6. Dynkin quivers, Euclidean quivers, wild quivers.

This last section is more sketchy, its aim is, on the one hand, to provide a short survey concerning the difference between the Dynkin quivers, the Euclidean quivers and the remaining ones, but also, on the other hand, to draw the attention to some important techniques not covered in the lectures (but note that some of the definitions are not given and several proofs are missing).

6.1. The theorems of Gabriel and Kac.

A finite dimensional algebra is said to be *representation-finite* provided there are only finitely many isomorphism classes of indecomposable representations. The starting result for the representation theory of quivers was Gabriel's theorem:

Theorem (Gabriel). (a) *A connected quiver is representation finite if and only if it is a Dynkin quiver.*

The number of the indecomposable representations for the different Dynkin types is as follows:

$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\frac{1}{2}n(n+1)$	$n(n-1)$	36	69	120

note that the numbers do not depend on the orientation! Actually, as observed by Tits, there is a bijection between the indecomposable representations and the positive roots of the corresponding simple complex Lie algebra $\mathbf{g}$. This bijection is furnished by the dimension vector $\mathbf{dim}$ (it will be introduced in section 6.2). Recall that the (finite-dimensional) simple complex Lie algebras have been classified by Cartan, they are labeled by the Dynkin diagrams (including also the types $\mathbb{B}_n, \mathbb{C}_n, \mathbb{F}_4, \mathbb{G}_2$, which do not play a role when dealing with representations of quivers).

(b) *If Q is a Dynkin quiver and $\mathbf{g}$ is the corresponding simple complex Lie algebra, then $\mathbf{dim}$ yields a bijection between the set of isomorphism classes of indecomposable representations of Q and the set of positive roots of $\mathbf{g}$.*

Again, this is an assertion which shows that some invariants for quiver representations do not depend on the orientation of the quiver, namely here the dimension vectors of the indecomposable representations. One special representation of the Dynkin quiver Q should be mentioned: there is a unique indecomposable representation of maximal dimension, it corresponds to the unique maximal root. For example, for type $\mathbb{E}_8$, the dimension vector of the maximal indecomposable representation is

$$\begin{matrix} & & 3 & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 \end{matrix}$$

In our list of the Dynkin diagrams we have added on the right side the corresponding maximal root. It plays an important role (not only in Lie theory, but also) in the representation theory of quivers.

Gabriel's theorem was extended to arbitrary finite quivers by Kac. Given any quiver Q without loops (or better just its underlying graph $\overline{Q}$), there is a corresponding (usually infinite-dimensional) complex Lie algebra $\mathbf{g}$, the Kac-Moody Lie algebra of type $\overline{Q}$, as well as a corresponding root system; here the roots are divided into two classes: the real roots and the imaginary roots (in the special case of dealing with a Dynkin quiver, the Kac-Moody Lie algebra of type $\overline{Q}$ is just the finite-dimensional simple Lie algebra of type $\overline{Q}$, and there are no imaginary roots). Kac has shown:

Theorem (Kac). *If Q is a finite quiver without loops and $\mathbf{g}$ the corresponding Kac-Moody Lie algebra, then* **dim** *yields a surjective map from the set of isomorphism classes of indecomposable representations of Q onto the set of positive roots of $\mathbf{g}$.*

If r is a positive real root of $\mathbf{g}$, then $\mathbf{dim}^{-1}(r)$ is a single isomorphism class. If r is a positive imaginary root, and k is infinite, then $\mathbf{dim}^{-1}(r)$ consists if infinitely many isomorphism classes.

Actually, there is a corresponding result also for quivers with loops, but one needs to define the corresponding Lie algebras, or, at least, the corresponding root systems.

6.2. The Euler form.

The exact sequence

$$0 \to \operatorname{Hom}(M,N) \to \bigoplus_x \operatorname{Hom}_k(M_x,N_x) \xrightarrow{\Xi} \bigoplus_a \operatorname{Hom}_k(M_{t(\alpha)},N_{h(\alpha)}) \to \operatorname{Ext}^1(M,N) \to 0$$

shows that the dimension difference

$$\dim_k \operatorname{Hom}(M,N) - \dim_k \operatorname{Ext}^1(M,N)$$

only depends on the dimensions of the various vector spaces M_x, N_x.

