G_1T-MODULES, AR-COMPONENTS, AND GOOD FILTRATIONS

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0. Motivation

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over an algebraically closed field k. The structure of $\mathfrak{g}$ and its modules is usually analyzed by considering weight space decompositions relative to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \quad ; \quad M = \bigoplus_{\lambda \in X(M)} M_\lambda.$$

Here $R \subset \mathfrak{h}^* \setminus \{0\}$ and $X(M) \subset \mathfrak{h}^*$ are finite subsets, and $\mathfrak{g}_\alpha, M_\lambda$ denote the root spaces and weight spaces of $\mathfrak{g}$ and M, respectively.

In the classical situation, that is, when $\mathfrak{g}$ is semisimple and $\mathrm{char}(k) = 0$, these decompositions define grading of $\mathfrak{g}$ and M relative to a finitely generated subgroup $Q \subset \mathfrak{h}^*$. This group is torsion free (and hence free), which has a number of consequences:

- If $\alpha \neq 0$, then there exists $n \in \mathbb{N}$ with $(n\alpha + R) \cap R = \emptyset$. Consequently, $(\mathrm{ad}\, x_\alpha)^n(\mathfrak{g}) = (0)$ for $x_\alpha \in \mathfrak{g}_\alpha$. Thus, these elements act via nilpotent transformations. By the same token, x_α acts nilpotently on M.
- By choosing postive roots corresponding to a Borel subalgebra one obtains a partial ordering on Q that is compatible with the addition.

If $\mathrm{char}(k) = p > 0$, then we have

$$p\,\lambda = 0 \quad \forall\ \lambda \in \mathfrak{h}^*,$$

so that one obtains a grading relative to a p-elementary abelian group rather than a torsion free group.

Example. We consider the p-dimensional Witt algebra $W(1) := \mathrm{Der}(k[X]/(X^p))$. Setting $x := X + (X^p)$, we denote by ∂ the derivation induced by the partial derivative of $k[X]$. Then $W(1) = \bigoplus_{i=-1}^{p-2} ke_i$, where $e_i := x^{i+1}\partial$. We have

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j} & -1 \leq i+j \leq p-2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\mathfrak{h} := ke_0$ is a Cartan subalgebra and $W(1)$ is $\mathbb{Z}$-graded.

Setting $y := x + 1$, we obtain another basis of $W(1)$ by defining $e'_i := y^i\partial$. This time we have

$$[e'_i, e'_j] = (j-i)e'_{i+j},$$

where the indices are to be interpreted $\mathrm{mod}(p)$. For instance, we have

$$[e'_1, e'_{p-2}] = [y^2\partial, y^{p-1}\partial] = (y^2\partial(y^{p-1}) - y^{p-1}\partial(y^2))\partial = (p-3)y^p\partial = (p-3)\partial = (p-3)e'_{-1}.$$

Thus, for $p \neq 3$, we obtain a $\mathbb{Z}/(p)$-grading. (For $p = 3$ we have $W(1) \cong \mathfrak{sl}(2)$).

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1. G_1T-modules

From now on we assume that $\mathrm{char}(k) = p > 0$ and let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of a reductive group G. Then $\mathfrak{g}$ shares many properties with a complex semisimple Lie algebra. For instance,

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$$

affords a triangular decomposition, with $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{g}^+$ being a Borel subalgebra. We know that $(\mathfrak{g}, [p])$ is a restricted Lie algebra and that the three constituents above are p-subalgebras. We consider the restricted enveloping algebra

$$U_0(\mathfrak{g}) := U(\mathfrak{g})/(\{x^p - x^{[p]} \ ; \ x \in \mathfrak{g}\}).$$

Given an algebra homomorphism $\lambda : U_0(\mathfrak{b}) \longrightarrow k$ one defines the corresponding *baby Verma module* via

$$Z(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} k_\lambda.$$

The idea was to prove results similar to those obtained by Bernstein-Gel'fand-Gel'fand in their famous 1976 paper on the category $\mathcal{O}$. When Jantzen tried to carry out this program in 1978, he encountered the aforementioned problems concerning the weights. He overcame these obstacles by defining the category of G_1T-modules.

