Übungsaufgaben 12.

Local rings

1. Show: The endomorphism ring of the 1-module $N(\infty)$ is a local ring.

2. Let V a vector space (over the field k). Define a multiplication on $k \oplus V$ as follows

$$(x, v)(x', v') = (xx', xv' + x'v),$$

for $x \in k$ and $v \in V$.

(a) Show: With this multiplication, $k \oplus V$ is a local ring with radical $I = 0 \oplus V$, and $I^2 = 0$. This ring is usually denoted by $k \ltimes V$.

(b) Let M be any module with $\operatorname{soc} M \subseteq \operatorname{rad} M$. Let $u\colon \operatorname{soc} M \to M$ be the inclusion map and $p\colon M \to M/\operatorname{rad}(M)$ be the canonical projection. Show that the subring of $\operatorname{End}(M)$ generated by the k-multiples of 1 and the elements ufp, where $f\colon M/radM \to \operatorname{soc} M$ is a homomorphism, is isomorphic as a ring to $k \ltimes \operatorname{Hom}(M/radM, \operatorname{soc} M)$.

(Remark: The definition $(x, v)(x', v') = (xx', xv' + x'v)$ can be used only in case that k is commutative, since otherwise this multiplication will not be associative! A better definition would be $(x, v)(x', v') = (xx', xv' + vx')$, meaning that we are allowed to multiply vectors both from the left as well as from the right by scalars. For k commutative, just let $vx = xv$ for $x \in k$ and $v \in V$; in general we need that V is what is called a k-k-bimodule.)

3. Let k be a field and $k(X)$ the field of rational funktions, its elements are of the form f/g, where f, g are polynomials with coefficients in k and $g = 0$.

Let $a \in k$. Show that the set of elements f/g with $g(a) \neq 0$ is a subring of $k(T)$ which is a local ring. (What is its maximal ideal?).

Remark. These kind of rings are the origin of considering local rings — here, "local" means that one fixes the position a and looks at the behaviour of functions (here rational functions) in the neighborhood of a.

4. Let k be a field.

(a) Let R be a k-algebra. Assume that R is a local ring with radical I such that $R = k \oplus I$. Show that any subring of R which contains $k = k \cdot 1_R$ is a local ring.

(b) Determine all local subrings of the ring of all upper triangular (3×3)-matrices with coefficients in k. (Remark: By definition, a subring of the ring R with identity element 1_R is supposed to contain the element 1_R.)

Zusatz-Aufgabe. Let k be an algebraically closed field. Determine for any indecomposable Kronecker module M its endomorphsim ring $\operatorname{End}(M)$.