Modules with injective cohomology

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(joint work with John Greenlees)

Let G be a finite group, and let k be an algebraically closed field of characteristic p. Then the cohomology ring $H^*(G,k) = \mathrm{Ext}^*_{kG}(k,k)$ is a Noetherian graded commutative k-algebra, so we can form the maximal ideal spectrum $V_G = \max\operatorname{spec} H^*(G,k)$. This is a closed homogeneous affine variety, and was studied extensively by Quillen [6, 7]. If M is a finitely generated kG-module then there is a ring homomorphism

$$H^*(G,k) \xrightarrow{M\otimes_k -} \mathrm{Ext}^*_{kG}(M,M),$$

and the *support variety* $V_G(M)$ is defined to be the subvariety of V_G determined by the kernel of this homomorphism. Support varieties have been investigated extensively by Carlson and others.

If $\mathfrak{p}$ is a homogeneous prime ideal in $H^*(G,k)$ corresponding to a closed homogeneous irreducible subvariety V of V_G, then there is a *kappa module* $\kappa_\mathfrak{p} = \kappa_V$, introduced by Benson, Carlson and Rickard [1], with the following properties:

(i) $V \subseteq V_G(M) \iff \kappa_V \otimes_k M$ is not projective,
(ii) κ_V is idempotent, in the sense that $\kappa_\mathcal{V} \otimes_k \kappa_\mathcal{V} \cong \kappa_V \oplus$ (projective), and
(iii) $\kappa_\mathcal{V}$ is usually not finite dimensional.

The modules κ_V were used by Benson, Carlson and Rickard in [1] to develop a theory of varieties for infinitely generated kG-modules. Instead of associating a single variety to M, we associate a collection of subvarieties of V_G:

$$\mathcal{V}_G(M) = \{V \subseteq V_G \mid \kappa_V \otimes_k M \text{ is not projective}\}.$$

For example, $\mathcal{V}_G(\kappa_V) = \{V\}$. One of the most important properties of this variety theory is the tensor product formula

$$\mathcal{V}_G(M \otimes_k N) = \mathcal{V}_G(M) \cap \mathcal{V}_G(N).$$

This, together with the statement that $\mathcal{V}_G(M) = \varnothing$ if and only if M is projective, are what make the variety theory useful.

The purpose of the joint work with Greenlees was to determine the cohomology of these modules κ_V. It turns out that it is more sensible to ask about Tate cohomology. The answer, together with some consequences, is given by the following theorem.

Theorem (Benson and Greenlees [2])**.** (i) *The Tate cohomology of the kappa modules is given by*

$$\hat{H}^*(G,\kappa_V) \cong I_\mathfrak{p}[d].$$

Here, $I_\mathfrak{p}$ denotes the injective hull of $H^*(G,k)/\mathfrak{p}$ in the category of graded modules over $H^*(G,k)$, and d is the dimension of the variety V (i.e., the Krull dimension of $H^*(G,k)/\mathfrak{p}$).

(ii) *The kappa modules are the representing objects for the Matlis dual of Tate cohomology:*

$$\underline{\mathrm{Hom}}_{kG}(M,\kappa_V)\cong \mathrm{Hom}_{H^*(G,k)}(\hat{H}^*(G,M),I_{\mathfrak{p}}[d]);$$

these representing objects were investigated in [3].

(iii) *The modules κ_V are pure injective—there are no phantom maps into them.*

(iv) $\widehat{\mathrm{Ext}}^*_{kG}(\kappa_V,\kappa_V)\cong H^*(G,k)^{\wedge}_{\mathfrak{p}}=\varprojlim_n H^*(G,k)/\mathfrak{p}^n.$

