Block representation type for groups and Lie algebras

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Group Algebras. Let k be an algebraically closed field of characteristic $p > 0$. Throughout, all algebras and modules are assumed to be finite dimensional. An associative k-algebra Λ decomposes into a direct sum $\Lambda = \mathcal{B}_1 \oplus \mathcal{B}_2 \oplus \cdots \oplus \mathcal{B}_s$ of two-sided ideals, that are indecomposable associative k-algebras. The relevance of this *block decomposition* for representation theory was first observed by Brauer and Nesbitt in their study of non-semisimple group algebras of finite groups.

Because of these historical origins, results on group algebras have often served as a paradigm for other classes of algebras, such as reduced enveloping algebras of restricted Lie algebras or distribution algebras of infinitesimal group schemes. In my talk, I will compare the representation theories of finite groups and restricted Lie algebras, focusing on the notion of representation type. In retrospect, most phenomena characteristic of infinitesimal group schemes already occur at the level of restricted Lie algebras [2, 3, 4].

Let me begin by collecting some of the methods and results from the modular representation theory of finite groups. We fix a finite group G, and recall that the unique block $\mathcal{B}_0(G) \subset k[G]$ containing the trivial $k[G]$-module k is the *principal block.*

Mackey Decomposition. If $H \subset G$ is a subgroup and M is an H-module, then

$$k[G] \otimes_{k[H]} M|_H \cong \bigoplus_{HgH} k[H] \otimes_{k[H \cap gHg^{-1}]} M^g.$$

In particular, M is always a direct summand of the restriction of the induced module. Mackey's result leads to the important notion of the defect: Each block $\mathcal{B} \subset k[G]$ gives rise to a p-subgroup $D_{\mathcal{B}} \subset G$ that measures the complexity of $\mathcal{B}$. Since the defect group of $\mathcal{B}_0(G)$ is a Sylow-p-subgroup, it is the most complicated block of $k[G]$.

The aforementioned facts together with Brauer correspondence imply that representation type behaves well under passage from the principal block to other blocks, or from a group to a subgroup.

Reduced Enveloping Algebras. Let $(\mathfrak{g}, [\,,\,])$ be a Lie algebra, $B \subset \mathfrak{g}$ a basis. If for every element $x \in B$ the p-th power of the inner derivation $\operatorname{ad} x : \mathfrak{g} \longrightarrow \mathfrak{g}\ ;\ \ y \mapsto [x, y]$ is again inner, then a theorem by Jacobson ensures the existence of a map $[p] : \mathfrak{g} \longrightarrow \mathfrak{g}\ ;\ \ x \mapsto x^{[p]}$ that enjoys the basic properties of the p-power operator of an associative algebra. In particular, we have

$$(\operatorname{ad} x)^p = \operatorname{ad} x^{[p]} \qquad \forall\ x \in \mathfrak{g}.$$

The pair $(\mathfrak{g}, [p])$ is then referred to as a *restricted Lie algebra.*

In the 1970's Kac and Weisfeiler noticed that much of the representation theory of $\mathfrak{g}$, or equivalently that of its universal enveloping algebra $U(\mathfrak{g})$, is captured by an algebraic family of $(U_\chi(\mathfrak{g}))_{\chi \in \mathfrak{g}^*}$ of associative algebras of dimension $p^{\dim \mathfrak{g}}$. The study of this family has since been one of the focal points in the representation theory of modular Lie algebras. By definition, we have $U_\chi(\mathfrak{g}) := U(\mathfrak{g})/I_\chi$, where $I_\chi := (\{x^p - x^{[p]} - \chi(x)^p 1 \; ; \; x \in \mathfrak{g}\})$. The algebra $U_\chi(\mathfrak{g})$ is a Frobenius algebra, though in general not symmetric. Contrary to finite groups, the Cartan matrix of $U_\chi(\mathfrak{g})$ may be singular. The example of the Steinberg module shows that one cannot expect to have good control of the composition of induction and restriction in the sense of Mackey. By analogy with finite groups, special attention is given to the principal block $\mathcal{B}_0(\mathfrak{g}) \subset U_0(\mathfrak{g})$. In a similar vein, the algebra $U_0(\mathfrak{g})$, being located at the generic point of the family, is thought of as the most complicated member of the family.

