## Übungsaufgaben 4.

Let $k$ be a field.

**1.** Determine all minimal generating sets for the 1-module

$$(k^3, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}).$$

**2.** Let $V = k^3$ with basis $e_1, e_2, e_3$. Define linear maps $\phi, \alpha, \beta \colon V \to V$ by

$$\begin{aligned} \phi(e_1) &= e_3, & \phi(e_2) &= e_3, & \phi(e_3) &= 0, \\ \alpha(e_1) &= e_3, & \alpha(e_2) &= 0, & \alpha(e_3) &= 0, \\ \beta(e_1) &= 0, & \beta(e_2) &= e_3, & \beta(e_3) &= 0. \end{aligned}$$

We are interested in the 1-module $(V, \phi)$ and the 2-module $(V, \alpha, \beta)$.

Show: (a) Every submodule of $(V, \alpha, \beta)$ is also a submodule of $(V, \phi)$.

(b) $(V, \phi)$ has additional submodules! For example, if $k = \mathbb{F}_p$ is the finite field with $p$ elements, then there are precisely $p$ subspaces of $V$, which are submodules of $(V, \phi)$, but not submodules $(V, \alpha, \beta)$.

(c) $(V, \alpha, \beta)$ is indecomposable, but $(V, \phi)$ is decomposable.

**3. The module $N(\infty)$.** Let $k[T]$ be the polynomial ring in one variable $T$ with coefficients in $k$. Let $k[T, T^{-1}]$ be the following ring: start with the vector space with basis $\{T^z \mid z \in \mathbb{Z}\}$, and define a multiplication on this basis as follows: $T^z T^{z'} = T^{z+z'}$. Using the multiplication map $\phi = T\cdot$ both $k[T]$ as well as $k[T, T^{-1}]$ are 1-modules. Of course, $(k[T], T\cdot)$ is a submodule of $(k[T, T^{-1}], T\cdot)$.

(a) Show: The factor module $(k[T, T^{-1}], T\cdot)/(k[T], T\cdot)$ is isomorphic to $N(\infty)$.

(b) Let $k = \mathbb{R}$ be the field of the reell numbers. Differentiation $p \mapsto \frac{\mathrm{d}}{\mathrm{d}\,T} p$ is a linear map $\frac{\mathrm{d}}{\mathrm{d}\,T} \colon k[T] \to k[T]$. Show: The 1-module $(k[T], \frac{\mathrm{d}}{\mathrm{d}\,T})$ is isomorphic to $N(\infty)$.

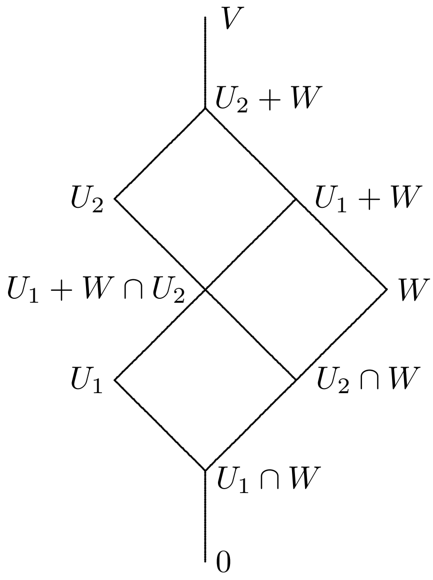
**4.** Let $U_1, U_2, W$ be submodules of the module $V$ such that $U_1 \subseteq U_2$. Show: The set of submodules

$$0,\ U_1,\ U_2,\ W,\ U_1 \cap W,\ U_2 \cap W,\ U_1 + W,\ U_2 + W,\ U_1 + W \cap U_2,\ V$$

is closed under $\cap$ and $+$ (thus, we obtain a “sublattice” of the lattice of all submodules of $V$.)

Hint (nothing is difficult, the only problem is to order the arguments conveniently):

(a) There are the following obvious inclusions:




Note that if $U, U'$ are submodules of $U$ with $U \subseteq U'$, then $U \cap U' = U$ und $U + U' = U'$, thus we only have to consider pairs of submodules $U, U'$ which are not necessarily comparable.

(b) There are only five pairs $(U, U')$ of submodules in the list which may be incomporable:

$$(U_2, U_1 + W), \quad (U_2, W), \quad (U_1 + W \cap U_2, W), \quad (U_1, W), \quad (U_1, U_2 \cap W).$$

Always, we have to determine $U + U'$ as well as $U \cap U'$. Thus, we have to show that any of these 10 submodules belongs to the given list. (In nearly all cases, nothing has to be shown!)