Übungsaufgaben 10.

The socle of a module.

Definition: Let M be a module. The socle $\operatorname{soc}(M)$ of M is by definition the sum of all simple submodules of M. Show:

(1). The socle $\operatorname{soc}(M)$ of M is semisimple and it contains any semisimple submodule of M. In particular, $\operatorname{soc}(\operatorname{soc}(M)) = \operatorname{soc}(M)$.

(2). If U is a submodule of M, then $\operatorname{soc}(U) = U \cap \operatorname{soc}(M)$.

(3). If U, U' are submodules of M, then $\operatorname{soc}(U) + \operatorname{soc}(U') \subseteq \operatorname{soc}(U + U')$.

(4). If V_i $(i \in I)$ are modules, then $\operatorname{soc}(\bigoplus_{i\in I} V_i) = \bigoplus_{i\in I} \operatorname{soc}(V_i)$.

(5). If $f\colon M \to N$ is a module homomorphism, then $f(\operatorname{soc}(M)) \subseteq \operatorname{soc}(N)$.

(6). There exists a module M such that $\operatorname{soc}(M/\operatorname{soc}(M)) \neq 0$.

(7). There exists a module M with submodules U, U' such that

$$\operatorname{soc}(U) + \operatorname{soc}(U') \neq \operatorname{soc}(U + U').$$

(8). There exists a module homomorphism $f\colon M \to N$, such that

$$f(\operatorname{soc}(M)) \neq \operatorname{soc}(N).$$