An equivalence between the homotopy categories of projectives and of injectives

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(joint work with Henning Krause)

Let R be a commutative noetherian ring with a dualizing complex D; thus D is a bounded complex of injective R-modules, with $H(D)$ finitely generated, and the natural morphism $R \to \operatorname{Hom}_R(D, D)$ is a homology isomorphism. The starting point of the work described in this talk was the realization that $\mathbf{K}(\operatorname{Proj} R)$ and $\mathbf{K}(\operatorname{Inj} R)$, the homotopy categories of complexes of projective and injective R-modules, respectively, are equivalent. This equivalence comes about as follows: D consists of injective modules and, R being noetherian, direct sums of injectives are injective, so $D \otimes_R -$ defines a functor from $\mathbf{K}(\operatorname{Proj} R)$ to $\mathbf{K}(\operatorname{Inj} R)$. This functor factors through $\mathbf{K}(\operatorname{Flat} R)$, the homotopy category of flat R-modules, and provides the lower row in the following diagram:

$$\mathbf{K}(\operatorname{Proj} R) \underset{\pi=\mathrm{inc}}{\overset{\pi_r}{\rightleftarrows}} \mathbf{K}(\operatorname{Flat} R) \underset{D\otimes_R -}{\overset{\operatorname{Hom}_R(D,-)}{\rightleftarrows}} \mathbf{K}(\operatorname{Inj} R)$$

The triangulated structures on the homotopy categories are preserved by π and $D \otimes_R -$. The functors in the upper row of the diagram are the corresponding right adjoints; π_r exists because π preserves coproducts and $\mathbf{K}(\operatorname{Proj} R)$ is compactly generated; the latter property was discovered by Jørgensen [3]. Then one has:

Theorem 1. *The functor $D \otimes_R - \colon \mathbf{K}(\operatorname{Proj} R) \to \mathbf{K}(\operatorname{Inj} R)$ is an equivalence of triangulated categories, with quasi-inverse $\pi_r \circ \operatorname{Hom}_R(D, -)$.*

This equivalence is closely related to, and may be viewed as an extension of, Grothendieck's duality theorem for $\mathbf{D}^f(R)$, the derived category of complexes whose homology is bounded and finitely generated. To see this connection one has to consider the commutative diagram of functors:

$$\begin{array}{ccc} \mathbf{K}^c(\operatorname{Proj} R) & \xrightarrow{D\otimes_R -} & \mathbf{K}^c(\operatorname{Inj} R) \\ {\scriptstyle \mathsf{P}}\downarrow\simeq & & \simeq\downarrow{\scriptstyle \mathsf{Q}} \\ \mathbf{D}^f(R) & \xrightarrow{\mathbf{R}\operatorname{Hom}_R(-,D)} & \mathbf{D}^f(R) \end{array}$$

where the top row consists of the compact objects in $\mathbf{K}(\operatorname{Proj} R)$ and $\mathbf{K}(\operatorname{Inj} R)$, respectively. The functor P is the composition of $\operatorname{Hom}_R(-, R) \colon \mathbf{K}(\operatorname{Proj} R) \to \mathbf{K}(R)$ with the canonical functor $\mathbf{K}(R) \to \mathbf{D}(R)$; it is a theorem of Jørgensen [3] that P is an equivalence of categories. The functor Q is induced by $\mathbf{K}(R) \to \mathbf{D}(R)$, and Krause [4] proves that it is an equivalence. Given these descriptions it is not hard to verify that $D \otimes_R -$ preserves compactness; this explains the top row of the diagram. Now, Theorem 1 implies that the $D \otimes_R -$ restricts to an equivalence between compact objects, so the diagram above implies $\mathbf{R}\operatorname{Hom}_R(-, D)$ is an equivalence; this is one form of the duality theorem; *cf.* Hartshorne [2].

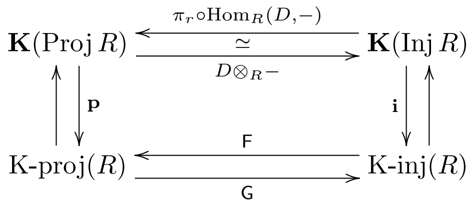
Conversely, given that $\mathbf{R}\mathrm{Hom}_R(-, D)$ is an equivalence, it follows that the top row of the diagram is an equivalence; this is the crux of the proof of Theorem 1.