In order to formulate this properly, let us consider the free abelian group $\mathbb{Z}Q_0$ with basis Q_0, its elements will be written in the form $d = (d_x)_x$ with integers d_x for all $x \in Q_0$. If M is a representation of Q, then we may consider the element $\mathbf{dim}\, M = (\dim_k M_x)_x$ as such an element, it is called the *dimension vector* of M

We define on $\mathbb{Z}Q_0$ a bilinear form depending on the quiver Q as follows: If $d, d' \in \mathbb{Z}Q_0$, let

$$\langle d, d' \rangle = \sum_{x\in Q_0} d_x d'_x - \sum_{\alpha\in Q_1} d_{t(\alpha)} d'_{h(\alpha)};$$

we are also interested in the corresponding quadratic form

$$q(d) = \langle d, d \rangle.$$

Proposition. *If M, M' are representations of Q, then*

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, M' \rangle = \dim \operatorname{Hom}(M,M') - \dim \operatorname{Ext}^1(M,M').$$

Corollary 1. *If M is an exceptional representation of Q with* $\operatorname{End}(M) = k$, *then* $q(\mathbf{dim}\, M) = 1$.

Remark: The condition $\operatorname{End}(M) = k$ is actually always satisfied. We know already that $\operatorname{End}(M)$ is a division ring, see section 5.4. Thus, in case k is algebraically closed, it follows directly that $\operatorname{End}(M) = k$. However, also in general one can show that $\operatorname{End}(M) = k$ for any exceptional representation of a quiver.

Corollary 2. *If M is a representation with* $\operatorname{End}(M)$ *a division ring and* $\operatorname{Ext}^1(M, M) \neq 0$, *then* $q(\mathbf{dim}\, M) \leq 0$.

Proof. Let $D = \operatorname{End}(M)^{\mathrm{op}}$. Since $\operatorname{Ext}^1(M, M)$ is a non-zero D-D-bimodule, the k-dimension of $\operatorname{Ext}^1(M, M)$ is at least $\dim_k D$, thus

$$q(\mathbf{dim}\, M) = \dim_k D - \dim_k \operatorname{Ext}^1(M, M) \leq 0.$$

In case we deal with a quiver without cyclic paths, the group $\mathbb{Z}Q_0$ can be identified with the Grothendieck group $K_0(\operatorname{mod} kQ)$ of finite-dimension representations of Q modulo all exact sequences. Namely, according to the Jordan-Hölder theorem, the Grothendieck group $K_0(\operatorname{mod} kQ)$ is the free abelian group with basis the set of isomorphism classes of simple kQ-modules. But if Q has no cyclic paths, then we know that the simple representations are of the form $S(x)$, with $x \in Q_0$, thus we may identify the basis vector of $\mathbb{Z}Q_0$ with index $x \in Q_0$ with the isomorphism class of $S(x)$. If we do so, then for every representation M of the quiver, its dimension vector $\mathbf{dim}\, M$ has as coordinate with index x just the Jordan-Hölder multiplicity of $S(x)$ in M.

6.3. The quadratic form of a quiver.

We have introduced in 6.2 a quadratic form $q = q_Q$ on the free abelian group $\mathbb{Z}Q_0$. By definition,

$$q(d) = \sum_{x \in Q_0} d_x^2 - \sum_{\alpha \in Q_1} d_{t(\alpha)} d_{h(\alpha)}.$$

Note that in contrast to the bilinear form $\langle -, - \rangle$, this quadratic form only depends on the underlying graph $\overline{Q}$ of Q, and not on the orientation of the edges.

Proposition. *Let Q be a finite connected quiver and q the corresponding quadratic form.*

(a) *If Q is a Dynkin quiver, then q is positive definite,*

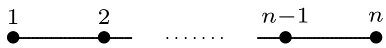
(b) *If Q is a Euclidean quiver, then q is positive semi-definite with radical of rank 1.semi*

(c) *If Q is neither a Dynkin quiver nor a Euclidean quiver, then q is indefinite.*

Proof. This is standard knowledge, say in Lie theory: the first assertions are used in order to classify the finite-dimensional semi-simple Lie algebras, see any such book. An elementary (and very nice) reference is the Bernstein-Gelfand-Ponomarev paper.

Here is an outline of the main steps: In the Dynkin case, one may consider the quadratic forms case by case. A good procedure seems to be to consider first the cases $\mathbb{A}_n$, and then trees with a unique branching vertex c such that c has precisely three neighbors (we may call such a graph a star with 3 arms).

Thus, let us start with the case $\mathbb{A}_n$, with $n \geq 1$:

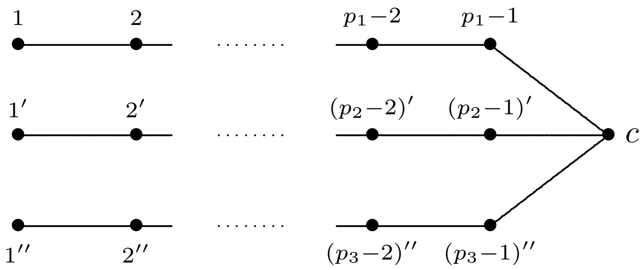


We may rewrite the quadratic form as follows:

$$q(d) = \quad \sum\nolimits_{i=1}^{n-1} \tfrac{i}{2(i+1)} \Big(\tfrac{i+1}{i} d_i - d_{i+1} \Big)^2 + \Big(1 - \tfrac{n-1}{2n} \Big) d_n^2.$$

Since $q(d)$ is written as a linear combination of squares with positive coefficients, it follows that q is positive semi-definite. But q is even positive definite, since the n linear forms $\frac{i+1}{i} d_i - d_{i+1}$ (with $1 \leq 1 \leq n-1$) and d_n are linearly independent.

