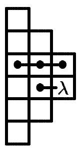
A remark by M. C. R. Butler on subgroup embeddings

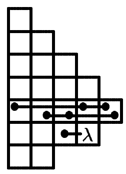
Markus Schmidmeier

An object in the submodule category $\mathcal{S}(\Lambda)$ is a pair $M = (M_0; M_1)$ which consists of a finitely generated Λ-module M_0 together with a Λ-submodule M_1 of M_0. A morphism $f : M \to N$ in $\mathcal{S}(\Lambda)$ is given by a Λ-linear map $f : M_0 \to N_0$ which preserves the submodules, that is, $f(M_1) \subseteq N_1$ holds. In this abstract, Λ usually will be a commutative local uniserial ring; we will call Λ uniserial for short. The radical factor field will be denoted by k and t will be a radical generator (thus $\Lambda/\langle t\rangle = k$).

We have two special cases in mind: In the first case, Λ is the ring $\mathbb{Z}/\langle p^n\rangle$ where p is a prime. Then we are dealing with the category of all possible embeddings of a subgroup in a p^n-bounded finite abelian group. The classification problem for the objects in $\mathcal{S}(\mathbb{Z}/\langle p^n\rangle)$ was raised by Birkhoff [1] in 1934. In the second case, Λ is the factor ring $k[T]/\langle T^n\rangle$ of the polynomial ring one variable T over the field k. Then we consider the possible invariant subspaces of a nilpotent operator: The objects in $\mathcal{S}(k[T]/\langle T^n\rangle)$ may be written as triples (V, ϕ, U), where V is a k-space, $\phi : V \to V$ is a k-linear transformation with $\phi^n = 0$ and U is a subspace of V with $\phi(U) \subseteq U$.

The type $t(B)$ of a finite length Λ-module B is the partition $\mu = (\mu_1, \ldots, \mu_t)$ such that $B \cong \bigoplus_{i=1}^t \Lambda/\langle t^{\mu_i}\rangle$. Thus the pair $(t(B); t(A))$ is an isomorphism invariant of a submodule embedding $(B; A)$. Birkhoff showed that the number of isomorphism classes of subgroup embeddings $(B; A) \in \mathcal{S}(\mathbb{Z}/\langle p^6\rangle)$ with $t(B) = (6, 4, 2)$ and $t(A) = (4, 2)$ tends to infinity with p; namely, for each value $0 < \lambda < p$, the following embeddings are pairwise nonisomorphic. The group B is generated by elements x, y, z of order p^6, p^4, p^2, respectively, and the subgroup A_λ is given by the generators $p^2x + py + z$ and $p^2y + p\lambda z$ of order p^4 and p^2, as pictured below.

$(A_\lambda \subseteq B)$: 

$(C_\lambda \subseteq D)$: 

Is this family $(B; A_\lambda)$ of pairwise nonisomorphic subgroup embeddings the first family which occurs? We know from [2] that the category $\mathcal{S}(\mathbb{Z}/\langle p^5\rangle)$ has finite type, and hence in Birkhoff's example the exponent of the big group, which is p^6, is minimal. However, the exponent of the subgroup, which is p^4, is not minimal, as the above examples of embeddings $(D; C_\lambda)$ in $\mathcal{S}(\mathbb{Z}/\langle p^7\rangle)$ shows, where this exponent is p^3.

For Λ a uniserial ring of length n, and $m \le n$, let $\mathcal{S}_m(\Lambda)$ be the full subcategory of $\mathcal{S}(\Lambda)$ of all pairs $(B; A)$ where $t^m A = 0$. For each pair (n, m) where $m \le n$, the representation type of the category $\mathcal{S}_m(k[T]/\langle T^n\rangle)$ has been determined in [5], see also [4] for several finite cases. Recall that the categories $\mathcal{S}_3(k[T]/\langle T^6\rangle)$ and $\mathcal{S}_2(k[T]/\langle T^7\rangle)$ have finite type, in fact, all categories of type $\mathcal{S}_2(k[T]/\langle T^n\rangle)$ are representation finite. It follows that in case $\Lambda = k[T]/\langle T^n\rangle$, the above two families are minimal in the following sense: If we fix the exponent of the submodule (or the big module) then the exponent of the big module (the submodule, respectively,) is as small as possible.