We develop Theorem 1 in two directions. The first one deals with the difference between $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\, R)$, the category of acyclic complexes in $\mathbf{K}(\mathrm{Proj}\, R)$, and $\mathbf{K}_{\mathrm{tac}}(\mathrm{Proj}\, R)$, its subcategory of totally acyclic complexes. We consider also the injective counterparts. The main new result in this context is summarized in:

Theorem 2. *The quotient triangulated categories* $\mathbf{K}_{\mathrm{ac}}(\mathrm{Proj}\, R)/\mathbf{K}_{\mathrm{tac}}(\mathrm{Proj}\, R)$ *and* $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj}\, R)/\mathbf{K}_{\mathrm{tac}}(\mathrm{Inj}\, R)$ *are compactly generated. The compact objects in each of these categories are equivalent to* $\mathrm{Thick}(R, D)/\,\mathrm{Thick}(R)$*, up to direct factors.*

The quotient $\mathrm{Thick}(R, D)/\,\mathrm{Thick}(R)$ is a subcategory of $\mathbf{D}^f(R)/\,\mathrm{Thick}(R)$, the stable category of R. Since D has finite projective dimension if and only if R is Gorenstein, we deduce: R is Gorenstein if and only if every acyclic complex of projectives is totally acyclic, if and only if every acyclic complex of injectives is totally acyclic. An interesting feature of Theorem 2 is, that it draws our attention to the (monogenic) category $\mathrm{Thick}(R, D)/\,\mathrm{Thick}(R)$ as a measure of the failure of a ring R from being Gorenstein. Its role is thus analogous to that of the full stable category with regards to regularity: $\mathbf{D}^f(R)/\,\mathrm{Thick}(R)$ is trivial if and only if R is regular. This observation, and others of this ilk, suggest that $\mathrm{Thick}(R, D)/\,\mathrm{Thick}(R)$ is an object worth investigating.

Next we turn to the functors induced on $\mathbf{D}(R)$ by the ones in Theorem 1. This involves two different realizations of the derived category as a subcategory of $\mathbf{K}(R)$, both obtained from the localization functor $\mathbf{K}(R) \to \mathbf{D}(R)$: one by restricting it to K-proj(R) the subcategory of K-projective complexes, and the other by restricting it to K-inj(R), the subcategory of K-injective complexes. The inclusion K-proj$(R) \subseteq \mathbf{K}(\mathrm{Proj}\, R)$ admits a right adjoint $\mathbf{p}$, the inclusion K-inj$(R) \subseteq \mathbf{K}(\mathrm{Proj}\, R)$ admits a left adjoint $\mathbf{i}$, and one obtains a diagram



where G is $\mathbf{i} \circ (D \otimes_R -)$ restricted to K-proj(R), and F is $\mathbf{p} \circ \pi_r \circ \mathrm{Hom}_R(D, -)$ restricted to K-inj(R). It follows that (G, F) is an adjoint pair of functors. However, the equivalence in the upper row of the diagram does not imply an equivalence in the lower one. Indeed, using Theorem 1, we prove:

The natural morphism $X \to \mathsf{FG}(X)$ *is an isomorphism if and only if the mapping cone of the morphism* $(D \otimes_R X) \to \mathbf{i}(D \otimes_R X)$ *is totally acyclic.*

The point being that the mapping cones of resolutions are, in general, only acyclic. Complexes in K-inj(R) for which the morphism $\mathsf{GF}(Y) \to Y$ is an isomorphism can be characterized in a similar fashion. This is the key observation that allows us to describe the subcategories of K-proj(R) and K-inj(R) where the functors G and F restrict to equivalences. A further extension of these results, when translated to the derived category, reads:

Theorem 3. *A complex X of R-modules has finite G-projective dimension if and only if the morphism $X \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X)$ in $\mathbf{D}(R)$ is an isomorphism and $H(D \otimes^{\mathbf{L}}_R X)$ is bounded on the left.*

This theorem, together with its counterpart for G-injective dimensions, recovers recent results of Christensen, Frankild, and Holm [1], who arrived at them from another route. In the talk I focused on commutative rings. However, the results carry over, with suitable modifications in the statements and with nearly identical proofs, to non-commutative rings that possess dualizing complexes. The details are given in our article, which we intend to post on the Math arXiv shortly; I am writing this on 26th February, 2005.

References

[1] L. W. Christensen, A. Frankild, H. Holm, *On Gorenstein projective, injective and flat dimensions—A functorial description with applications.* Preprint (2004).

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