Übungsaufgaben 6.

Linear Relations.

Recall that a linear relation on a vector space W is a subspace of $W \times W$. We consider linear relations as Q_0-free Kronecker modules (for example we call ρ a band provided $\pi_1 u$ and $\pi_2 u$ are both bijective, where π_i are the canonical projections $W \times W \to W$ und $u\colon \rho \to W \times W$ is the inclusion map). The composition of two relations ρ, ρ' on W will be denoted by $\rho \circ \rho'$ or just by $\rho\rho'$. Thus ρ^t is the t-fold composition of ρ.

1. Let W be a vector space.

(a) Given subspaces W', W'' of W and a linear transformation $f\colon W'' \to W/W'$, define

$$\rho(f) = \{(w_1, w_2) \mid f(w_2) = w_1 + W' \text{ with } w_2 \in W'', w_1 \in W\}.$$

Show: The map $f \mapsto \rho(f)$ defines a bijection of sets

$$\{\text{triples } (W', W'', f\colon W'' \to W/W')\} \longrightarrow \{\text{relations on } W\}.$$

(b) Show that this implies: There is a bijection between the set of relations on W and five-tuples $(W_1', W_2', W_1'', W_2''; g)$ where $W_1' \subseteq W_2'$ and $W_1'' \subseteq W_2''$ are subspaces of W and $g\colon W_2''/W_1'' \to W_2'/W_1'$ is an isomorphism.

2. Show that the following conditions are equivalent:

(*i*) ρ is a band.

(*ii*) $\rho \circ \rho^{-1} = \{(w, w) \mid w \in W\} = \rho^{-1} \circ \rho$.

3. Show

(a) If $\rho = P_t$ with $t \geq 0$, then ρ^{t+1} is the direct sum of $t+1$ copies of P_0.

(b) If $\rho = Q_t$ with $t \geq 1$, then ρ^t is the direct sum of t copies of Q_0.

4. If ρ is any relation, then

$$\rho \circ \rho^{-1} \circ \rho = \rho.$$