove the first sentence. We have to understand what it means that the exact sequences $\epsilon(M,N,e)$ and $\epsilon(M,N,e')$ are equivalent: there has to exist a map $h \colon W(M,N,e) \to W(M,N,e')$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & W(M,N,e) & \xrightarrow{[0\ 1]} & M & \longrightarrow & 0 \\
 & & \| & & \downarrow h & & \| & & \\
0 & \longrightarrow & N & \xrightarrow[\begin{bmatrix}1\\0\end{bmatrix}]{} & W(M,N,e') & \xrightarrow[{[0\ 1]}]{} & M & \longrightarrow & 0
\end{array}$$

For such a map $h = (h_x)_x$, the maps $h_x \colon W(M,N,e)_x \to W(M,N,e')_x$ have to be of the form

$$h_x = \begin{bmatrix} 1 & f_x \\ 0 & 1 \end{bmatrix} : h_x \colon W(M,N,e)_x = N_x \oplus M_x \to N_x \oplus M_x = W(M,N,e')_x,$$

with $f_x \colon M_x \to N_x$. Now for every arrow $\alpha \colon x \to y$, we must have $h_y W_\alpha = W'_\alpha h_x$, or, written in matrices:

$$\begin{bmatrix} 1 & f_y \\ & 1 \end{bmatrix} \begin{bmatrix} N_\alpha & e_\alpha \\ & M_\alpha \end{bmatrix} = \begin{bmatrix} N_\alpha & e'_\alpha \\ & M_\alpha \end{bmatrix} \begin{bmatrix} 1 & f_x \\ & M_\alpha \end{bmatrix},$$

thus

$$e'_\alpha - e'_\alpha = N_\alpha f_x - f_y M_\alpha = \Xi(f)_\alpha.$$

Of course, also conversely, if $e'_\alpha - e'_\alpha = \Xi(f)_\alpha$, then the extensions $\epsilon(M,N,e)$ and $\epsilon(M,N,e')$ are equivalent.

Remark. Since the cokernel is a vector space, we may (and will) consider also $\mathrm{Ext}^1(M,N)$ as a vector space.

> As we have mentioned, there is a direct way to define an addition (the Baer addition) and scalar multiplication an the set of equivalence classes of extensions. If one uses the Baer addition on $\mathrm{Ext}^1(M,N)$, one has to show that the bijection established in the Theorem is in fact a vector space isomorphism. Below we will see that the vector space operations on $\mathrm{Ext}^1(M,N)$ as defined here coincide with those which we obtain when we calculate $\mathrm{Ext}^1(M,N)$ using a projective presentation of M, thus with the standard definition.

5.7. The standard resolution of a quiver representation.

The standard guide Ξ_{MN} can be obtained from a certain projective presentation of M, namely the standard presentation, by applying the functor $\mathrm{Hom}(-,N)$. In order to define the standard presentation of M, we define the following two projective modules:

$$P^s(M) = \bigoplus_{x \in Q_0} P(x) \otimes_k M_x,$$
$$\Omega^s(M) = \bigoplus_{\alpha \in Q_1} P(h(\alpha)) \otimes_k M_{t(\alpha)}.$$

The tensor product $\otimes_k$ which we use here, means just the following: if V is a vector space of dimension v, then $P(x) \otimes_k V$ is the direct sum of v copies of $P(x)$; if we choose a basis of V, we may think of the copies being indexed by the elements of the basis. Of course, with $P(x)$ also $P(x) \otimes_k V$ is projective, for any vector space V.

The *standard resolution* of M is given as follows:

$$0 \to \Omega^s(M) \xrightarrow{d} P^s(M) \xrightarrow{p} M \to 0,$$

where the maps are defined as follows:

$$\begin{aligned} p(w \otimes a) &= wa && \text{for} \quad w \in P(x),\ a \in M_x, \\ d(w \otimes a) &= w\alpha \otimes a - w \otimes \alpha a && \text{for} \quad w \in P(h(\alpha)),\ a \in M_{t(\alpha)}. \end{aligned}$$

Proposition 1. *The standard resolution of any representation M is an exact sequence.*

The proof is just a direct calculation, fiddling around with linear combinations of paths, see the Lecture Notes by Crawley-Boevey.

