Übungsaufgaben 7.

Representations of gentle quivers.

The structure theorem for representations of gentle quivers asserts that every indecomposable representation is a string or a band. Use this theorem in order to answer the following questions:

1. Homology. Let $Q = (Q_0, Q_1, Q_2)$ be a complete gentle quiver. If M is a finite-dimensional representation of Q, define

$$H(M) = \bigoplus\nolimits_{\alpha \in Q_1} (\alpha')^{-1} 0 / \alpha 1.$$

Claim: If M is a band module, then $H(M) = 0$. If M is a string module, then $\dim H(M) = 2$.

2. The complete gentle tree. Let $Q = (Q_0, Q_1, Q_2)$ be the complete gentle tree. Show that every indecomposable representation $M = (M_x, M_\alpha)$ of Q is thin and for any α, the map M_α is injective or surjective.

3. Commutativity relations. Let $Q = (Q_0, Q_1, Q_2)$ be a complete gentle quiver, let $x, y \in Q_0$. Let $\alpha_r \cdots \alpha_2 \alpha_1$ and $\beta_s \cdots \beta_2 \beta_1$ be words starting in x and ending in y, such that all the α_i, β_j are arrows. Assume in addition that $\alpha_1 \neq \beta_1$ and $\alpha_r \neq \beta_s$. Let $w = \alpha_r \cdots \alpha_2 \alpha_1 \beta_1^{-1} \beta_2^{-1} \cdots \beta_s^{-1}$. Show

(a) The only indecomposable representation $M = (M_x, M_\alpha)$ of Q which satisfies $\alpha_r \cdots \alpha_2 \alpha_1 = \beta_s \cdots \beta_2 \beta_1 \neq 0$ is the band module $M(w, \phi)$ with $\phi = 1 \colon k \to k$.

(a) The only indecomposable representation $M = (M_x, M_\alpha)$ of Q which satisfies $\alpha_r \cdots \alpha_2 \alpha_1 = -\beta_s \cdots \beta_2 \beta_1 \neq 0$ is the band module $M(w, \phi)$ with $\phi = -1 \colon k \to k$.

4. Double complex. A double complex is by definition of the form $(M_{ij}, d^h_{ij}, d^v_{ij})$ with index set $\mathbb{Z} \times \mathbb{Z}$, such that all M_{ij} are vector spaces, $d^h_{ij} \colon M_{ij} \to M_{i-1,j}$ and $d^v_{ij} \colon M_{ij} \to M_{i,j-1}$ are linear transformations and the following relations are satisfied:

$$d^h \circ d^h = 0, \quad d^v \circ d^v = 0, \quad d^h \circ d^v + d^v \circ d^h = 0.$$

Use 3 in order to determine all indecomposable double complexes.