Calabi-Yau varieties and reflexive polytopes

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1. Calabi-Yau varieties in $\mathbb{P}^n$

Let k be the field of complex numbers and $\mathbb{P}^n$ the projective n-space over k. The anticanonical sheaf ω^{-1} is isomorphic to $\mathcal{O}(n+1)$, and we can identify a global section in ω^{-1} with a homogeneous polynomial of degree $n+1$ in the $n+1$ variables $x_0, \ldots, x_n$.

a) Let $n = 2$, then a generic polynomial f of degree 3 defines an elliptic curve E in $\mathbb{P}^2$.

b) Let $n = 3$, then a generic polynomial of degree 4 defines a K3-surface in $\mathbb{P}^4$.

c) Let $n = 4$, then a generic polynomial of degree 5 defines a 3–dimensional Calabi-Yau variety in $\mathbb{P}^5$.

All these varieties X are Calabi-Yau varieties (see definition below), in particular, $\omega_X \simeq \mathcal{O}$ (the canonical sheaf is trivial) and the Serre duality is of the form $\mathrm{Ext}^l(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^{n-l}(\mathcal{G}, \mathcal{F})^*$.

We can also (using the action of the torus k^n on $\mathbb{P}^n$) identify the space of polynomials of degree n with all formal linear combinations of elements in a lattice polytope $\Delta(n)$. (The elements in $\Delta(n)$ correspond to a torus invariant basis of the space of homogeneous polynomials of degree $n+1$ in $n+1$ variables, for the torus action $(\lambda_1, \ldots, \lambda_n)(x_0, x_1, \ldots, x_n) := (x_0, \lambda_1 x_1, \ldots, \lambda_n x_n)$ this basis consists just of the monomials.) So we get

$$\Delta(n) := \{a \in \mathbb{Z}^{n+1} \mid \sum_{i=1} a_i = n+1, a_i \geq 0\},$$

a simplex in the lattice $\mathbb{Z}^{n+1}$. This is a polytope which has precisely one inner lattice point (a lattice point not on the boundary of $\Delta(n)$), it is $(1, 1, \ldots, 1)$.

On the projective n–space, there exists a sequence of line bundles $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n)$ without any self extensions ($\mathrm{Ext}^l_{\mathbb{P}^n}(\mathcal{O}(i), \mathcal{O}(j)) = 0$ for all $0 \leq i, j \leq n$ and all l) generating the derived category of coherent sheaves on $\mathbb{P}^n$. Classical results on the derived category of coherent sheaves on $\mathbb{P}^n$ allow us to describe it using derived categories of modules over the endomorphism ring of $\bigoplus_{i=0}^n \mathcal{O}(i)$.

2. Calabi-Yau Varieties

Definition. A *Calabi-Yau variety* X is a smooth projective variety satisfying

(1) $\omega_X \simeq \mathcal{O}$ (the canonical sheaf is trivial), and

(2) $\mathrm{H}^l(X; \mathcal{O}_X) = 0$ for all $1 \leq l \leq \dim X - 1$.

The definition above can be generalised, sometimes one only wants X to be complete, and in dimension greater or equal to 4, one often allows some mild singularities. Calabi-Yau varieties can be constructed in Fano varieties, we explain the construction in more detail below.

Example. a) *Let $X \subset \mathbb{P}^n$ be a hyper surface defined by a generic homogeneous polynomial of degree $n+1$ (as in section 1), then X is a Calabi-Yau variety.*

b) *Let F be a smooth Fano variety satisfying $\mathrm{H}^l(F;\mathcal{O}_F) = 0$ for all $1 \leq l \leq \dim X$. Take a generic element f in $\mathrm{H}^0(F;\omega_F^{-1})$, then the hyper surface X defined by f is a Calabi-Yau variety. Condition* 1) *follows from the adjunction formula and condition* 2) *from the long exact cohomology sequence applied to*

$$0 \longrightarrow \omega_F \longrightarrow \mathcal{O}_F \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

To find Calabi-Yau varieties, we need to find Fano varieties F satisfying the condition $\mathrm{H}^l(F,\mathcal{O}_F) = 0$ for all $1 \leq l \leq \dim X$. The conditions on F can be chosen weaker at several places. E. g., it is sufficient that F has only isolated singularities (a generic section does not meet these singularities), and one can also take partial resolutions $\widetilde{F}$ of singular Fano varieties F satisfying $\omega_{\widetilde{F}} \simeq \mathcal{O}_{\widetilde{F}}$. There exists a large class of those varieties that can be constructed using so-called reflexive polytopes, the class of toric (possibly singular) Fano varieties (see [5]).