Now we look at a star with three arms; such a graph may be obtained by starting with three graphs of type $\mathbb{A}_n$ where $n = p_1, p_2, p_3$ and identifying the vertices say on the right to get one vertex $c = p_1 = (p_2)' = (p_3)''$, here is a picture:



Using our knowledge about the graphs of type $\mathbb{A}_n$, we may rewrite the quadratic form for our star as

$$\begin{aligned} q(d) = \quad & \sum\nolimits_{i=1}^{p_1-1} \tfrac{i}{2(i+1)} \Big(\tfrac{i+1}{i} d_i - d_{i+1} \Big)^2 \\ & + \sum\nolimits_{i=1}^{p_2-1} \tfrac{i}{2(i+1)} \Big(\tfrac{i+1}{i} d_{i'} - d_{(i+1)'} \Big)^2 \\ & + \sum\nolimits_{i=1}^{p_3-1} \tfrac{i}{2(i+1)} \Big(\tfrac{i+1}{i} d_{i''} - d_{(i+1)''} \Big)^2 \\ & + \Big(1 - \frac{p_1-1}{2p_1} - \frac{p_2-1}{2p_2} - \frac{p_3-1}{2p_3} \Big) d_c^2. \end{aligned}$$

We see that we deal with a linear combination of squares, and the decisive coefficient is the coefficient

$$\lambda \;=\; 1 - \frac{p_1-1}{2p_1} - \frac{p_2-1}{2p_2} - \frac{p_3-1}{2p_3}$$

of d_c^2, which can be positive, zero or negative (depending on the numbers p_1, p_2, p_3), whereas all the other coefficients are of the form $\frac{i}{2(i+1)}$, thus positive. Now

$$\begin{aligned}\lambda &= 1 - \frac{p_1-1}{2p_1} - \frac{p_2-1}{2p_2} - \frac{p_3-1}{2p_3} \\ &= \frac{2p_1p_2p_3 - (p_1-1)p_2p_3 - (p_2-1)p_1p_3 - (p_3-1)p_1p_2}{2p_1p_2p_3} \\ &= \frac{-p_1p_2p_3 + p_2p_3 + p_1p_3 + p_1p_2}{2p_1p_2p_3} \\ &= \tfrac{1}{2}\Big(-1 + \tfrac{1}{p_1} + \tfrac{1}{p_2} + \tfrac{1}{p_3}\Big)\end{aligned}$$

We see that

$$\begin{aligned}\lambda > 0 &\quad\Longleftrightarrow\quad \tfrac{1}{p_1} + \tfrac{1}{p_2} + \tfrac{1}{p_3} > 1 \\ \lambda = 0 &\quad\Longleftrightarrow\quad \tfrac{1}{p_1} + \tfrac{1}{p_2} + \tfrac{1}{p_3} = 1 \\ \lambda < 0 &\quad\Longleftrightarrow\quad \tfrac{1}{p_1} + \tfrac{1}{p_2} + \tfrac{1}{p_3} < 1\end{aligned}$$

The following is easy to see:

Lemma. *The triples $p_1 \le p_2 \le p_3$ with $\sum_i \frac{1}{p_i} > 1$ are the following:*

$$(1, p_2, p_3), \quad (2, 2, p_3), \quad (2, 3, 3), \quad (2, 3, 4), \quad (2, 3, 5).$$

(the corresponding graphs are the Dynkin diagrams $\mathbb{A}_{p_2+p_3-1}, \mathbb{D}_{p_3+2}, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$).

There are precisely three triples $p_1 \le p_2 \le p_3$ with $\sum_i \frac{1}{p_i} = 1$, namely the triples

$$(3, 3, 3), \quad (2, 4, 4), \quad (2, 3, 6).$$

(the corresponding graphs are the Euclidean diagrams $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$).

> The reader may wonder whether the convention which we use here (and which seems to be widely accepted) is reasonable: For example, looking at the graph $\mathbb{E}_7$, we say that it has an $\mathbb{A}_2$-arm, an $\mathbb{A}_3$-arm and an $\mathbb{A}_4$-arm, thus we draw the attention to the triple of numbers $(2, 3, 4)$ and not to $(1, 2, 3)$ which would correspond to the optical impression of having arms of length 1, 2, and 3. The formulae presented above, as well as many other ones which express properties of stars with 3 arms seem to be sufficient justification.
>
> The importance of these triples of numbers was stressed already by Felix Klein in his book *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade* (1884).