In the classical case where $\Lambda = \mathbb{Z}/\langle p^n\rangle$, the results in [5] per se do not answer the question whether or not the above two families are minimal. This is the point of Butler's remark. In fact, it is not surprising that the special case $\Lambda = k[T]/\langle T^n\rangle$ is better understood, since in this case many powerful techniques are available (in particular covering theory). In the following we describe two example classes where the representation theory is independent of the underlying commutative uniserial ring Λ. Our last theorem can be used to answer the question in the positive.

Controlled wildness

Let $\mathcal{A}$ be an additive category and $\mathcal{C}$ a class of objects (or a full subcategory) in $\mathcal{A}$. Given objects A, A' in $\mathcal{A}$, we will write $\operatorname{Hom}(A, A')_{\mathcal{C}}$ for the set of maps $A \to A'$ which factor through a (finite) direct sum of objects in $\mathcal{C}$. Here we attach to $\mathcal{C}$ the ideal $\langle\mathcal{C}\rangle$ in $\mathcal{A}$ generated by the identity morphisms of the objects in $\mathcal{C}$. The same convention will apply to a single object C in $\mathcal{A}$: We denote by $\operatorname{Hom}(A, A')_C$ the set of maps $A \to A'$ which factor through a (finite) direct sum of copies of C. Given an ideal $\mathcal{I}$ of $\mathcal{A}$, we write $\mathcal{A}/\mathcal{I}$ for the corresponding factor category, as usual. It has the same objects as $\mathcal{A}$ and for any two objects A, A' of $\mathcal{A}$, the group $\operatorname{Hom}_{\mathcal{A}/\mathcal{I}}(A, A')$ is defined as $\operatorname{Hom}_{\mathcal{A}}(A, A')/\mathcal{I}(A, A')$. In particular, the category $\mathcal{A}/\langle\mathcal{C}\rangle$ has the same objects as $\mathcal{A}$ and $\operatorname{Hom}_{\mathcal{A}/\langle\mathcal{C}\rangle}(A, A') = \operatorname{Hom}_{\mathcal{A}}(A, A')/\operatorname{Hom}(A, A')_{\mathcal{C}}$.

Definition. We say that $\mathcal{A}$ is *controlled k-wild* provided there are full subcategories $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}/\langle\mathcal{C}\rangle$ is equivalent to $\operatorname{mod} k\langle X, Y\rangle$ where $k\langle X, Y\rangle$ is the free k-algebra in two generators. We will call $\mathcal{C}$ the *control class*, and in case $\mathcal{C}$ is given by a single object C then this object C will be the *control object.*

Theorem 1. ([3, Theorem 2]) *Let Λ be a uniserial ring of length $n \geq 7$ and let k be its radical factor. Then the category $\mathcal{S}_4(\Lambda)$ is controlled k-wild.*

Auslander-Reiten quivers in the representation finite case

For $\mathcal{P}$ a (finite) poset, let $\operatorname{sub}_\Lambda \mathcal{P}$ denote the category of Λ-linear subspace representations of $\mathcal{P}$. For example, if $\mathcal{P}$ is the one point poset then $\operatorname{sub}_\Lambda \mathcal{P} = \mathcal{S}(\Lambda)$. We construct Auslander-Reiten sequences which are not split exact in each component.

Notation. For X a Λ-module and S a subset of $\mathcal{P}$ denote by X^S the representation which has the space X in each component labelled by a point in S and which is zero otherwise.

Suppose that $0 \to X \to Y \to Z \to 0$ is an Auslander-Reiten sequence in $\operatorname{mod}\Lambda$ and $i \in \mathcal{P}$ and let $X \to E$ be the injective envelope for X. Then there is an Auslander-Reiten sequence in the category $\operatorname{sub}_\Lambda \mathcal{P}$ of the following type.

$$(*) \qquad 0 \to X^{\leq i} \oplus E^{\not\leq i} \to Y^{=i} \oplus X^{<i} \oplus E^{\not\leq i} \oplus Z^{>i} \to Z^{\geq i} \to 0.$$

This sequence is split exact in each component different from the i-th.

