1. Introduction

Given an artin algebra \( A \), the finitistic dimension of \( A \) is defined to be the supremum of the projective dimensions of the finitely generated left \( A \)-modules of finite projective dimension. The famous finitistic dimension conjecture says that for any artin algebra \( A \) the finitistic dimension of \( A \) is finite. This conjecture was proposed 45 years ago and still remains open, and has been related to at least five other homological conjectures (see the last 6 conjectures of the total 13 conjectures in the book [4]):

**Strong Nakayama conjecture** [7]: If \( M \) is a non-zero module over an artin algebra \( A \), then there is an integer \( n \geq 0 \) such that \( \text{Ext}^n_A(M, A) \neq 0 \).

**Generalized Nakayama conjecture** [2]: If \( 0 \to A \to I_0 \to I_1 \to \ldots \) is a minimal injective resolution of an artin algebra \( A \), then any indecomposable injective is a direct summand of some \( I_j \). Equivalently, if \( M \) is a finitely generated \( A \)-module such that \( \text{add}(A) \subseteq \text{add}(M) \) and \( \text{Ext}^i_A(M, M) = 0 \) for all \( i \geq 1 \), then \( M \) is projective.

**Nakayama conjecture** [15]: If all \( I_j \) in a minimal injective resolution of an artin algebra \( A \), say \( 0 \to A \to I_0 \to I_1 \to \ldots \), are projective, then \( A \) is self-injective.

**Gorenstein symmetry conjecture**: Let \( A \) be an artin algebra. If the injective dimension of \( A \) is finite, then the injective dimension of \( A \) is finite.

In general, all the above conjectures are still open. They have the following well-known relationship: The finitistic dimension conjecture \( \implies \) the strong Nakayama conjecture \( \implies \) the generalized Nakayama conjecture \( \implies \) the Nakayama conjecture. And, the finitistic dimension conjecture \( \implies \) the Gorenstein symmetry conjecture.

In this talk I shall report on some new developments attacking the finitistic dimension conjecture. Our idea to approach the conjecture is to use a chain of subalgebras with certain radical conditions. Let us introduce the following notion:

**Definition.**

1. Given an artin algebra \( A \), we say that the left representation distance of \( A \), denoted by \( \text{rep.dis}(A) \), is the minimum of the lengths of chains of subalgebras \( A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_s \) such that \( \text{rad}(A_i) \) is a left ideal in \( A_{i+1} \) for all \( i \) and that \( A_s \) is representation-finite. Here we have denoted the Jacobson radical of \( A \) by \( \text{rad}(A) \).

2. A homomorphism \( f : B \to A \) between two algebras \( A \) and \( B \) is said to be radical-full if \( \text{rad}(A) = \text{rad}(B) \).

Note that the left representation distance of an artin algebra is always finite by [19] and invariant under Morita equivalences. Every surjective homomorphism is radical-full. Note that if \( B \) is a subalgebra of an artin algebra \( A \), the inclusion map being radical-full does not imply that \( \text{rad}(B) \) is a left ideal in \( A \).
2. Main results

In this section we shall summarize some new results in the recent papers [19, 20]. For some known results on finitistic dimension conjecture we refer to [3, 9, 10, 8, 11, 16] and many other papers. (I apologize that I could not display all literature here.)

Our main results are the following.

**Theorem 1.** Let $A$ be an artin algebra.

1. If $\text{lreps} \dim(A) \leq 2$, then the finitistic dimension conjecture is true for $A$.
2. Let $B$ be a subalgebra of $A$ such that $\text{rad}(B)$ is a left ideal of $A$ and that the inclusion map is radical-full. If the global dimension of $A$ is at most 4, then the finitistic dimension conjecture is true for $B$.

We may also use a chain of factor algebras to bound the finitistic dimension. In this direction, we have the following result.

**Theorem 2.** Let $A$ be an artin algebra, and let $I$ and $J$ be two ideals in $A$ with $IJ \text{rad}(A) = 0$. If $A/I$ and $A/J$ are representation-finite, then the finitistic dimension conjecture is true for $A$.

The proofs of Theorem 1 and Theorem 2 are based on the following lemmas.

**Lemma 1.** Suppose $B$ is a subalgebra of $A$ such that $\text{rad}(B)$ is a left ideal in $A$. Then, for any $B$-module $X$ and integer $i \geq 2$, there is a projective $A$-module $Q$ and an $A$-module $Z$ such that $\Omega_B^i(X) \cong \Omega_A(Z) \oplus Q$ as $A$-modules, where $\Omega_B$ stands for the first syzygy over the algebra $B$.

**Lemma 2** ([11]). For any artin algebra $A$ there is a function $\Psi$ from the finitely generated $A$-modules to the non-negative integers such that

1. $\Psi(M) = \text{proj.dim}(M)$ if $M$ has finite projective dimension.
2. For any natural number $n$, $\Psi(\bigoplus_{j=1}^n M) = \Psi(M)$.
3. For any $A$-modules $X$ and $Y$, $\Psi(X) \leq \Psi(X \oplus Y)$.
4. If $0 \to X \to Y \to Z \to 0$ is an exact sequence in $A$-mod such that the projective dimension of $Z$ is finite, then $\Psi(Z) \leq \Psi(X \oplus Y) + 1$.

Based on the above results, there are many elementary questions, for example, if the left representation distance of $A$ is 3, could we prove the finitistic dimension conjecture for $A$? For more information and the details of the proofs of the above main results we refer to the papers [19, 20]. Preprints can be downloaded from http://math.bnu.edu.cn/~ccxi/.

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**References**


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