Let us turn the attention to the radical of the quadratic form q. By definition, the *radical* of a quadratic form q is the set (indeed subgroup) of all elements r with $q(d+r) =$

$q(d)$ for all vectors d. In particular, any vector r in the radical of q satisfies $q(r) = 0$ (but we stress that the converse is not true). Now in our case

$$q(d + r) = q(d) + q(r) + \langle d, r\rangle + \langle r, d\rangle.$$

It follows that r belongs to the radical if and only if

$$\langle e(x), r\rangle + \langle r, e(x)\rangle = 0,$$

for all vertices $x \in Q_0$ (here, $e(x)$ denotes the canonical basis vector in $\mathbb{Z}Q_0$, with coefficients $(e(x))_x = 1$ and $(e(x))_y = 0$ for $y \neq x$). But clearly:

$$\langle e(x), r\rangle + \langle r, e(x)\rangle = 2r_x - \sum_{t(\alpha)=x} r_{h(\alpha)} - \sum_{h(\alpha)=x} r_{t(\alpha)}.$$

Thus we see that r *belongs to the radical of* q *provided* $2r_x$ *is equal to the sum of the neighboring values* r_y. For example, if $Q = \widetilde{E}_8$, then there is the following radical vector:

$$\begin{array}{cccccccc} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

and it generates the radical (as a subgroup of $\mathbb{Z}Q_0$). In our list of the Euclidean diagrams we have added on the right side a vector r which is positive and turns out to generate the radical. Note that in all cases we see: if we delete a vertex x with $r_x = 1$ (one such vertex is encircled, but usually there are several such vertices), then the graph which we obtain is the corresponding Dynkin diagram and the restriction of r is precisely the maximal root for the Dynkin diagram. This shows quite clearly, that for the study of the Euclidean diagrams, the maximal root of the corresponding Dynkin diagram plays a decisive role.

6.4. Dynkin quivers.

Proposition. *Let Q be a Dynkin quiver and M a representation of Q. If* $\operatorname{End}(M)$ *is a division ring, then* $\operatorname{Ext}^1(M, M) = 0$.

Proof: Assume that $\operatorname{End}(M)$ is a division ring and that $\operatorname{Ext}^1(M, M) \neq 0$. Then, according to section 6.2, we have $q(\mathbf{dim}\, M) \leq 0$. However, q is positive definite, by 6.3.

Proposition. *Let Q be a Dynkin quiver and M an indecomposable representation of Q. Then* $\operatorname{End}(M)$ *is a division ring.*

Proof: See for example the Lectures by Crawley-Boevey, section 2.

Corollary. *If k is an algebraically closed field and M is an indecomposable representation of a Dynkin quiver, then* $q(\mathbf{dim}\, M) = 1$.

Note that for a positive definite quadratic form on a finitely generated free abelian group, there are only finitely many vectors d with $q(d) = 1$. Thus it follows that *a Dynkin* *quiver is* of bounded representation type (this means that the indecomposable representations are of bounded length), and therefore *representation-finite* (according to Rojter who proved the first Brauer-Thrall conjecture).

We have not yet shown that actually **dim** provides (for a Dynkin quiver) a bijection between the isomorphism classes of the indecomposable representations and the positive vectors d with $q(d) = 1$ (for the Dynkin graphs, the vectors d with $q(d) = 1$ are precisely the positive roots of the corresponding Lie algebra). It still remains to show: For any positive root r, there is an indecomposable, and there is up to isomorphism only one.

Definition: We say that a representation M of Q is *in general position* provided $\dim_k \operatorname{End}(M) \leq \dim_k \operatorname{End}(M')$ for all representations M' with $\mathbf{dim}\, M = \mathbf{dim}\, M'$.

Lemma. *Assume that M is a representation in general position and let $M = M' \oplus M''$ be a direct decomposition. Then* $\operatorname{Ext}^1(M', M'') = 0$.

Proof: This is a direct consequence of Proposition 5.1. Namely, if $0 \to M' \to Y \to M'' \to 0$ is a non-split exact sequence, then

$$\dim_k \operatorname{End}(Y) < \dim_k \operatorname{End}(M' \oplus M'') = \dim_k \operatorname{End}(M),$$

but of course $\mathbf{dim}\, Y = \mathbf{dim}\, M$.

Thus, if M is in general position and $M = \bigoplus M_i$ with indecomposable representations M_i, then $\operatorname{Ext}^1(M_i, M_j) = 0$ for all $i \neq j$.

