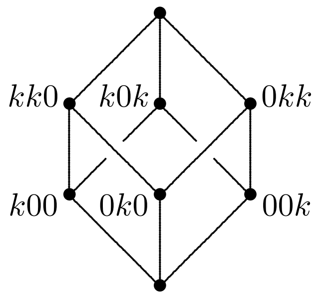
Remark to Exercise 3.1.: There are many possibilities! Let us look at the following easier problem:

Problem. Let k be a field and let $V = (k^3, \phi, \psi)$ be a module such that

$$\phi = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$$

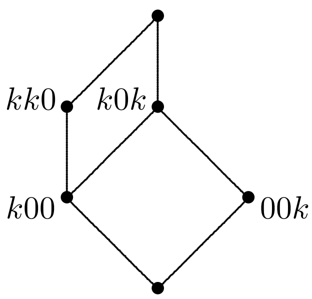
with pairwise different $\lambda_i \in k$. How can the lattice of submodules of V look like?

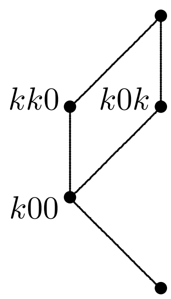
Solution. There are 9 essentially different cases. The submodule lattice L of $V = (k^3, \phi)$ itself looks as follows:

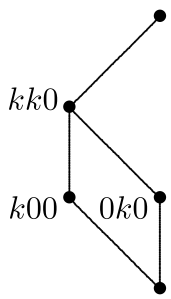


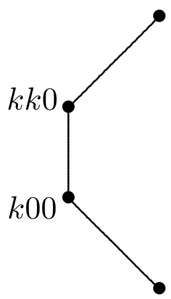
All the sublattices L' of L which include 0 and V can be realized by chosen a suitable map ψ. Since the “length” of the lattice L is 3, the lattice L' can have length $1, 2$ or 3.

Length of L' is 3. There are 4 essentially different cases where L' has length 3. We display cases where both $k00, kk0$ belong to L'; in the first three cases we still have made an additional choice:



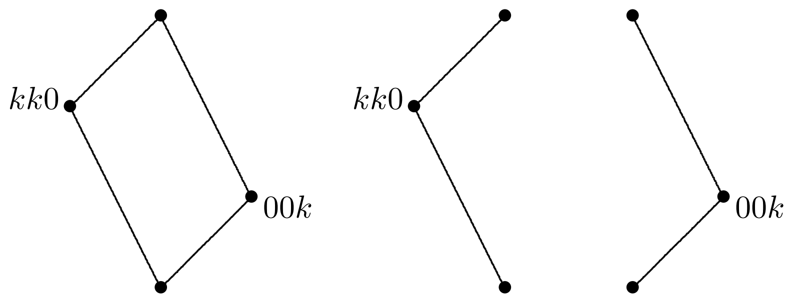






$$\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 1 \\ & 0 & \\ & & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & & 1 \\ & 0 & 1 \\ & & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

Length of L' is 2. There are three essentially different cases with L' being of length 2. We display the cases where $k00$ or $kk0$ belong to L':



$$\psi = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix} \quad \psi = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & 1 \\ & & 0 \end{bmatrix} \quad \psi = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & 1 & 0 \end{bmatrix}$$

Length of L' is 1. In order to achieve that L' has length 1 (thus (V, ϕ, si) is simple), we may take for example $\psi = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & 1 \\ & 1 & 0 \end{bmatrix}$