The use of the tensor product $\otimes_k$ has the following advantage: We have seen in section 4.6 that the evaluation map $f \mapsto f_x(e_x)$ yields an isomorphism $\mathrm{Hom}(P(x), N) \to N_x$. Of course, there is also a corresponding isomorphism $\mathrm{Hom}_k(k, N_x) \to N_x$ which sends $\phi\colon k \to N_x$ to $\phi(1)$, thus we may combine these isomorphisms (or better, the first isomorphism which the inverse of the second) in order to obtain a canonical isomorphism

$$\mathrm{Hom}(P(x), N) \to \mathrm{Hom}_k(k, N).$$

Using the tensor product $\otimes_k$, this yields an isomorphism

$$\eta\colon \mathrm{Hom}(P(x) \otimes_k V, N) \to \mathrm{Hom}_k(V, N)$$

for any vector space V, we call it the *evaluation map.*

Consider now representations M, N of the quiver Q and take the standard resolution of M. If we apply the functor $\mathrm{Hom}(-, N)$ to the standard resolution of M, we obtain the following exact sequence:

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(P^s(M), N) \xrightarrow{\mathrm{Hom}(d,N)} \mathrm{Hom}(\Omega^s(M), N) \to \mathrm{Ext}^1(M, N) \to 0.$$

Let us consider the evaluation maps

$$\eta_0\colon \mathrm{Hom}(P^s(M), N) \longrightarrow \bigoplus_x \mathrm{Hom}_k(M_x, N_x)$$

$$\eta_1\colon \mathrm{Hom}(\Omega^s(M), N) \longrightarrow \bigoplus_a \mathrm{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)})$$

Here, the index x runs through the set Q_0 of the vertices, the index α through the set Q_1 of arrows; also, we wrote $\otimes$ instead of $\otimes_k$.

Proposition 2. *Let M, N be representations of the quiver Q. The standard guide Ξ_{MN} is just* $\mathrm{Hom}(d, N)$*, where u is the standard presentation of M,* namely the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}(\bigoplus_x P(x)\otimes M_x, N) & \xrightarrow{\mathrm{Hom}(d,N)} & \mathrm{Hom}(\bigoplus_\alpha P(h(\alpha))\otimes M_{t(\alpha)}, N) \\ \eta_0\big\downarrow & & \big\downarrow\eta_1 \\ \bigoplus_x \mathrm{Hom}_k(M_x, N_x) & \xrightarrow{\quad\Xi_{MN}\quad} & \bigoplus_a \mathrm{Hom}_k(M_{t(\alpha)}, N_{h(\alpha)}) \end{array}$$

Proof. Let $f = (f_x)_x \in \bigoplus_x \mathrm{Hom}_k(M_x, N_x)$. Under the inverse of the middle map η, we obtain the homomorphism defined by the maps $P(x)\otimes_k M_x \to N_x$ with $w\otimes a \mapsto wf_x(a)$ for $w \in P(x)$ and $a \in M_x$, let us call it f'. Under $\mathrm{Hom}(d, N)$, we get the homomorphism $\mathrm{Hom}(d, N)(f'))$, let us look at its restriction to $P(y)\otimes M_x$ (where $t(\alpha) = x$, and $h(\alpha) = y$, thus $\alpha\colon x \to y$), it maps $e_y \otimes a$ first (under d) to $\alpha\otimes a - e_y \otimes \alpha a$ and then under f' to $\alpha f_x(a) - f_y(\alpha a)$, thus to $N_\alpha f_x(a) - f_y M_\alpha(a)$.

But Ξ_{MN} also sends $f = (f_x)_x$ to the element of $D(M, N)$ whose component indexed by $\alpha\colon x \to y$ is $N_\alpha f_x - f_y M_\alpha$.

Corollary. *The maps* $\mathrm{Hom}(d, N)$ *and* Ξ_{MN} *have the same kernel, namely* $\mathrm{Hom}(M, N)$ *and the same cokernel, namely* $\mathrm{Ext}^1(M, N)$*.*

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(M,N) & \to & \mathrm{Hom}(P^s(M),N) & \xrightarrow{\mathrm{Hom}(d,N)} & \mathrm{Hom}(\Omega^s(M),N) & \to & \mathrm{Ext}^1(M,N) \to 0 \\ & & \| & & \eta_0\big\downarrow & & \big\downarrow\eta_1 & & \| \\ 0 & \to & \mathrm{Hom}(M,N) & \to & \bigoplus_x \mathrm{Hom}_k(M_x,N_x) & \xrightarrow{\Xi_{MN}} & \bigoplus_a \mathrm{Hom}_k(M_{t(\alpha)},N_{h(\alpha)}) & \to & \mathrm{Ext}^1(M,N) \to 0 \end{array}$$

It seems to be of interest to compare the cokernel maps: we either may start (in the upper row) with an element ϕ in $\mathrm{Hom}(\Omega^s(M), N)$ and form the induced exact sequence with respect to ϕ, or else (in the lower row) we may take an element $e \in D(M, N) = \bigoplus_\alpha(M_{t(\alpha)}, N_{h(\alpha)}$ and form the extension $\epsilon(M, N, e)$. What we obtain, for $\eta_1(\phi) = e$, are exact sequences which are equivalent.