A construction of maximal 1-orthogonal modules for preprojective algebras

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(joint work with Bernard Leclerc and Jan Schröer)

For a Dynkin quiver $Q = (Q_0, Q_1, t, h)$ we consider its double $\bar{Q}$, which is obtained from Q by adding an extra arrow $a^*\colon h(a) \to t(a)$ for each arrow $a\colon t(a) \to h(a)$ in Q_1. The preprojective algebra $\Lambda = \mathrm{k}\bar{Q}/\langle \sum_{a \in Q_1} [a, a^*] \rangle$ is in this situation a finite dimensional, selfinjective algebra.

Let $F\colon \widetilde{\Lambda} \to \Lambda$ be the universal covering of Λ. Consider moreover an embedding $J\colon \Gamma_Q \to \widetilde{\Lambda}$ where Γ_Q is the Auslander algebra of $\mathrm{k}Q$. To be precise, we should replace here our algebras by locally bounded categories, and consider contravariant functors instead of right modules.

Consider now $I'_Q = \mathrm{Hom}_{\mathrm{k}}(\Gamma_Q, \mathrm{k})$ as a right Γ_Q-module and define the Λ-module $I_Q := F_\lambda J^{\cdot}(I_Q)$. Here, $F_\lambda\colon \text{mod-}\widetilde{\Lambda} \to \text{mod-}\Lambda$ is the usual push down functor associated to F and $J^{\cdot}\colon \text{mod-}\Gamma_Q \to \text{mod-}\widetilde{\Lambda}$ is the "extension by 0" associated to J. Clearly, I_Q is a direct sum of $|\Pi_Q|$ pairwise non-isomorphic indecomposable summands, where Π_Q is the set of positive roots associated to Q.

We can describe $\mathcal{E}_Q = \mathrm{End}_\Lambda(I_Q)$ as a quiver with relations: The quiver $\bar{A}_Q$ of $\mathrm{End}_\Lambda(I_Q)$ is obtained from the Auslander-Reiten quiver A_Q of $\mathrm{k}Q$ by inserting an additional arrow $\rho_x\colon x \to \tau x$ for each non-projective vertex x of A_Q. The relations are the usual mesh relations for A_Q and, moreover, for each arrow $\beta\colon x \to y$ with y not a projective vertex there is a relation $\tau(\beta)\rho_x - \rho_y\beta$ (interpret this as $\rho_y\beta$ if x is projective). In other words, precisely for each arrow $\alpha\colon u \to v$ in $\bar{A}_Q$ with not both u and v injective there is a homogeneous relation of length 2 from v to u.

Now, a slightly tricky calculation shows that

$$\dim_{\mathrm{k}} \mathcal{E}_Q Q = q_Q(\underline{\dim}\, I_Q), \tag{1}$$

where q_Q is the quadratic form associated to Q.

Remarks. (1) *For $X \in \text{mod-}\Lambda$ let $\mathbf{v} = \underline{\dim}\, X$. If we denote by x the corresponding point in the preprojective variety $\Lambda_{\mathbf{v}}$, one has*

$$\dim \mathrm{Ext}^1_\Lambda(X, X) = 2\,\mathrm{codim}_{\Lambda_{\mathbf{v}}}(\mathrm{Gl}_{\mathbf{v}} \cdot x) = 2(\dim \mathrm{End}_\Lambda(X) - q_Q(\mathbf{v})).$$

We conclude from (1) *that I_Q is rigid, i.e.* $\mathrm{Ext}^1_\Lambda(I_Q, I_Q) = 0$.

(2) *In* [4] *it was shown that $|\Pi_Q|$ is an upper bound for the number of pairwise non-isomorphic direct summands of a rigid module. As we have seen for I_Q, this upper bound is reached. We call such modules* complete rigid.

(3) *In* [3] *we show that if there exists a complete rigid module T such that the quiver of* $\mathrm{End}_\Lambda(T)$ *has no loops, then each complete rigid module is even maximal 1-orthogonal in the sense of Iyama* [5].

(4) *Let now $\mathrm{k} = \mathbb{C}$ and N be a maximal unipotent subgroup of a simple (complex) Lie group of type $|Q|$. It follows from* [1] *that the coordinate ring* $\mathbb{C}[N]$ *has the structure of an (upper) cluster algebra. The exchange matrix*

for the initial seed constructed there can be codified in a quiver which coincides with the quiver of our $\mathcal{E}_Q$ (for a proper reduced word for the longest element in the corresponding Weyl group).

References

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