Corollary. *Let Q be a Dynkin quiver. Let $r \in \mathbb{Z}Q_0$ with $q(r) = 1$. Then any representation M of Q with* $\mathbf{dim}\, M = r$ *which is in general position is indecomposable and has endomorphism ring k.*

Proof: Let M be a representation of Q with $\mathbf{dim}\, M = r$ which is in general position and write it as $M = \bigoplus M_i$ with indecomposable representations M_i. As we just have seen, $\operatorname{Ext}^1(M_i, M_j) = 0$ for all $i \neq j$. But we know that we also have $\operatorname{Ext}^1(M_i, M_i) = 0$ for all i, thus $\operatorname{Ext}^1(M, M) = 0$. But then

$$1 = q(\mathbf{dim}\, M) = \dim_k \operatorname{End}(M)$$

shows that $\operatorname{End}(M) = k$, thus M is indecomposable.

In particular, there exists an indecomposable representation M with $\mathbf{dim}\, M$ the maximal root!

6.5. More about the Dynkin quivers.

As is the last section, let us consider again a Dynkin quiver Q. We have seen that in case k is algebraically closed, then $\operatorname{End}(M) = k$ for any indecomposable representation. This is true for k an arbitrary field, but this needs some further considerations. There are several possible proofs available, anyone provides further information.

(a) Knitting the **Auslander-Reiten quiver** (case by case). If one could show from the beginning that we deal with a quiver of finite representation type, then it would be sufficient to know that we deal with a preprojective component (because for M in a preprojective component, $\mathrm{End}(M) \simeq \mathrm{End}(P)$ for some indecomposable projective module, and if P is an indecomposable projective module and Q has no cyclic paths, then $\mathrm{End}(P) = k$).

(b) Use of the **Coxeter transformation.** Here, one uses only knowledge which concerns the quadratic form. This method also shows directly that all the indecomposable modules are determined by the dimension vector.

(c) Use of **reflection functors.** Here one relates the representations of quivers with the same underlying graph but may-be different orientation to each other.

(d) **Schofield induction.** We know that all indecomposables are exceptional, thus are obtained by Schofield induction from the simple kQ-modules $S(x)$. Inductively, we see that $\mathrm{End}(M) = k$ for all exceptional modules.

6.6. Euclidean quivers.

As for the Dynkin quivers, also for the Euclidean quivers the full classification of all the indecomposable representations is known and is quite easy to overlook.