3. Reflexive Polytopes

Definition. Let M be a lattice in $M_{\mathbb{R}} \simeq \mathbb{R}^n$. A *lattice polytope* Δ in $M_{\mathbb{R}}$ is the convex hull in $M_{\mathbb{R}}$ of a finite number of lattice points (that is points in M). We assume $\dim \Delta = n$ and 0 be an interior lattice point of Δ. The polytope Δ is *reflexive* if its dual polytope

$$\Delta^\circ := \{n \in M_{\mathbb{R}}^* \mid n(m) \geq -1 \quad \forall m \in \Delta\}$$

is also a lattice polytope. A lattice polytope is *smooth* if for each vertex v the cone spanned by $\Delta - v$ (we shift the polytope so that v becomes the zero point and consider the cone with apex in 0 generated by the shifted elements in Δ) is generated by a $\mathbb{Z}$–basis of $M_{\mathbb{R}}$.

To each lattice polytope Δ one can associate a toric variety F_Δ. If Δ is smooth, then F_Δ is smooth, and if Δ is reflexive, then F_Δ is a Fano variety. Conversely, each toric Fano variety also comes from a reflexive polytope, the sections in ω_F^{-1} form a reflexive polytope (similar to the example in section 1).

Let Δ be a lattice polytope. We define a cone $C(\Delta)$ as the cone with apex in 0 generated by $\Delta \times \{1\} \subset M_{\mathbb{R}} \times \mathbb{R}$. The lattice points $C(\Delta)_{\mathbb{Z}}$ in $C(\Delta)$ form a semi-group, and we consider the semi-group ring $S(\Delta)$ of $C(\Delta)_{\mathbb{Z}}$. It is a graded ring, the degree comes from the additional element, so $\deg(x,a) := a$ for $x \in a\Delta$. Then we define the projective algebraic variety F_Δ as $\mathrm{Proj}(S(\Delta))$. This variety is of dimension n, and it comes with an action of an n–dimensional algebraic torus $T \simeq k^n$, the torus acts with a dense orbit. If we consider the T–action on $\mathrm{H}^0(F_\Delta, \mathcal{O}_{F_\Delta}(1))$ (where $\mathcal{O}_{F_\Delta}(1)$ is taken with respect to the given embedding in $\mathbb{P}^N$, where N is the number of lattice points in Δ), then the T–invariant points form the lattice points of the n–dimensional lattice polytope Δ.

We conclude this section with an overview of the classification of reflexive polytopes.

$n = 1$: There exists precisely one reflexive simplex, it is the convex hull of -1 and 1 in $\mathbb{R}$.

$n = 2$: It is an exercise to classify them, there exist precisely 16 reflexive polytopes and 5 of them are smooth. These five smooth ones correspond to the five toric del Pezzo surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and the blow up of $\mathbb{P}^2$ in one, two or three points (the three points must not lie on a common line).

$n = 3$: A classification of the smooth reflexive polytopes can be found in [10], there exist 18. They can be classified using certain double weighted triangulations of the plane. The classification of all reflexive polytopes is done by a computer, the algorithm can be found in [8], there exist $4,319$ of them (see [11]).

$n = 4$: The classification of 4–dimensional smooth reflexive polytopes was done by Batyrev in [5], there exist 124 of them. The classification of all reflexive polytopes is mainly a problem on hard disc space (as one of the authors told me), there exist $473,800,776$ of them (see [9, 11]).

For reflexive simplices the classification is much simpler and consists essentially of the classification of so-called weight systems. These weight systems also appear for weighted projective spaces in the sense of Baer, Geigle and Lenzing ([3]).

4. Quivers and Reflexive Polytopes

Surprisingly, one can construct some reflexive polytopes using quivers, however the class of these polytopes is not very large (see [1, 6]). On the other hand, a smooth reflexive polytope constructed from a quiver comes always with a sequence of line bundles without any self extension (see [1]). There exist also several other approaches to construct exceptional sequences of line bundles on toric varieties. It is an open problem (see [2, 7]) whether there exists on any smooth toric variety a full strong exceptional sequence of line bundles, (similar to the one on $\mathbb{P}^n$). This problem is even open for toric surfaces.

References

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