By general theory, one can find a maximal torus $T \subset G$ such that $\mathfrak{h} = \mathrm{Lie}(T)$. Recall that T acts on $\mathfrak{g}$ via the adjoint representation

$$\mathrm{Ad} : T \longrightarrow \mathrm{GL}(\mathfrak{g}).$$

In fact, T operates via automorphisms of the restricted Lie algebra $(\mathfrak{g}, [p])$ and the subalgebras $\mathfrak{h}$, $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are T-invariant. Consequently, T also acts on the corresponding restricted enveloping algebras, so that we obtain an operation

$$\mathrm{Ad} : T \longrightarrow \mathrm{Aut}_k(U_0(\mathfrak{g}))$$

of T on $U_0(\mathfrak{g})$

Let $X(T) := \mathrm{Hom}(T, k^\times)$ be the character group of T. Since T is a torus, $X(T)$ is a finitely generated torsion free abelian goup. If V is a finite dimensional T-module, then

$$V = \bigoplus_{\lambda \in X(T)} V_\lambda,$$

where $V_\lambda = \{v \in V \ ; \ t \cdot v = \lambda(t)v \ \ \forall\, t \in T\}$. Here is Jantzen's definition of a G_1T-module:

Definition. A finite dimensional k-vector space V is a G_1T*-module* if

(a) V is a $U_0(\mathfrak{g})$-module,

(b) V is a T-module,

(c) we have
$$t(uv) = \mathrm{Ad}(t)(u)(tv) \quad \ \forall\, t \in T,\, u \in U_0(\mathfrak{g}),\, v \in V,$$

(d) the differential $\mathfrak{h} \longrightarrow gl(V)$ of the T-action coincides with the action of $\mathfrak{h}$ coming from (a).

Remarks. (i) Recall that $\mathfrak{g}$ corresponds to the first Frobenius kernel $G_1 \lhd G$, so that $\mathrm{mod}\, U_0(\mathfrak{g}) = \mathrm{mod}\, G_1$. By (a) and (b) the vector space V is a T-module and a G_1-module. Condition (d) ensures that the restrictions of these two actions to the first Frobenius kernel $T_1 = G_1 \cap T$ coincide.

(ii) Thanks to condition (c) we have an action of the semidirect product $G_1 \rtimes T$ on V. In view of (d), this action is trivial on

$$T_1 \xrightarrow{\sim} \{(t^{-1}, t) \ ; \ t \in T_1\}.$$

Consequently, the action factors through to

$$G_1T \cong (G_1 \rtimes T)/T_1.$$

Since T acts on $U_0(\mathfrak{g})$ via automorphisms, we have a decomposition

$$U_0(\mathfrak{g}) = \bigoplus_{\alpha\in X(T)} U_0(\mathfrak{g})_\alpha \quad \text{with} \quad U_0(\mathfrak{g})_\alpha U_0(\mathfrak{g})_\beta \subset U_0(\mathfrak{g})_{\alpha+\beta}.$$

In other words, $U_0(\mathfrak{g})$ is an $X(T)$-graded algebra.

A vector space V fulfills conditions (a)-(c) if and only if it is a $X(T)$-graded $U_0(\mathfrak{g})$-module. Let us disregard condition (d) for the time being and focus on the category $\operatorname{mod}_{X(T)} U_0(\mathfrak{g})$ of $X(T)$-graded modules and degree 0 homomorphisms. This is a *Frobenius category*, that is, it has enough projectives and enough injectives, and the projectives coincide with the injectives. We thus have all the tools from the theory of self-injective algebras at our disposal.

Recall that $\operatorname{mod}_{X(T)} U_0(\mathfrak{g})$ decomposes into blocks: Two simples S, S' belong to the same block, if there exist finitely many simple modules $S = S_1, S_2, \ldots, S_n = S'$ such that $\operatorname{Ext}^1(S_i, S_{i+1}) \oplus \operatorname{Ext}^1(S_{i+1}, S_i) \neq (0)$ for $1 \leq i \leq n-1$. Modules belong to a block if all their composition factors do.

By definition, $\operatorname{mod} G_1T \subset \operatorname{mod} G_1 \rtimes T$ is the full subcategory of $\operatorname{mod}_{X(T)} U_0(\mathfrak{g})$, whose objects are the G_1T-modules.