The special case of the Kronecker quiver

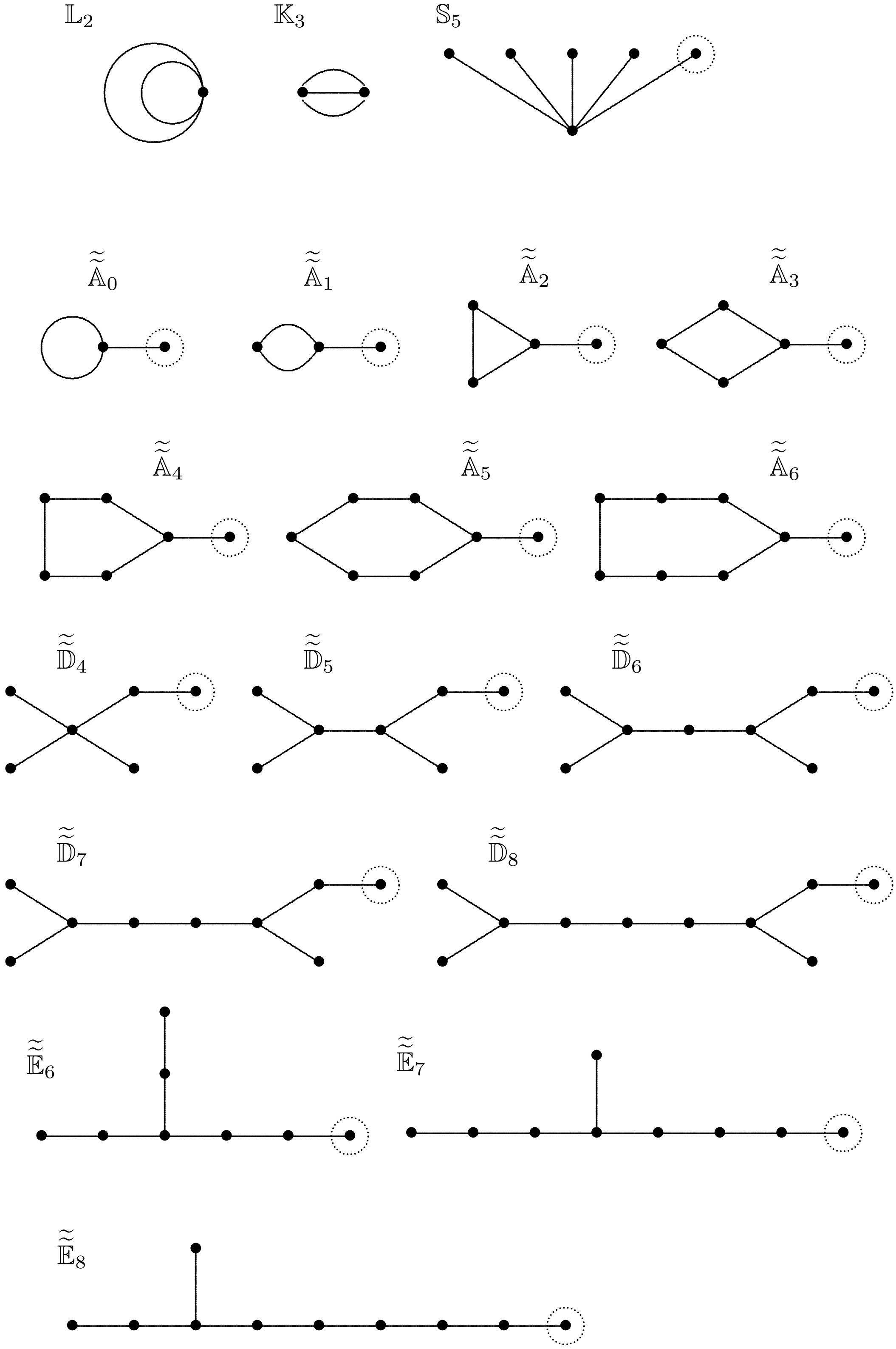
$$\circ \rightrightarrows \circ$$

was investigated already by Weierstrass and then solved by Kronecker in 1890.

The next case which was studied was the 4-subspace quiver $\widetilde{\mathbb{D}}_4$, see Gelfand-Ponomarev, 1970. The solution for arbitrary Euclidean quivers is due to Donovan-Freislich and Nazarova (1973).

6.7. Wild quivers.

Let us deal with the following list of graphs.



Note that any of these graphs has at most 10 vertices.

Proposition. *These are the minimal graphs with indefinite quadratic form q. Always, there exists a vector d with positive integer coefficients such that $q(d) = -1$.*

Proof. First, let us show the existence of the vector d. For the graph $\mathbb{L}_2$, we take $d = (1)$, of course $q(1) = 1^2 - 2 = -1$. Similarly, for $\mathbb{K}_3$, take $d = (1, 1)$, we have $q(d) = 1^2 + 1^2 - 3 = -1$. The remaining graphs are obtained from a Euclidean graph E by adding a vertex ω (see the encircled vertex) and an edge connecting ω to say x. For E there exists a vector d' with positive integer coefficients such that $d'_x = 2$. Namely, in the case $\mathbb{S}_5$ take for d' the positive radical generator, whereas in all the other cases take for d' twice the positive radical generator. Let d be defined by $d_y = d'_y$ for the vertices y of E and $d_\omega = 1$. Then $q(d) = q(d') + d_\omega^2 - d_x d_\omega = 0 + 1 - 2 = -1$. Thus, always we have found d such that $q(d) = -1$, in particular we see that q is indefinite.

It remains to show that any graph $\overline{Q}$ with indefinite quadratic form contains a subgraph in the list. Of course, as a minimal graph with indefinite quadratic form, $\overline{Q}$ has to be connected.

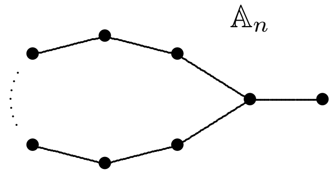
If there is a vertex with at least two loops, then $\mathbb{L}_2$ is a subgraph. Thus, we can assume that there is no vertex with more than one loop.

If there is a vertex x with one loop, then there have to be additional vertices, thus there is vertex ω connected to x, thus $\tilde{\tilde{\mathbb{A}}}_0$ is a subgraph. Now we can assume that there are no loops.

If there are multiple edges, then $\mathbb{K}_3$ or $\tilde{\tilde{\mathbb{A}}}_0$ has to be a subgraph. Thus we can assume that there are no multiple edges.

If there is a cycle, then there is an elementary cycle, as well as a vertex x on this cycle with a neighbor say ω outside the cycle, let us denote the corresponding subgraph by $\tilde{\tilde{\mathbb{A}}}_n$ provided the cycle consists of $n + 1$ vertices (here, ≥ 2):

$\tilde{\tilde{\mathbb{A}}}_n$

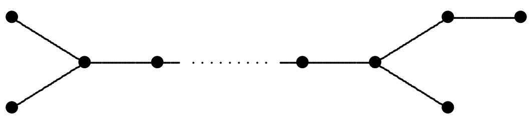


For $n \geq 7$, the graph $\tilde{\tilde{\mathbb{A}}}_n$ contains $\tilde{\tilde{\mathbb{E}}}_7$ as a subgraph, the remaining graphs $\tilde{\tilde{\mathbb{A}}}_n$ with $2 \leq n \leq 6$ occur in the list. Thus we can assume that $\overline{Q}$ is a tree.

