Parabolic group actions and tilting modules

Lutz Hille

1. Parabolic group actions

Let k be an algebraically closed field and $V_0 = \{0\} \subset V_1 \subset V_2 \subset \ldots V_{t-1} \subset V_t$ be a flag of finite dimensional vector spaces with $d_i := \dim V_i - \dim V_{i-1}$. The stabiliser of this flag is a parabolic subgroup in $\mathrm{GL}(V_t)$ denoted by $P(d)$. It is also the group of invertible elements in $\mathrm{End}_{k\mathbb{A}_t}(\bigoplus P(i)^{d_i})$, where the modules $P(i)$ are the indecomposable projective modules over the path algebra $k\mathbb{A}_t$ of a directed quiver of type $\mathbb{A}_t$. Let further I be a subset in $\{(i,j) \mid 1 \leq i < j \leq t\}$ so that for all (i,j) in I also $(i,j+1)$, for $j < t$, and $(i-1,j)$, for $i > 1$, are both in I. The set I can be seen as a root ideal in the positive roots of the root system of type $\mathbb{A}_t$ or simply as a subset closed under taking right and upper neighbours. To such a subset we associate a function $h : \{1,\ldots,t\} \longrightarrow \{0,\ldots,t-1\}$ defined by $h(j) := \max\{i \mid (i,j) \in I\}$ if such an i exists and $h(j) = 0$ otherwise. Using this notation we can define the group $P(d)$ and several Lie algebras in $P(d)$.

$$\begin{array}{rcl} P(d) & := & \{f \in \mathrm{Aut}(V_t) \mid f(V_i) \subseteq V_i\} \\ \mathfrak{p}_u(d) & := & \{f \in \mathrm{End}(V_t) \mid f(V_i) \subseteq V_{i-1}\} \\ \mathfrak{n}(I,d) & := & \{f \in \mathrm{End}(V_t) \mid f(V_i) \subseteq V_{h(i)}\} \\ \mathfrak{p}_u(d)^{(l)} & := & \{f \in \mathrm{End}(V_t) \mid f(V_i) \subseteq V_{i-1-l}\} \end{array}$$

Main Question: When does $P(d)$ act with a dense orbit on $\mathfrak{n}(I,d)$ and $\mathfrak{p}_u(d)^{(l)}$?

Example 1. a) *By a classical result of Richardson, it is well-known that $P(d)$ acts always with a dense orbit on $\mathfrak{p}_u(d)$.*

b) *Let I be the set generated by $\{(1,2),(3,4),(5,6)\}$ and $d = (1,1,1,1,1,1)$, where $t = 6$, then $P(d)$ does not act wit a dense orbit on $\mathfrak{n}(I,d)$.*

c) *Let $l = 1$, $t = 9$ and $d = (2,1,2,2,1,2,2,1,2)$, then $P(d) \subset \mathrm{GL}_{15}$ does not act with a dense orbit on $\mathfrak{p}_u(d)^{(1)}$.*

The aim of the talk is to present an equivalent problem to the questions above for the existence of certain modules without self extensions over certain subalgebras of the Auslander algebra of $k[T]/T^t$. Moreover, we present several partial results concerning the question above.

2. Subalgebras of the Auslander algebra of $k[T]/T^t$

To the ideals defined above one can associate certain subquotient algebras of the Auslander algebra of $k[T]/T^t$. We denote by $\mathcal{A}_t$ the algebra $\mathrm{End}(\bigoplus_{i=1}^t k[T]/T^i)$ (it is the Auslander algebra of $k[T]/T^t$). The quiver of this algebra consists of t vertices and arrows α_i (corresponding to the inclusion of $k[T]/T^i$ into $k[T]/T^{i+1}$) and of arrows β_i (corresponding to the projection of $k[T]/T^{i+1}$ onto $k[T]/T^i$. The subquotient algebras $\mathcal{A}(I)$ and $\mathcal{A}_{t,l}$ corresponding to the ideals $\mathfrak{n}(I,d)$ and

