Auslander-Reiten sequences, locally free sheaves and Chebysheff polynomials

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Let R be the exterior algebra in $n+1$ variables, and let S denote the symmetric algebra in $n+1$ variables. It is well known that R is a selfinjective Koszul algebra and S is its Koszul dual. By $\mathcal{K}_R$ and $\mathcal{K}_S$ we denote the categories of linear R-modules (S-modules respectively) where the morphisms are the degree zero homomorphisms. The Koszul duality can be then used to obtain mutually inverse dualities between the category of linear R-modules and that of the linear S-modules:

$$\mathcal{K}_R \underset{\mathcal{F}}{\overset{\mathcal{E}}{\rightleftarrows}} \mathcal{K}_S$$

By $\mathrm{coh}(\mathbf{P}^n)$ we denote the category of coherent sheaves over the projective n-space, and if M is a finitely generated graded S-module, we set $\widetilde{M} \in \mathrm{coh}(\mathbf{P}^n)$ to be its sheafification. A theorem of Avramov and Eisenbud [1] states that for every finitely generated graded S-module M, there exists an integer k such that the shifted truncation $M_{\geq k}[-k] = M_k \oplus M_{k+1} \oplus \ldots [-k]$ is a linear S-module. This means that every indecomposable coherent sheaf $\mathcal{F}$, is the sheafification of the graded shift of some linear R-module, and using the Koszul duality we can write $\mathcal{F} \cong \widetilde{\mathcal{E}(M)}[i]$ for some linear R-module M and some integer i.

The main ingredient is the following theorem from [2]:

Theorem 1. *Let M be an indecomposable linear nonfree R-module. Then there exists an exact sequence $0 \to A \to B \to M \to 0$ that is an Auslander-Reiten sequence in $\mathcal{K}_R$. Moreover the Loewy length of A is precisely 2.*

It turns out that if we denote by J the radical of R, then the module M/J^2M is indecomposable, and the induced sequence $0 \to A \to B/J^2B \to M/J^2M \to 0$ is an Auslander-Reiten sequence in the category $\mathrm{gr}_0\, R/J^2$ of graded R/J^2-modules generated in degree zero. Since the algebra R/J^2 is stably equivalent to the generalized Kronecker algebra we can use this algebra to describe the Auslander-Reiten quiver of $\mathcal{K}_R$:

Theorem 2. *Let R denote the exterior algebra in $n+1$ variables, where $n > 1$. The A-R quiver of $\mathcal{K}_R$ has a connected component that coincides with the preinjective component of $\mathrm{gr}_0\, R/J^2$, a component consisting only of the module R, and all the remaining connected components are quivers of the type $\mathbf{N}^- A_\infty$.*

We can use now the Koszul duality and Serre's theorem to show that certain subcategories of the category of coherent sheaves over the projective n-space have one sided Auslander-Reiten sequences, and describe the shapes of their Auslander-Reiten quivers. First, for each integer i, denote by $\mathcal{K}_S[i]$ the graded shifts of the category of linear S-modules and by $\widetilde{\mathcal{K}_S[i]}$ their sheafifications. We have the following:

Theorem 3. *For each integer i, the category $\widetilde{\mathcal{K}_S[i]}$ has left Auslander-Reiten sequences, that is, given an indecomposable coherent sheaf $\mathcal{F}$ in $\widetilde{\mathcal{K}_S[i]}$, there exist an exact sequence*

$$0 \to \mathcal{F} \to \mathcal{B} \to \mathcal{C} \to 0$$

that is almost split in $\widetilde{\mathcal{K}_S[i]}$. Moreover, the Auslander-Reiten quiver of the subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^n)$, where $n > 1$, has one preprojective component, and the remaining components are all quivers of the type $\mathbf{N}A_\infty$.

We can use the fact that every coherent sheaf is the sheafification of a linear S-module, to compute the rank of a locally free sheaf. Namely, if $\mathcal{F}$ is locally free, then we can write $\mathcal{F} = \widetilde{\mathcal{E}(M)}$ for some linear R-module M, and then we have $\mathrm{rk}\, F = \sum_{i=0}^{p}(-1)^i \dim M_i$ where the M_i are the graded pieces of M. Using the Koszul duality we can prove that if a component of the Auslander-Reiten quiver of some $\widetilde{\mathcal{K}_S[i]}$ contains a locally free sheaf, then all the sheaves in that component are locally free. Then it is easy to show that each sheaf lying in the preprojective component of one of the $\widetilde{\mathcal{K}_S[i]}$ is locally free. We have the following:

Proposition. *Let $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2 \ldots$ be the locally free sheaves lying in the preprojective component of the subcategory $\widetilde{\mathcal{K}_S[i]}$ of $\mathrm{coh}(\mathbf{P}^n)$. Denote by*

$$T_k(x) = \sum_{m=0}^{[k/2]} (-1)^m \binom{k-m}{m} (2x)^{k-2m}$$

the Chebysheff polynomials of the second kind. Then for each $k \geq 1$, we have

$$\mathrm{rk}\, \mathcal{F}_k = T_k(\frac{n+1}{2}) - T_{k-1}(\frac{n+1}{2}).$$

In addition, if $n > 1$, then for each k, $\mathrm{rk}\, \mathcal{F}_{k+1} > \mathrm{rk}\, \mathcal{F}_k$.

It is a long standing problem to determine whether there are indecomposable locally free sheaves of small ranks over $\mathbf{P^n}$. In this regard we have the following theorem.

Theorem 4. *Let $n > 1$. For each integer i, each A-R component of $\widetilde{\mathcal{K}_S[i]}$ contains at most one locally free sheaf of rank less than n. Moreover, in each component the ranks increase exponentially.*

References

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