If there is a vertex with at least 4 neighbors, then $\mathbb{S}_5$ or $\tilde{\tilde{\mathbb{D}}}_4$ is a subgraph. Thus we can assume that any vertex has at most three neighbors.

If there are two vertices both having three neighbors, there has to be a subgraph of the form $\tilde{\tilde{\mathbb{D}}}_n$ (with $n + 2$ vertices):

$\tilde{\tilde{\mathbb{D}}}_n$



If $n \geq 9$, then $\tilde{\tilde{\mathbb{D}}}_n$ contains $\tilde{\tilde{\mathbb{E}}}_7$ as a subgraph. The remaining graphs $\tilde{\tilde{\mathbb{D}}}_n$ with $5 \leq n \leq 8$ are in the list.

It remains to deal with the stars with 3 arms, say with arms $\mathbb{A}_{p_1}, \mathbb{A}_{p_2}, \mathbb{A}_{p_3}$ as considered in section 6.2, where $2 \leq p_1 \leq p_2 \leq p_3$. If $p_1 \geq 4$, or if $p_1 = 3$ and $p_2 = 3$, then $\tilde{\tilde{\mathbb{E}}}_6$ has to be a subgraph. Thus $p_1 = 2$. If $p_2 \geq 4$, then $\tilde{\tilde{\mathbb{E}}}_7$ has to be a subgraph. On the other hand, if $p_1 = 2$ and $p_2 = 2$, then we deal with a Dynkin diagram of type $\mathbb{D}$, impossible. The cases $p_1 = 2, p_2 = 3$ remain: since the quadratic form is indefinite, we must have $p_3 \geq 7$, thus $\tilde{\tilde{\mathbb{E}}}_8$ is a subgraph. This completes the proof.

A finite-dimensional k-algebra Λ is called *strictly wild,* provided there is a full exact embedding of the category of finite-dimensional representations of the quiver $\mathbb{L}_2$ into the category $\operatorname{mod}\Lambda$. A quiver is said to be strictly wild provided its path algebra is strictly wild.

Theorem. *If Q is a connected quiver which is neither a Dynkin quiver nor a Euclidean quiver, then Q is strictly wild.*

Sketch of proof. We use the **process of simplification** as outlined in Ringel, *Representations of K-species and bimodules,* J.Algebra 1976. This amounts to the following inductive procedure: given any quiver Q in the list, we have to find a finite set $\mathcal{N} = \{N_1, \dots, N_t\}$ of representations of Q such that $\operatorname{End}(N_i) = k$, $\operatorname{Hom}(N_i, N_j) = 0$ for all $i \neq j$ in $\{1, \dots, t\}$ (such a set may be called a *set of orthogonal bricks*) such that the Ext-quiver $\Delta(\mathcal{N})$ of $\mathcal{N}$ is already known to be strictly wild (by definition, the Ext-*quiver* $\Delta(\mathcal{N})$ has t vertices labeled $[N_1], \dots, [N_t]$ and the number of arrows $[N_i] \to [N_j]$ is given by $-\dim_k \operatorname{Ext}^1(N_i, N_j)$). We may use the Euler form in order to determine this number: Since $\operatorname{End}(N_i) = k$, we have

$$\begin{aligned} \dim_k \operatorname{Ext}^1(N_i, N_i) &= -q(\mathbf{dim}\, N_i) + 1, \\ \dim_k \operatorname{Ext}^1(N_i, N_j) &= -\langle \mathbf{dim}\, N_i, \mathbf{dim}\, N_j \rangle \qquad \text{for} \quad i \neq j. \end{aligned}$$

We distinguish five cases.

(1) $\mathbb{K}_3$. Let Q be a quiver of type $\mathbb{K}_3$, let N be any two-dimensional indecomposable representation of Q and $\mathcal{N} = \{N\}$. Since $q(\mathbf{dim}\, N) = -1$, we see that $\Delta(\mathcal{N}) = \mathbb{L}_2$.

(2) $\tilde{\tilde{\mathbb{A}}}_0$, say with subspace arm, thus we consider the quiver

$$Q \qquad \begin{tikzcd} \circ \arrow[loop left] & \circ \arrow[l] \end{tikzcd}$$

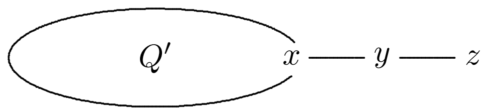
Let N be the following representation of Q

$$\begin{tikzcd} k^2 \arrow[loop left, "\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}"] & & k \arrow[ll, "\begin{bmatrix} 0 \\ 1 \end{bmatrix}"'] \end{tikzcd}$$

One easily checks that $\operatorname{End}(N) = k$. Also here, let $\mathcal{N} = \{N\}$. Then $\dim_k \operatorname{Ext}^1(N, N) = 2$, thus again $\Delta(\mathcal{N}) = \mathbb{L}_2$. In case the arm has factor space orientation, we proceed similarly.