$\mathfrak{p}_u(d)^{(l)}$ have arrows α_i and γ_j consisting of certain compositions of the arrows β_i: the algebra $\mathcal{A}_{t,l}$ has arrows α_i (starting in i and ending in $i+1$) for $i = 1, \ldots, t-1$ and arrows γ_j (starting in $j+l+1$ and ending in i) for $j = 1, \ldots, t-1-l$ defined as $\gamma_j := \beta_j\beta_{j+1}\ldots\beta_{j+l}$. The algebra $\mathcal{A}(I)$ has arrows α_i (starting in i and ending in $i+1$) for $i = 1, \ldots, t-1$ and arrows γ_j (starting in $h(j)$ and ending in j) for all pairs $(h(j), j)$ in I with $(h(j)+1, j+1) \in I)$ defined as $\gamma_j := \beta_j\beta_{j+1}\ldots\beta_{h(j)}$ (see [1] for details).

Note that the constructions coincide in the special cases when $I = \{(i,j) \mid 1 \leq i < j - l \leq t - l\}$.

Theorem 2.1. *The algebras $\mathcal{A}(I)$ are quasi-hereditary and the category of Δ–good modules coincides with the set of all modules M satisfying one of the following equivalent conditions*

a) *the maps $M(\alpha_i)$ are injective,*
b) *the projective dimension of M is at most 1, and*
c) *the restriction of M to the subalgebra generated by α_i (it is the path algebra of a directed quiver of type $\mathbb{A}_t$) is projective.*

It is proven in [6, 2] that the orbits for the $P(d)$ action are in bijection with the isomorphism classes of modules over the corresponding algebra $\mathcal{A}(I)$.

3. Richardson's result and tilting modules
(joint work with T. Brüstle, C. M. Ringel, and G. Röhrle)

Richardson's result (see Example 1,a), [7]) implies that for each dimension vector e with $e_i - e_{i-1} \geq 0$ there exists precisely one good module $M(e)$ of dimension vector e without self extensions (it is not indecomposable in general). We construct this module explicitly (so we also construct all indecomposable modules without self extensions explicitly). Moreover, we can use an analogous construction to get so called *standard* modules over the algebra $\mathcal{A}_{t,l}$ and also over the algebra $\mathcal{A}(I)$, which also do not have self extensions. For $l = 1$ and $t \geq 6$ there exist modules without self extensions which are not standard (see Example 2).

Let P be the largest indecomposable finite dimensional projective $\mathcal{A}_t$–module (it is the projective cover of the simple module $S(t)$). Note that P has all finite dimensional indecomposable projective modules $P(i)$ as a submodule, they are generated by one element say $p(i)$ in P. Let $A = \{a_1, \ldots, a_r\}$ for $1 \leq a_1 < a_2 < \ldots < a_r \leq t$ be an ordered set of natural numbers. For each such set A we define a unique submodule $\Delta(A)$ of P which is Δ–good as the module generated by the elements $\alpha^{a_i - i}(p(i))$ (where α^{a_i-i} denotes the composition of $a_i - i$ arrows α_i, so that $\alpha^{a_i-i}(p(i))$ makes sense).

Theorem 3.1. *An indecomposable Δ–good module without self extensions is isomorphic to $\Delta(A)$ for some subset A. Each basic tilting module (note that a module of projective dimension at most one is already Δ–filtered) is isomorphic to the direct sum $\bigoplus_{i=1}^{t} \Delta(\sigma(1), \ldots, \sigma(i))$ for some element σ in the symmetric group S_t.*

