Abelian Varieties

Amnon Neeman

In representation theory one is interested in Calabi–Yau triangulated categories. These few lectures were an attempt to survey the classical analogue in algebraic geometry and complex analysis. In this abstract I treat the case of abelian varieties. Much more detail on everything I say may be found in [2, 3].

For the sake of definiteness, we begin with the definitions.

Definition 0.1. *A* Calabi–Yau manifold *is a connected, compact, complex manifold with trivial sheaf of top differential forms.*

In other words a connected, compact, complex manifold X of dimension g will be Calabi–Yau if the sheaf Ω^g_X has a nowhere vanishing holomorphic section. We recall

Theorem 0.2. (Serre Duality). *Let X be a connected, compact, complex manifold of dimension g. If $\mathcal{D}$ is the bounded derived category of chain complexes of coherent analytic sheaves on X, then there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{D}}(A,B)^* \ \simeq\ \mathrm{Hom}_{\mathcal{D}}\big(B, A\otimes\Omega^g_X[g]\big).$$

In the language of [1] the category $\mathcal{D}$ has a Serre functor S, given by the formula

$$S(-)\ =\ (-)\otimes\Omega^g_X[g].$$

$\mathcal{D}$ is a Calabi–Yau triangulated category if and only if $\Omega^g_X \simeq \mathcal{O}_X$, that is if and only if the manifold X is Calabi–Yau. The dimension of the Calabi–Yau triangulated category $\mathcal{D}$ agreed with the complex dimension of the manifold X. One very classical case of this is complex tori. We recall the definition

Definition 0.3. *A* complex torus *is a connected, compact, complex Lie group.*

We note that every complex torus is automatically Calabi–Yau. The point is that the line bundle Ω^g_X has a unique trivialisation by a left invariant g–form. Take any non–vanishing g–form at the identity, and extend it (uniquely) to a left invariant g–form on all of X.

Let us say a little more about connected, compact, complex Lie groups. We observe

Theorem 0.4. *Any connected, compact, complex Lie group is commutative.*

Proof. The result is well–known but we include a proof. Let X be a connected, compact, complex Lie group. Consider the map

$$f : X\times X \longrightarrow X$$

given by

$$f(x,y) = xyx^{-1}y^{-1}.$$

If $e\in X$ is the identity, then $f(e,y)=e$ for all $y\in X$. Now let $V\subset X$ be a small ball around e. Then $f^{-1}V$ must contain an open set of the form $U\times X$, with U

an open neighbourhood of the identity $e \in X$. For any $u \in U$, the map f induces a holomorphic map from the compact manifold $\{u\} \times X$ to the ball V, and any such map is constant. But then

$$f(u, y) = f(u, e) = e;$$

that is f sends all of $U \times X$ to the singleton e. Now analytic continuation tells us that f collapses all of $X \times X$ to e. $\square$

It follows that the Lie algebra of X is commutative; it is just the trivial Lie algebra $\mathbb{C}^g$. Furthermore, the exponential map $\mathbb{C}^g \longrightarrow X$ is a group homomorphism, which is locally a diffeomorphism. The image is an open subgroup of the connected group X, and hence the exponential map is surjective. This means that X is isomorphic to a quotient group $\mathbb{C}^g/\Lambda$, where Λ is a discrete closed subgroup of $\mathbb{C}^g$. Since X is compact, Λ must be a lattice. That is the natural map

$$\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \longrightarrow \mathbb{C}^g$$

is an isomorphism. We summarise:

Theorem 0.5. *Any connected, compact, complex Lie group is $\mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice.*

Remark 0.6. Theorem 0.5 justifies the terminology of Definition 0.3. By Theorem 0.5 a connected, compact, complex Lie group is $\mathbb{C}^g/\Lambda$, which is nothing other than a $2g$–dimensional real torus with a complex structure. Hence we call these complex tori.