(3) $\widetilde{\widetilde{\mathbb{A}}}_n$ with $n \geq 1$. Let Q be such a quiver, thus Q is obtained form a quiver Q' of type $\widetilde{A}_n$ by adding an $\mathbb{A}_2$-arm. Let ω be the vertex outside Q'. Let N be any indecomposable thin representation of Q' with $N_y = k$ for all vertices y of Q', and $\mathcal{N} = \{N, S(\omega)\}$. Then $\Delta(\mathcal{N})$ is a quiver of type $\widetilde{\widetilde{\mathbb{A}}}_0$.

(4) The cases $\widetilde{\widetilde{\mathbb{D}}}_n$ and $\widetilde{\widetilde{\mathbb{E}}}_m$ with $4 \leq n \leq 8$ and $6 \leq m \leq 8$. These quivers are obtained from a Dynkin quiver Q' by adding an $\mathbb{A}_3$-arm in a vertex x of Q', say

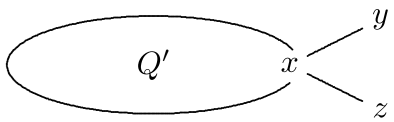


Let N be the maximal indecomposable representation of Q' and note that in all cases $\dim_k N_x = 2$. It follows that $\Delta(\mathcal{N})$ is of the following form

$$\begin{tikzcd}
{[N]} \arrow[r, dash, bend left] \arrow[r, dash, bend right] & {[S(y)]} \arrow[r, dash] & {[S(z)]}
\end{tikzcd}$$

thus of type $\widetilde{\widetilde{\mathbb{A}}}_1$.

(5) $\mathbb{S}_5$. Here we deal with a quiver obtained from a Dynkin quiver Q' by adding two $\mathbb{A}_2$-arms in a vertex $x \in Q'_0$.



In our case $\mathbb{S}_5$, the subquiver Q' is of type $\mathbb{D}_4$. Again, we consider the maximal indecomposable representation N of Q' and note that in our case $\dim_k N_x = 2$. It follows that $\Delta(\mathcal{N})$ is of the following form

$$\begin{tikzcd}
 & {[S(y)]} \\
{[N]} \arrow[ur, dash, bend left] \arrow[ur, dash, bend right] \arrow[dr, dash, bend left] \arrow[dr, dash, bend right] & \\
 & {[S(z)]}
\end{tikzcd}$$

thus it contains a subquiver of type $\widetilde{\widetilde{\mathbb{A}}}_1$.

Appendix

4.7. Review of some known results from the theory of rings and modules.

We want to recall two basic results which concern modules of finite length over any ring R. The modules to be considered are R-modules.

Let M be a module. A *composition series* of M is a chain

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t = M$$

of submodules (a "filtration" of M) such that all the factors M_i/M_{i-1} with $1 \leq i \leq t$ are simple. The number t is called the *length* of the composition series.

Jordan-Hölder Theorem. *Assume that two composition series of M are given:*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_s = M,$$
$$0 = M'_0 \subseteq M'_1 \subseteq M'_2 \subseteq \cdots \subseteq M'_t = M.$$

Then $s = t$ and there is a permutation π of the set $\{1, 2, \ldots, s\}$ such that the modules M_i/M_{i-1} and $M'_{\pi(i)}/M'_{\pi(i)-1}$ are isomorphic, for $1 \leq i \leq s$.

In addition, for any filtration

$$0 = M''_0 \subseteq M''_1 \subseteq M''_2 \subseteq \cdots \subseteq M''_r = M.$$

with proper inclusions $M''_{i-1} \subset M''_i$ for all $1 \leq i \leq r$, we have $r \leq t$.

On the basis of this result, one introduces the following definitions: If M has a composition of length t, then one calls t the *length* of M and one calls the factors of a composition series the *composition factors* of M. If there is given a composition series

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_s = M$$

of M, and S a simple module, then one calls the number of factors M_i/M_{i-1} which are isomorphic to S the *Jordan-Hölder multiplicity of S in M.*

Theorem of Krull-Remak-Schmidt. *Let M be a module of finite length, and assume that there are given two direct decompositions:*

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_s, \qquad \text{and} \qquad M = M'_1 \oplus M'_2 \oplus \cdots \oplus M'_t,$$

such that all the modules M_i, M'_j with $1 \leq i \leq s, 1 \leq j \leq t$ are indecomposable. Then $s = t$ and there is a permutation π of the set $\{1, 2, \ldots, s\}$ such that the modules M_i and $M'_{\pi(i)}$ for $1 \leq i \leq s$ are isomorphic.

The proof is based on the following Lemma which is of independent interest:

Fitting Lemma. *Let M be an indecomposable module of finite length. Then the endomorphism ring of M is a local ring with nilpotent radical.*