Lemma 1.1. *The full subcategory* $\operatorname{mod} G_1T \subset \operatorname{mod}_{X(T)} U_0(\mathfrak{g})$ *is a sum of blocks of* $\operatorname{mod}_{X(T)} U_0(\mathfrak{g})$.

Example. We consider $\operatorname{mod} \mathrm{SL}(2)_1T$, where $T := \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \; ; \; t \in k\}$. Then we have $X(T) \cong \mathbb{Z}$, where $i \in \mathbb{Z}$ corresponds to $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^i$. For $i \equiv -1 \mod(p)$, the module $\hat{P}(i) = \hat{L}(i)$ is simple. Otherwise the Loewy factors of $\hat{P}(i)$ are given by

$$\hat{P}(i) = \begin{array}{c} \hat{L}(i) \\ \hat{L}(-i-2) \oplus \hat{L}(-i-2+2p) \\ \hat{L}(i) \end{array}.$$

Hence we have blocks $\mathcal{B}(i)$ containing simple modules $\{\hat{L}(i+2np), \hat{L}(2np-i-2) \; ; \; n \in \mathbb{Z}\}$ for $0 \leq i \leq p-2$, and blocks $\mathcal{B}(np-1)$ containing one simple module. The non-simple blocks can be described as trivial extensions of radical square zero hereditary algebra of type A_∞^∞.

2. Auslander-Reiten Components

Let $\mathcal{F} : \operatorname{mod}_{X(T)} U_0(\mathfrak{g}) \longrightarrow \operatorname{mod} U_0(\mathfrak{g})$ be the forgetful functor. Since $X(T) \cong \mathbb{Z}^n$ is a free abelian group, we can apply the results by Gordon and Green on modules over graded Artin algebras:

Theorem 2.1 (Gordon-Green, 1982)**.** (1) *The category* $\operatorname{mod}_{X(T)} U_0(\mathfrak{g})$ *has almost split sequences.*

(2) *The functor* $\mathcal{F} : \operatorname{mod}_{X(T)} U_0(\mathfrak{g}) \longrightarrow \operatorname{mod} U_0(\mathfrak{g})$ *sends indecomposables to indecomposables and almost split sequences to almost split sequences.*

In view of (1.1), the foregoing statement remains true if $\operatorname{mod}_{X(T)} U_0(\mathfrak{g})$ is replaced by $\operatorname{mod} G_1T$. We can thus speak of the stable Auslander-Reiten quiver $\Gamma_s(G_1T)$ of $\operatorname{mod} G_1T$. We want to study $\Gamma_s(G_1T)$ and the stable AR-quiver $\Gamma_s(\mathfrak{g})$ of $U_0(\mathfrak{g})$ by means of rank varieties.

Recall that

$$\mathcal{V}_{\mathfrak{g}} := \{x \in \mathfrak{g} \ ; \ x^{[p]} = 0\}$$

is the *nullcone* of $\mathfrak{g}$. Given $M \in \operatorname{mod} U_0(\mathfrak{g})$, we define the *rank variety* of M via

$$\mathcal{V}_{\mathfrak{g}}(M) := \{x \in \mathcal{V}_{\mathfrak{g}} \ ; \ M|_{U_0(kx)} \text{ is not projective}\} \cup \{0\}.$$

If $M \in \operatorname{mod} G_1T$, then we put $\mathcal{V}_{\mathfrak{g}}(M) := \mathcal{V}_{\mathfrak{g}}(\mathcal{F}(M))$.

Here are some facts concerning rank varieties and AR-components:

- A module $M \in \operatorname{mod} G_1T$ is projective if and only if $\dim \mathcal{V}_{\mathfrak{g}}(M) = 0$.
- If $\Theta \subset \Gamma_s(G_1T)$ is a component, then we have

  $$\mathcal{V}_{\mathfrak{g}}(M) = \mathcal{V}_{\mathfrak{g}}(N) \qquad \forall \ [M], [N] \in \Theta.$$

  Accordingly, we can speak of the variety $\mathcal{V}_{\mathfrak{g}}(\Theta)$ of the AR-component Θ.
- Using rank varieties one can show that Webb's Theorem holds for the components of $\Gamma_s(G_1T)$.