To this date, the most promising replacement of a defect appears to be given by Carlson's concepts of support varieties and rank varieties, that were transferred to our context by Friedlander-Parshall [7]. Let $\mathcal{V}_\mathfrak{g} := \{x \in \mathfrak{g} \; ; \; x^{[p]} = 0\}$ be the *nullcone* of $\mathfrak{g}$. Given a $U_\chi(\mathfrak{g})$-module M the *rank variety* $\mathcal{V}_\mathfrak{g}(M)$ is defined via

$$\mathcal{V}_\mathfrak{g}(M) := \{x \in \mathcal{V}_\mathfrak{g} \; ; \; M|_{U_{\chi|_{kx}}(kx)} \text{ is not free}\} \cup \{0\}.$$

If $\mathcal{B} \subset U_\chi(\mathfrak{g})$ is a block with simple modules $S_1, \ldots, S_n$, then we put

$$\mathcal{V}_\mathcal{B} := \bigcup_{i=1}^{n} \mathcal{V}_\mathfrak{g}(S_i) \subset \mathcal{V}_\mathfrak{g}.$$

This is our replacement of a defect. Again, $\mathcal{V}_\mathcal{B} \subset \mathcal{V}_{\mathcal{B}_0(\mathfrak{g})} = \mathcal{V}_\mathfrak{g}$, so that $\mathcal{B}_0(\mathfrak{g})$ has the largest defect.

Facts. Let $\mathcal{B} \subset U_\chi(\mathfrak{g})$ be a block.

(1) $\mathcal{B}$ is representation-finite if and only if $\dim \mathcal{V}_\mathcal{B} \leq 1$.
(2) If $\mathcal{B}$ is tame (and representation-infinite), then $\dim \mathcal{V}_\mathcal{B} = 2$.

From now on we assume that $p \geq 3$. In the early eighties, Drozd, Rudakov and Fischer independently showed that $\mathcal{B}_0(\mathfrak{sl}(2))$ is Morita equivalent to the trivial extension of the Kronecker algebra. It turns out that for Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ of algebraic groups, all tame blocks of $U_0(\mathfrak{g})$ are of this type [1].

Examples. We consider the Lie algebra $\mathfrak{g} := \mathfrak{sl}(2) \oplus kz$, where $[z, \mathfrak{sl}(2)] = (0)$. Using the standard basis $\{e, h, f\} \subset \mathfrak{sl}(2)$, we introduce two p-maps on $\mathfrak{g}$:

(1) The algebra $\mathfrak{sl}(2)_n$ is defined via $e^{[p]} = 0 \; ; \; h^{[p]} = h \; ; \; f^{[p]} = z \; ; \; z^{[p]} = 0$.
(2) The algebra $\mathfrak{sl}(2)_s$ is defined via $e^{[p]} = 0 \; ; \; h^{[p]} = h + z \; ; \; f^{[p]} = 0 \; ; \; z^{[p]} = 0$.

Let $C(\mathfrak{g}) := \{x \in \mathfrak{g} \; ; \; [x, \mathfrak{g}] = (0)\}$ be the *center* of $\mathfrak{g}$. By general theory, we have a "Fitting decomposition"

$$(*) \qquad C(\mathfrak{g}) = \mathfrak{t} \oplus \mathfrak{u}$$

of $C(\mathfrak{g})$ into its toral and unipotent parts. Here is a recognition criterion for tameness:

Theorem ([2]). *Let $\mathfrak{g}$ be a restricted Lie algebra.*

(1) *Then $\mathcal{B}_0(\mathfrak{g})$ is tame if and only if $\mathfrak{g}/C(\mathfrak{g})^{[p]} \cong \mathfrak{sl}(2), \mathfrak{sl}(2)_s$.*
(2) *If $\mathcal{B}_0(\mathfrak{g})$ is tame and $C(\mathfrak{g})$ is unipotent or toral, then $U_0(\mathfrak{g})$ is tame.*

In particular, the block $\mathcal{B}_0(\mathfrak{sl}(2)_n)$ is wild, while the algebra $U_0(\mathfrak{sl}(2)_s)$ is tame. Moreover, $\mathfrak{h} := ke \oplus kz$ is a p-subalgebra of $\subset \mathfrak{sl}(2)_s$ with $U_0(\mathfrak{h}) \cong k[X,Y]/(X^p, Y^p)$. Thus, $U_0(\mathfrak{h}) \subset U_0(\mathfrak{sl}(2)_s)$ is wild, while $U_0(\mathfrak{sl}(2)_s)$ is tame.