Lemma. *Each other Auslander-Reiten sequence $0 \to A \to B \to C \to 0$ in $\operatorname{sub}_\Lambda \mathcal{P}$ is split exact in each component, that is, for each $i \in \mathcal{P}$, $B_i = A_i \oplus C_i$ holds.*

Corollary. *If Λ is a uniserial ring then the type detects:*

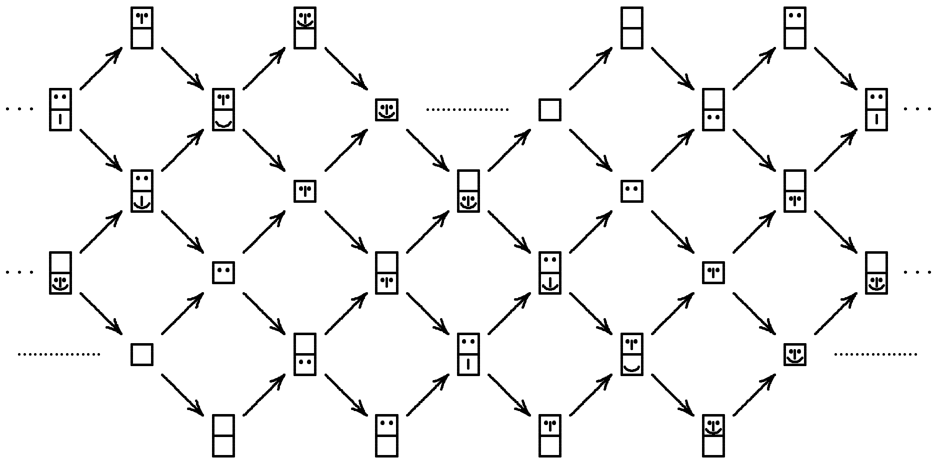
- *projective modules in* $\operatorname{sub}_\Lambda \mathcal{P}$ *and their radicals,*
- *injective modules in* $\operatorname{sub}_\Lambda \mathcal{P}$ *and the end term of their source maps, and*
- *starting terms and end terms of AR-sequences in* $\operatorname{sub}_\Lambda \mathcal{P}$ *of type* $(*)$.

Theorem 2. *Suppose* Λ, Δ *are commutative uniserial rings of the same length,* $\mathcal{C}$ *and* $\mathcal{D}$ *are connected components of their Auslander-Reiten quiver and* $\mathcal{K}$ *and* $\mathcal{L}$ *are slices in* $\mathcal{C}$ *and* $\mathcal{D}$*, respectively. Suppose*

(1) $\mathcal{K}$ *and* $\mathcal{L}$ *are isomorphic as graphs,*
(2) *their points correspond to objects of the same type and*
(3) *certain objects in* $\mathcal{K}$ *are determined uniquely by their type.*

Then the connected components $\mathcal{C}$ *and* $\mathcal{D}$ *are isomorphic as graphs and condition* (2) *holds for all points.*

Example. *Let* $\mathcal{P}$ *be the chain of three points and* Λ *any uniserial ring of length 2. We obtain the following Auslander-Reiten quiver. In fact, the AR-sequences starting at a module with simple total space are exactly the sequences of type* $(*)$*; these sequences form a slice.*



References

[1] G. Birkhoff, *Subgroups of abelian groups*, Proc. Lond. Math. Soc., II. Ser. **38** (1934), 385–401.

[2] F. Richman and E. A. Walker, *Subgroups of p^5-bounded groups,* in: Abelian groups and modules, Trends Math., Birkhäuser, Basel, 1999, pp. 55–73.

[3] C. M. Ringel and M. Schmidmeier, *Submodule Categories of Wild Representation Type*, manuscript, 1–13, see `http://arxiv.org/abs/math/0409417`.

[4] W. Rump, *Irreduzible und unzerlegbare Darstellungen klassischer Ordnungen,* Bayreuther Mathematische Schriften **32**, Bayreuth, 1990.

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