4. A reduction theorem

The classification of all pairs (I, d), so that $P(d)$ acts with a dense orbit on $\mathfrak{p}_u(I, d)$ seems to be more difficult than the classification of all pairs (t, l), so that $P(d)$ acts with a dense orbit on $\mathfrak{p}_u(d)^{(l)}$. In this part we claim, that both classifications are equivalent. One direction is obvious, so we concentrate on the non-obvious one. We show two results. First, if we allow d to have entries 0, then one can show, that $\mathfrak{p}_u(I, d)$ is isomorphic (together with the group action) to some $\mathfrak{p}_u(\overline{d})^{(l)}$ (where $\overline{d}$ is a certain dimension vector obtained from d by filling in some zeros) by [5], Theorem 1.4.2. Using [5], Theorem 1.4.1 we can even replace $\overline{d}$ by a dimension vector $\underline{d}$ without an entry zero, so that $P(\underline{d})$ acts with a dense orbit on $\mathfrak{p}_u(\underline{d})^{(l)}$ precisely when $P(\overline{d})$ acts with a dense orbit on $\mathfrak{p}_u(\overline{d})^{(l)}$.

5. Actions of the Borel subgroup
(joint work with S. Goodwin)

In this section we consider the special case, when $d_i \leq 1$ (for simplicity we allow $d_i = 0$, instead of working with the subset I, however, both approaches are equivalent by Section 4). If we consider an ideal $\mathfrak{b}_u^{(1)} \subseteq \mathfrak{n} \subseteq \mathfrak{b}_u$, then we can ask whether B acts with a dense orbit (it corresponds to a good module over $\mathcal{A}_{t,1}$ without self extensions and dimension vector e with $e_i - e_{i-1} \leq 1$) and whether this orbit is indecomposable (it corresponds to an indecomposable good module). A dimension vector d as above consists of certain strings of entries 1 of some length, say $a_0, \ldots, a_r$ with $a_i > 0$. The strings not in the beginning and the end are called *intermediate*, so the length of the intermediate strings is $a_1, \ldots, a_{r-1}$.

Theorem 5.1. *Let d be a dimension vector with $d_i \leq 1$. Then B acts with a dense orbit on $\mathfrak{p}_u(d)^{(1)}$ precisely when one of the following conditions is satisfied:*

a) *if $d_i = 1$ for some i, then $d_{i-1} = 0$ and $d_{i+1} = 0$ (d is* standard*) or*
b) *there is at most one intermediate string of entries 1 of even length.*

The orbit above is indecomposable precisely when d is either standard or there exists precisely one intermediate string of even length.

Example 2. a) *The dimension vector in Example 1,b) is $(1, 0, 1, 1, 0, 1, 1, 0, 1)$, so it is the minimal dimension vector d, so that B does not act with a dense orbit on $\mathfrak{p}_u(d)^{(1)}$.*
b) *The minimal non-standard dimension vector, so that B acts with a dense orbit and the orbit is indecomposable is $(1, 0, 1, 1, 0, 1)$.*

References

[1] T. Brüstle, L. Hille, *Matrices over upper triangular bimodules and Δ-filtered modules over quasi-hereditary algebras.* Colloq. Math. **83** (2000), no. 2, 295–303.

[2] T. Brüstle, L. Hille, *Actions of parabolic subgroups in GL_n on unipotent normal subgroups and quasi-hereditary algebras.* Colloq. Math. **83** (2000), no. 2, 281–294.

[3] T. Brüstle, L. Hille, C. M. Ringel, G. Röhrle, *The Δ-filtered modules without self-extensions for the Auslander algebra of $k[T]/\langle T^n \rangle$.* Algebr. Represent. Theory **2** (1999), no. 3, 295–312.

[4] S. Goodwin, L. Hille, . . . , in preparation.

[5] L. Hille: *Actions of Algebraic Groups, Geometric Quotients and Quivers (German)*, Habilitationsschrift, Hamburg (2003)

[6] L. Hille, G. Röhrle, *Variation on a theme of Richardson.* Special issue on linear algebra methods in representation theory. Linear Algebra Appl. **365** (2003), 239–246.

[7] R. W. Richardson, *Conjugacy classes in parabolic subgroups of semisimple algebraic groups.* Bull. London Math. Soc. **6** (1974), 21–24.