Recall that

$$\mathfrak{g} = \bigoplus_{\alpha \in R \cup \{0\}} \mathfrak{g}_\alpha \quad ; \quad R \subset X(T) \setminus \{0\}$$

is the root space decomposition of $\mathfrak{g}$ relative to T. Since G is reductive, we have $\mathfrak{g}_0 = \operatorname{Lie}(T) = \mathfrak{h}$ as well as $\dim_k \mathfrak{g}_\alpha = 1$ for all $\alpha \in R$.

Theorem 2.2. *The following statements hold:*

(1) *Let M be a G_1T-module. Then $\mathcal{V}_{\mathfrak{g}}(M)$ is a T-invariant conical subvariety of $\mathcal{V}_{\mathfrak{g}}$.*

(2) *If $\mathcal{V} \subset \mathcal{V}_{\mathfrak{g}}$ is a conical, closed, T-invariant subset, then there exists $M \in \operatorname{mod} G_1T$ such that $\mathcal{V} = \mathcal{V}_{\mathfrak{g}}(M)$.*

(3) *Let M be an indecomposable G_1T-module such that $\dim \mathcal{V}_{\mathfrak{g}}(M) = 1$. Then there exists a root $\alpha_M \in R$ such that*

  (a) *$\mathcal{V}_{\mathfrak{g}}(M) = \mathfrak{g}_{\alpha_M}$, and*

  (b) *$\tau_{G_1T}(M) \cong M \otimes_k k_{p\alpha_M}$.*

Since $X(T)$ is torsion free, it follows from (3) that $\Gamma_s(G_1T)$ has no τ_{G_1T}-periodic vertices.

Theorem 2.3. *Let $\Theta \subset \Gamma_s(G_1T)$ be a component.*

(1) *We have $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[A_\infty^\infty]$, $\mathbb{Z}[D_\infty]$.*

(2) *If $\dim \mathcal{V}_{\mathfrak{g}}(\Theta) \neq 2$, then $\Theta \cong \mathbb{Z}[A_\infty]$.*

Remarks. (1) If $\Theta \subset \Gamma_s(\mathrm{SL}(2)_1T)$ has a rank variety of dimension 2, then $\Theta \cong \mathbb{Z}[A_\infty^\infty]$.

(2) I do not know whether components of tree class D_∞ can occur.

3. Modules with a good filtration

The main advantage of working in $\operatorname{mod} G_1T$ rather than $\operatorname{mod} U_0(\mathfrak{g})$ rests on $\operatorname{mod} G_1T$ being a highest weight category in the sense of Cline-Parshall-Scott. The projective indecomposable objects in $\operatorname{mod} G_1T$ are indexed by elements of $X(T)$: Given $\lambda \in X(T)$ we let $\hat{P}(\lambda)$ and $\hat{L}(\lambda)$ be the projective indecomposable and the simple G_1T-module of highest weight λ, respectively. We also consider the G_1T-module

$$\hat{Z}(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} k_\lambda,$$

where $t(u \otimes \alpha) := \operatorname{Ad}(t)(u) \otimes \lambda(t)\alpha$ and $\mathfrak{h}$ acts on k_λ via the differential $d\lambda \in X(T)/pX(T) = X(T_1)$. We have

- $\operatorname{Top}(\hat{Z}(\lambda)) = \hat{L}(\lambda)$, and
- $[\hat{Z}(\lambda) : \hat{L}(\lambda)] = 1$.

The following result, often referred to as BGG duality or Brauer-Humphreys reciprocity, was one of Jantzen's main objectives:

Theorem 3.1 (Jantzen, 1979)**.** *Given $\lambda \in X(T)$, the module $\hat{P}(\lambda)$ has a $\hat{Z}$-filtration and*

$$(\hat{P}(\lambda) : \hat{Z}(\mu)) = [\hat{Z}(\mu) : \hat{L}(\lambda)].$$

Let R^+ be the set of positive roots of $\mathfrak{g}$, corresponding to a choice of a Borel subgroup $B \subset G$. We define a partial ordering on $X(T)$ via