Now we come to the question of how many different complex tori are there. The answer is clear. Two complex tori $\mathbb{C}^g/\Lambda$ and $\mathbb{C}^g/\Lambda'$ will agree if there is a linear transformation in $GL(g, \mathbb{C})$ taking Λ to Λ'. If we choose a basis for Λ we can always, up to a linear transformation in $GL(g, \mathbb{C})$, assume that g elements of this basis are the standard basis vectors for $\mathbb{C}^g$. Our freedom in varying Λ amounts to the freedom in selecting the other g basis vectors. The space of choices is an open subset of $\{\mathbb{C}^g\}^g = \mathbb{C}^{g^2}$. There are g^2 "degrees of freedom" in choosing a g–dimensional complex torus.

Definition 0.7. *A complex torus is called an* abelian variety *if it can be given the structure of an algebraic variety. Equivalently, this means it can be embedded as a complex analytic submanifold of projective space.*

How many complex tori are abelian varieties? One classical way to answer the problem is using Theta functions. We briefly explain.

If X admits an embedding into projective space then it must have a line bundle on it, with plenty of sections. Pulling back the line bundle by the exponential map $\mathbb{C}^g \longrightarrow X$ we get a holomorphic line bundle on $\mathbb{C}^g$, but all such bundles are trivial. The sections of the line bundle on X pull back to sections of the trivial bundle (that is, functions) on $\mathbb{C}^g$, with certain periodicity properties. These functions have been studied classically as Theta functions.

Without giving much detail, Theta functions are constructed as infinite sums. If $z \in \mathbb{C}^g$ and Ω is a symmetric $g \times g$ matrix over $\mathbb{C}$ with a positive definite imaginary part, we can form the sum

$$\Theta(\Omega, z) \;=\; \sum_{n \in \mathbb{Z}^g} \exp \pi i \big({}^t n \Omega n + 2\, {}^t n z \big)$$

If we fix Ω and view this as a function in z we get one of our sections of holomorphic line bundles on $\mathbb{C}^g$. The point we want to make is that, as we vary the parameter Ω over the symmetric $g \times g$ matrices, the dimension of the parameter space is only $g(g+1)/2$. There is only a $g(g+1)/2$–dimensional space of g–dimensional abelian varieties. Therefore most complex tori do not admit the structure of algebraic varieties.

The physics literature is divided on whether abelian varieties should be admitted as Calabi–Yau manifolds. From the point of representation of quivers, some of the most interesting examples come from elliptic curves, which are 1–dimensional abelian varieties. Undoubtedly the quiver theoretic statements one can make about the categories of sheaves over elliptic curves (equivariant with respect to the action of suitable automorphisms) all generalise to higher dimensional abelian varieties.

An elliptic curve admits an involution, which is nothing other than the map taking $x \in X$ to $-x \in X$. Much has been made of the quiver representations giving the category of equivariant sheaves on X. There is no reason why this should not generalise to higher dimension.

If $\sigma : X \longrightarrow X$ is the involution taking $x \in X$ to $-x \in X$, one can study the variety X/σ. If X is a curve then X/σ is nothing other than $\mathbb{P}^1$, in particular X/σ is smooth. In higher dimensions X/σ is singular. But the singularities of X/σ are not too bad and are well understood. For example if X is a surface (that is, 2–dimensional) then X/σ has exactly 16 singular points. A minimal resolution of these 16 points gives an Enriques surface. It is not quite Calabi–Yau, but almost. The sheaf Ω^g_X is not trivial, but $\{\Omega^g_X\}^2 = \Omega^g_X \otimes \Omega^g_X$ is. That is, there is an isomorphism $\{\Omega^g_X\}^2 \simeq \mathcal{O}_X$. In other words the Serre functor

$$S(-) \;=\; (-) \otimes \Omega^g_X[g].$$

is not a shift, but

$$S^2(-) \;=\; (-) \otimes \{\Omega^g_X\}^2[2g]$$

is a shift.

References

[1] Alexei I. Bondal and Mikhael M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337.

[2] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay, 1970.

[3] ———, *Tata lectures on theta. I*, Progress in Mathematics, vol. 28, Birkhäuser Boston Inc., Boston, MA, 1983, With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman.