Double Poisson algebras

Michel Van den Bergh

Let k be a field. If Q is a finite quiver and $\bar{Q}$ is its associated double quiver, then $k\bar{Q}/[k\bar{Q}, k\bar{Q}]$ is equipped with a natural Lie bracket $\{-,-\}$, the so-called necklace bracket [1, 5, 6].

The necklace bracket has a connection with representation spaces as follows. Let α be a dimension vector. Then $\mathrm{Rep}(\bar{Q}, \alpha)$ is the cotangent bundle of $\mathrm{Rep}(Q, \alpha)$, and as such it comes equipped with a classical Poisson bracket. The trace map

$$\mathrm{Tr} : k\bar{Q}/[k\bar{Q}, k\bar{Q}] \to \mathcal{O}(\mathrm{Rep}(Q, \alpha))^{Gl(\alpha)}$$

is a Lie algebra homomorphism.

This theory is somewhat unsatisfying since

- As $k\bar{Q}/[k\bar{Q}, k\bar{Q}]$ has no algebra structure, it cannot itself be regarded as a kind of (non-commutative) Poisson algebra.
- The above trace map only explains the Poisson bracket between *invariant* functions on $\mathrm{Rep}(Q, \alpha)$.

To solve these problems we introduce the notion of a *double Poisson* structure on a non-commutative algebra A [7]. This is by definition a bilinear map

$$\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$$

satisfying suitable analogues of the axioms of a Poisson algebra. If A is a double Poisson algebra then $A/[A, A]$ carries an induced Lie bracket $\{-,-\}$ and furthermore all representation spaces of A carry an induced Poisson bracket.

We show that $k\bar{Q}$ has a natural double Poisson structure whose associated Lie bracket is the necklace bracket and which induces the standard Poisson structure on $\mathrm{Rep}(\bar{Q}, \alpha)$.

The algebra DA of double poly-vector fields associated to A is defined as $T_A \,\mathrm{Der}(A, A \otimes A)$ [2]. This definition can be motivated by showing that the elements of DA induce poly-vector fields on all representation spaces. We show that DA has a natural (super) double Poisson structure which induces the Schouten bracket on all representation spaces. If A is quasi-free, then a double Poisson bracket on A can be described as an element P of $(DA/[DA, DA])_2$ such that $\{P, P\} = 0$.

For more information on non-commutative Poisson geometry, and in particular an application to the multiplicative preprojective algebras recently introduced by Crawley-Boevey and Shaw [4], see [7]. For a related approach see [3].

References

[1] R. Bocklandt and L. Le Bruyn, *Necklace Lie algebras and noncommutative symplectic geometry*, Math. Z. **240** (2002), no. 1, 141–167.

[2] W. Crawley-Boevey, *Preprojective algebras, differential operators and a Conze embedding for deformations of Kleinian singularities*, Comment. Math. Helv. **74** (1999), no. 4, 548–574.

[3] W. Crawley-Boevey, P. Etingof, and V. Ginzburg, *Noncommutative geometry and quiver algebras*, available as `math.AG/0502301`.

[4] W. Crawley-Boevey and P. Shaw, *Multiplicative preprojective algebras, middle convolution and the Deligne-Simpson problem*, available as `math.RA/0404186`.

[5] V. Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett. **8** (2001), no. 3, 377–400.

[6] M. Kontsevich, *Formal (non)commutative symplectic geometry*, The Gel'fand Mathematical Seminars, 1990–1992 (Boston, MA), Birkhäuser Boston, Boston, MA, 1993, pp. 173–187.

[7] M. Van den Bergh, *Double Poisson algebras*, available as `math.QA/0410528`.