$$\lambda \leq \mu :\Leftrightarrow \mu - \lambda \in \mathbb{N}_0 R^+.$$

Relative to this ordering we have

$$\hat{Z}(\lambda) \cong \hat{P}(\lambda)/M(\lambda),$$

where $M(\lambda) = \sum_{\mu > \lambda} \operatorname{im}(\hat{P}(\mu) \longrightarrow \hat{P}(\lambda))$. This means that the $\hat{Z}(\lambda) = \Delta(\lambda)$ are the standard modules in the highest weight category $\operatorname{mod} G_1T$, and that Jantzen's $\hat{Z}$-filtration is a Δ-good filtration. The costandard modules are given by

$$\bigtriangledown(\lambda) := \hat{Z}'(\lambda) := U_0(\mathfrak{g}) \otimes_{U_0(\mathfrak{b}^-)} k_{\lambda - 2(p-1)\varrho},$$

where $\mathfrak{b}^- := \mathfrak{h} \oplus \mathfrak{g}^-$ is the opposite Borel subalgebra, and $\varrho := \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. We denote by $F(\Delta)$ and $F(\bigtriangledown)$ the full subcategories of $\operatorname{mod} G_1T$, whose objects afford a Δ-filtration and a $\bigtriangledown$-filtration, respectively.

Lemma 3.2. *Let M be a G_1T-module.*
1. *$M \in F(\Delta)$ if and only if $\mathcal{V}_{\mathfrak{g}}(M) \cap \mathfrak{b}^- = \{0\}$.*
2. *$M \in F(\bigtriangledown)$ if and only if $\mathcal{V}_{\mathfrak{g}}(M) \cap \mathfrak{b} = \{0\}$.*

In particular, $F(\Delta)$ is closed under extensions, direct summands, and tensor products.

Theorem 3.3 (Ringel, 1991)**.** *The subcategory $F(\Delta)$ has relative almost split sequences.*

Strictly speaking, Ringel showed this for quasi-hereditary algebras, but his arguments also apply in our context. Since $\mathcal{V}_{\mathfrak{g}}(\hat{Z}(\lambda)) \subset \mathcal{V}_{\mathfrak{b}}$ and $\mathcal{V}_{\mathfrak{g}}(\hat{Z}'(\lambda)) \subset \mathcal{V}_{\mathfrak{b}^-}$, we obtain

$$\mathcal{V}_{\mathfrak{g}}(M) \subset \mathcal{V}_{\mathfrak{b}} \cap \mathcal{V}_{\mathfrak{b}^-} = \{0\}$$

for every $M \in F(\Delta) \cap F(\bigtriangledown)$. Thus, our tilting modules are projective, and the infinite-dimensional characteristic tilting module does not determine $F(\Delta)$.

The following result can be viewed as an interpretation of (3.3). In our particular context the relative almost split sequences are almost split within the category $\operatorname{mod} G_1T$:

Proposition 3.4. *Let M be an indecomposable G_1T-module, $\Theta \subset \Gamma_s(G_1T)$ and $\Psi \subset \Gamma_s(\mathfrak{g})$ the stable AR-components containing M and $\mathcal{F}(M)$, respectively.*

(1) *Every vertex of Ψ has a G_1T-structure.*

(2) *If $M \in F(\Delta)$, then every vertex of Θ belongs to $F(\Delta)$.*

(3) *If $\mathcal{F}(M)$ has a Z-filtration, so does every vertex of Ψ.*

The third statement illustrates the utility of $\operatorname{mod} G_1T$ in the study of $\operatorname{mod} U_0(\mathfrak{g})$.

Theorem 3.5. *The following statements hold:*

(1) *The module $\hat{L}(\lambda)$ is either projective, quasi-simple, or it belongs to a component of type $\mathbb{Z}[A_\infty^\infty]$.*

(2) *The module $\hat{Z}(\lambda)$ is either projective or quasi-simple.*

(3) *Every component of $\Gamma_s(G_1T)$ contains at most one $\hat{L}(\lambda)$ and at most one $\hat{Z}(\lambda)$, but not both.*

Remark. The proof of (3) employs formal characters and relies on the fact that $\mathbb{Z}[X(T)]$ is an integral domain. Working in $\operatorname{mod} U_0(\mathfrak{g})$ would involve $\mathbb{Z}[X(T)/pX(T)] \cong \mathbb{Z}[X_1, \ldots, X_n]/(X_1^p, \ldots, X_n^p)$. However, using the functor $\mathcal{F}$ one obtains the analogous result for $\Gamma_s(\mathfrak{g})$.

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