The extraordinary thing about the theorem is that its proof involves translating to the context of modules over E_∞ ring spectra and solving the problem there. The context is as follows. Let BG be the classifying space of G, so that $\Omega BG\simeq G$. The Rothenberg–Steenrod construction gives for any space X a quasiisomorphism between the differential graded algebras $\mathbb{R}\,\mathrm{End}_{C_*(\Omega X)}(k)$ and $C^*(X;k)$. In particular, for a finite group G this gives $\mathbb{R}\,\mathrm{End}_{kG}(k)\simeq C^*(BG;k)$. Writing $\mathcal{R}$ for $\mathbb{R}\,\mathrm{End}_{kG}(k)$ and $\mathcal{C}$ for $C^*(BG;k)$, the following diagram of categories and functors explains the route we took:

$$\begin{tikzcd}
\mathsf{Mod}(kG) \arrow[r] \arrow[dr] & D(kG) \arrow[d] \arrow[r, shift right, "\mathbb{R}\,\mathrm{Hom}_{kG}(k,-)"'] & D(\mathcal{R}^{\mathrm{op}}) \arrow[l, shift right, "-\otimes^{\mathbb{L}}_{\mathcal{R}}k"'] \arrow[r, "\simeq"] & D(\mathcal{C}) \\
& \mathsf{StMod}(kG) & &
\end{tikzcd}$$

Here, $D(kG)$ stands for the derived category of *all* chain complexes of kG-modules. Similarly, $D(\mathcal{R}^{\mathrm{op}})$ is the derived category obtained from the homotopy category of differential graded *right* $\mathcal{R}$-modules by inverting quasi-isomorphisms. Since $\mathcal{R}\simeq\mathcal{R}^{\mathrm{op}}$, this is equivalent to the derived category formed from the differential graded *left* $\mathcal{R}$-modules. We regard $\mathcal{C}$ (or rather, the Eilenberg–Mac Lane spectrum of $\mathcal{C}$) as an E_∞ ring spectrum; here, E_∞ means "commutative and associative up to all higher homotopies." This allows us, for example, to take two objects A and B in $D(\mathcal{C})$ and regard $A\otimes^{\mathbb{L}}_{\mathcal{C}}B$ as another object in $D(\mathcal{C})$, just as we can regard the tensor product of two modules over a commutative ring as another module over the same ring. For this purpose, it is essential to be working in a category of spectra in which the smash product is commutative and associative up to coherent natural isomorphism, and not just up to all higher homotopies; there are nowadays a number a candidates for such a category, and we chose to work in the framework of Elmendorf, Kříž, Mandell and May [5].

Another construction requiring the E_∞ structure is localization at a prime ideal in the homotopy. Since $\pi_*\mathcal{C}=H^{-*}(G,k)$, we can form the localization $\mathcal{C}_{\mathfrak{p}}$, and then use tensor products to apply a stable Koszul type construction with respect to a homogeneous system of parameters in $\mathfrak{p}$. This construction gives the image in $D(\mathcal{C})$ of a suitable lift to $D(kG)$ of the kappa module $\kappa_{\mathfrak{p}}$ in $\mathsf{StMod}(kG)$. This construction can therefore be regarded as a sort of local cohomology object in $D(\mathcal{C})$ for the prime $\mathfrak{p}$. The statement that its cohomology is injective is a sort of Gorenstein duality for $\mathcal{C}_{\mathfrak{p}}$.

The statement that $\mathcal{C}$ is Gorenstein in the appropriate sense appeared in the work of Dwyer, Greenlees and Iyengar [4]. The usual proof that localization at a prime ideal of a Gorenstein ring gives a Gorenstein ring no longer works in this context, because it relies on the characterization of Gorenstein via finite injective dimension, which doesn't make much sense in this context. So proving that $\mathcal{C}_{\mathfrak{p}}$ is Gorenstein went via a different route. We applied Grothendieck duality with respect to a normalization coming from an embedding of G in $SU(n)$, and proved the corresponding dual statement.

To summarize, the proof involves translating the original problem from modular representation theory into the language of modules over an E_∞ ring spectrum from algebraic topology, and then using methods from commutative algebra to solve the problem there. The level of machinery involved is formidable, but the hope is that other problems in modular representation theory will succumb to a similar route.

References

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