Using rank varieties and schemes of tori one first shows that $\mathfrak{g}/C(\mathfrak{g}) \cong \mathfrak{sl}(2)$, with $\mathfrak{u} \subset C(\mathfrak{g})$ being generated by one element [5, 6]. Let P be a principal indecomposable $U_0(\mathfrak{g})$-module, $\mathcal{B} \subset U_0(\mathfrak{g})$ the block belonging to P, and set $H_P := \operatorname{Rad}(P)/\operatorname{Rad}^3(P)$.

Proposition 1 ([2]). *The block $\mathcal{B}$ is tame if and only if H_P is decomposable.*

Filtrations by Verma modules and Auslander-Reiten Theory then yield the list of decomposable hearts. Let me illustrate one technical aspect. By general theory, the central extension $\mathfrak{g}$ is given by a p-semilinear map $\psi : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{g})$. The decomposition $(*)$ of $C(\mathfrak{g})$ provides a p-semilinear map $\psi_{\mathfrak{t}} : \mathfrak{sl}(2) \longrightarrow \mathfrak{t}$. One then has

$$U_0(\mathfrak{g}) \cong \bigoplus_{\gamma \in X(\mathfrak{t})} U_{\chi_\gamma}(\mathfrak{g}/\mathfrak{t}),$$

where $X(\mathfrak{t})$ is the character group of $\mathfrak{t}$, and $\chi_\gamma(x+u)^p = \gamma(\psi_{\mathfrak{t}}(x)) \quad \forall\, x \in \mathfrak{sl}(2),\ u \in \mathfrak{u}$.

The map ψ also gives rise to a p-semilinear form $\hat{\psi} : \mathfrak{sl}(2) \longrightarrow C(\mathfrak{g})/C(\mathfrak{g})^{[p]} \subset k$. For $\chi \in \mathfrak{sl}(2)^* \subset \mathfrak{g}^*$, we define

$$d(\psi, \chi) := \dim \mathcal{V}_{\mathfrak{sl}(2)} \cap \ker \hat{\psi} \cap \ker \chi.$$

A linear form $\chi \in \mathfrak{sl}(2)^*$ is *nilpotent* if it corresponds via the Cartan-Killing form to a nonzero nilpotent element of $\mathfrak{sl}(2)$.

Proposition 2 ([2]). *Let $\mathfrak{g}$ be a central extension of $\mathfrak{sl}(2)$ with $\hat{\psi} \neq 0$.*

(1) *If $C(\mathfrak{g})$ is unipotent, χ is nilpotent, and $d(\psi, \chi) \neq 0$, then $U_\chi(\mathfrak{g})$ is wild.*

(2) *If $d(\psi, \chi_\gamma) \neq 0$ for some nilpotent χ_γ, then $U_0(\mathfrak{g})$ possesses a wild block.*

Examples. (1) Let $\chi \in \mathfrak{sl}(2)_s^*$ be defined via $\chi(e) = 0 = \chi(h)$; $\chi(f) = 1$; $\chi(z) = 0$. Then $U_0(\mathfrak{sl}(2)_s)$ is tame, while $U_\chi(\mathfrak{sl}(2)_s)$ is wild.

(2) Let $\mathfrak{g} := \mathfrak{sl}(2) \oplus kz \oplus kt$, $[kz \oplus kt, \mathfrak{g}] = (0)$, $e^{[p]} = 0$; $h^{[p]} = h + z$; $f^{[p]} = t$; $z^{[p]} = 0$; $t^{[p]} = t$. Then $\mathcal{B}_0(\mathfrak{g})$ is tame, while $U_0(\mathfrak{g})$ is wild.

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