The Gabriel-Serre category, the Tate-Vogel category, and Koszul duality

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(joint work with Roberto Martínez-Villa)

The BGG correspondence [1] establishes an equivalence

$$\underline{\mathrm{gr}}(\wedge^{\bullet} V^{n+1}) \simeq D^b(\mathrm{coh}(P^n))$$

between the stable category of finitely generated graded modules over the exterior algebra on $n+1$ letters and the bounded derived category of coherent sheaves on the n-dimensional projective space. Using Koszul duality, this result can be significantly generalized [2]. In this paper, we provide a more transparent and less technically involved proof of that result. Our approach is based on Koszul duality and universal constructions.

Let Λ be a finite-dimensional graded algebra such that Λ_1 is a semisimple Λ_0-module. The Yoneda algebra of Λ will be denoted Γ. It is naturally graded by the cohomological degree. Let $\mathrm{gr}(\Lambda)$ be the category of finitely generated graded Λ-modules. Our first goal is to produce, for an arbitrary $M \in \mathrm{gr}(\Lambda)$, a complex of finitely generated projective graded Γ-modules. This was done in [2]. We shall now review that construction. Starting with the multiplication map $\Lambda_1 \otimes_{\Lambda_0} M_n \to M_{n+1}$ and applying the functor $D(-) := \mathrm{Hom}_{\Lambda_0}(-, \Lambda_0)$, we have a homomorphism of Γ_0-modules $D(M_{n+1}) \to D(M_n) \otimes_{\Gamma_0} \Gamma_1$. To this map, we can apply the functors $- \otimes_{\Gamma_0} \Gamma_l$, which results, for each n, in a degree zero homomorphism of graded Γ-modules

$$d_{n+1} : D(M)_{-n-1} \otimes_{\Gamma_0} \Gamma \to D(M)_{-n} \otimes_{\Gamma_0} \Gamma[1].$$

Lemma. $d^2 = 0$.

Thus the above construction yields a (linear) complex of projectives in $\mathrm{gr}(\Gamma)$. In fact, the construction is functorial and we have a contravariant functor

$$\lambda : \mathrm{gr}(\Lambda) \to \mathcal{LCP}^b(\mathrm{gr}(\Gamma)),$$

where the target is the category of bounded linear complexes of finitely generated projective graded Γ-modules.

Proposition. *If Λ is quadratic, then λ is a duality. If λ is Koszul, then M is (a shift of) a Koszul module if and only if $\lambda(M)$ is exact at the non-minimal degrees (i. e., $\lambda(M)$ is a (shifted) projective graded resolution of the Γ-module Koszul-dual to M).*

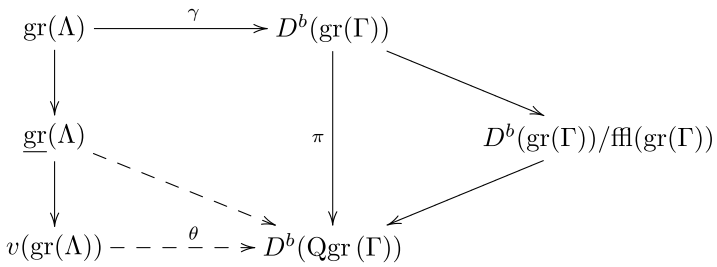
Composing λ with the tautological functor $\mathcal{LCP}^b(\mathrm{gr}(\Gamma)) \to D^b(\mathrm{gr}(\Gamma))$, we have a functor

$$\gamma : \mathrm{gr}(\Lambda) \to D^b(\mathrm{gr}(\Gamma)).$$

Taking the Verdier quotient of the target category by the subcategory of all complexes isomorphic to finite complexes of graded modules of finite length and identifying the result with the bounded derived category $D^b(\mathrm{Qgr}\,(\Gamma))$ of finitely generated graded Γ-modules modulo modules of finite length, we have a composite functor

$$\pi : D^b(\mathrm{gr}(\Gamma)) \to D^b(\mathrm{Qgr}\,(\Gamma)).$$

If $M \in \mathrm{gr}(\Lambda)$ is projective, then $\lambda(M)$ is semisimple and $\pi\gamma(M)$ is a zero object in $D^b(\mathrm{Qgr}\,(\Gamma))$. Therefore the composition $\pi\gamma$ factors through the stable category $\underline{\mathrm{gr}}(\Lambda)$. On that category, the syzygy operation Ω becomes an endofunctor. Applying $\pi\gamma$ to the short exact sequence $0 \to \Omega M \to P \to M \to 0$, where P is projective, we see that $\pi\gamma(\Omega M) \simeq \pi\gamma(M)[-1]$. In other words, $\pi\gamma$ "inverts" Ω and therefore factors, in a unique way, through the Tate-Vogel category $v(\mathrm{gr}(\Lambda))$.[1] The above observations are codified in the following commutative diagram:



We can now state our main result.

Theorem. *Suppose Λ is a finite-dimensional Koszul algebra such that Λ_1 is a semisimple Λ_0-module and the Yoneda algebra Γ is noetherian. Then the functor θ is a (contravariant) equivalence of triangulated categories.*

If Λ is an exterior algebra, then we recover the BGG correspondence.

References

[1] I. N. Bernstein, I. M. Gel'fand, and S. I. Gel'fand, *Algebraic vector bundles on P^n and problems of linear algebra.* (Russian) Funktsional. Anal. i Prilozhen. **12** (1978), no. 3, 66–67.

[2] R. Martínez Villa and M. Saorín, *Koszul equivalences and dualities.* Pacific J. Math. **214** (2004), no. 2, 359–378.

[1] This category is a universal solution to the problem of inverting an endofunctor, in our case, Ω.